Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 15 (1913)

Heft: 1: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE PRINCIPLES OF MATHEMATICS IN RELATION TO ELEMENTARY TEACHING

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DOI: https://doi.org/10.5169/seals-14852

THE PRINCIPLES OF MATHEMATICS IN RELATION TO ELEMENTARY TEACHING[1]

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We are concerned not with the advanced teaching of a few specialist mathematical students, but with the mathematical education of the majority of boys in our secondary schools. Again these boys must be divided into two sections; one section consists of those who desire to restrict their mathematical education, the other section is made up of those who will require some mathematical training for their subsequent professional careers, either in the form of definitive mathematical results or in the form of mathematically trained minds.

I shall call the latter of these two sections the mathematical section, and the former the non-mathematical section. But I must repeat that by the mathematical section is meant that large number of boys who desire to learn more than the minimum amount of mathematics. Furthermore most of my remarks about these sections of boys apply also to elementary classes of our university students.

The subject of this paper is the investigation of the place which should be occupied by modern investigations respecting Mathematical Principles in the education of both of these sections of schoolboys.

To find a foothold from which to start the enquiry, let us ask, why the non-mathematical section should be taught any mathematics at all beyond the barest elements of arithmetic. What are the qualities of mind which a mathematical training is designed to produce when it is employed as an element in a liberal education?

My answer, which applies equally to both sections of students,

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[1] Conférence présentée à la Section IV (Philosophie et enseignement) du 5e Congrès international des mathématiciens. Cambridge, août 1912.

is that there are two allied forms of mental discipline which should be acquired by a well designed mathematical course. These two forms though closely allied are perfectly distinct.

The first form of discipline is not in its essence logical at all. It is the power of clearly grasping abstract ideas, and of relating them to particular circumstances. In other words, the first use of mathematics is to strengthen the power of abstract thought. I repeat that in its essence this has nothing to do with logic, though as a matter of fact a logical discipline is the best method of producing the desired effect. It is not the philosophical theory of abstract ideas which is to be acquired, but the habit and the power of using them. Now there is one and only one way of acquiring the habit and the power of using anything, that is by the simple commonplace method of habitually using it. There is no other short cut. If in education we desire to produce a certain conformation of mind, we must day by day, and year by year, accustom the students' minds to grow into the desired structural shape. Thus to teach the power of grasping abstract ideas and the habit of using them, we must select a group of such ideas, which are important and are also easy to think about because they are clear and definite.

The fundamental mathematical truths concerning Geometry, Ratio, Quantity, and Number, satisfy these conditions as do no others. Hence the fundamental universal position held by mathematics as an element of a liberal education.

But what are the Fundamental Mathematical truths concerning Geometry, Quantity and Number? At this point we come to the great question of the relation between the modern science of the Principles of Mathematics and a Mathematical Education.

My answer to the question as to these Fundamental Mathematical truths is that, in any absolute sense, there are none. There is no unique small body of independent primitive unproved propositions which are the necessary starting-points of all mathematical reasoning on these subjects. In mathematical reasoning the only absolute necessary presuppositions are those which make logical deduction possible. Between these absolute logical truths and so-called Fundamental truths concerning Geometry, Quantity and Number, there is a whole new world of mathematical subjects concerning the logic of propositions, of classes, and of relations. But this subject is too abstract to form an elementary training ground in the difficult art of abstract thought.

It is for this reason that we have to make a compromise and start from such obvious general ideas as naturally occur to all men when they perceive objects with their senses.

In Geometry, the ideas elaborated by the Greeks and presented by Euclid are, roughly speaking, those adapted for our purpose, namely, ideas of volumes, surfaces, lines, of straightness and of curvature, of intersection and of congruence, of greater and less, of similarity, shape, and scale. In fact, we use in education those general ideas of spatial properties which must be habitually present in the mind of anyone who is to observe the world of phenomena with understanding.

Thus we come back to Plato's opinion that for a liberal education, Geometry, as he knew it, is the queen of sciences.

In addition to Geometry, there remain the ideas of quantity, ratio, and of number. This in practice means Elementary Algebra. Here the prominent ideas are those of "any number", in other words, the use of the familiar x, y, z, and of the dependence of variables on each other, or otherwise, the idea of functionality. All this is to be gradually acquired by the continual use of the very simplest functions which we can devise, of linear functions, graphically represented by straight lines, of quadratic functions graphically represented by parabolas, and of those simple implicit functions graphically represented by conic sections. Thence, with good fortune and a willing class, we can advance to the ideas of rates of increase, still confining ourselves to the simplest possible cases.

I wish here emphatically to remind you that both in Geometry and in Algebra a clear grasp of these general ideas is not what the pupil starts from, it is the goal at which he is to arrive. The method of progression is continual practice in the consideration of the simplest particular cases, and the goal is not philosophical analysis but the power of use.

But how is he to practise himself in their use. He cannot simply sit down and think of the relation $y = x + 1$, he must employ it in some simple obvious way.

This brings us to the second power of mind which is to be produced by a mathematical training, namely, the power of logical reasoning. Here again, it is not the knowledge of the philosophy of logic which it is essential to teach, but the habit of thinking logically. By logic, I mean deductive logic.

Deductive logic is the science of certain relations, such as implication, etc., between general ideas. When logic begins, definite particular individual things have been banished. I cannot relate logically this thing to that thing, for example this pen to that pen, except by the indirect way of relating some general idea which applies to this pen to some general idea which applies to that pen. And the individualities of the two pens is quite irrelevant to the logical process. This process is entirely concerned with the two general ideas. Thus the practice of logic is a certain way of employing the mind in the consideration of such ideas; and an elementary mathematical training is in fact nothing else but the logical use of the general ideas respecting Geometry and Algebra which we have enumerated above. It has therefore, as I started this paper by stating, a double advantage. It makes the mind capable of abstract thought, and it achieves this object by training the mind in the most important kind of abstract thought, namely, deductive logic.

I may remind you that other choices of a type of abstract thought might be made. We might train the children to contemplate directly the beauty of abstract moral ideas, in the hope of making them religious mystics. The general practice of education has decided in favour of logic, as exemplified in elementary mathematics.

We have now to answer the further question, what is the *rôle* of Logical Precision in the Teaching of Mathematics ? Our general answer to the implied question is obvious, Logical Precision is one of the two objects of the teaching of mathematics, and it is the only weapon by which the teaching of mathematics can achieve its other object. To teach mathematics is to teach logical precision. A mathematical teacher who has not taught that has taught nothing.

But having stated this thesis in this unqualified way, its meaning must be carefully explained ; for otherwise its real bearing on the problem of education will be entirely misunderstood.

Logical precision is the faculty to be acquired. It is the quality of mind which it is the object of the teaching to impart. Thus the habit of reading great literature is the goal at which a literary education aims. But we do not expect a child to start its first lesson by reading for itself Shakespeare. We recognize that reading is impossible till the pupil has learnt its alphabet and can spell, and then we start it with books of one syllable.

In the same way, a mathematical education should grow in logical precision. It is folly to expect the same careful logical analysis at the commencement of the training as would be appropriate at the end. It is an entire misconception of my thesis to construe it as meaning that a mathematical training should assume in the pupil a power of concentrated logical thought. My thesis is in fact the exact opposite, namely that this power cannot be assumed, and has got to be acquired, and that a mathematical training is nothing else than the process of acquiring it. My whole groundwork of assumption is that this power does not initially exist in a fully developed state. Of course like every other power which is acquired, it must be developed gradually.

The various stages of development must be guided by the judgment and the genius of the teacher. But what is essential is, that the teacher should keep clearly in his mind that it is just this power of logical precise reasoning which is the whole object of his efforts. If his pupils have in any measure gained this, they have gained all.

We have not yet however fully considered this part of our subject. Logical precision is the full realization of the steps of the argument. But what are the steps of the argument? The full statement of all the steps is far too elaborate and difficult an operation to be introduced into the mathematical reasoning of an educational curriculum. Such a statement involves the introduction of abstract logical ideas which are very difficult to grasp because there is so rarely any need to make them explicit in ordinary thought. They are therefore not a fit subject-ground for an elementary education.

I do not think that it is possible to draw any theoretical line between those logical steps which form a theoretically full logical investigation, and those which are full enough for most practical purposes, including that of education. The question is one of psychology, to be solved by a process of experiment. The object to be attained is to gain that amount of logical alertness which will enable its possessors to detect fallacy and to know the types of sound logical deduction. The objects of going further are partly philosophical, and also partly to lay bare abstract ideas whose investigation is in itself important. But both these objects are foreign to education.

My own opinion is that, on the whole, the type of logical precision handed down to us by the Greek mathematicians is roughly speaking what we want. In Geometry, this means the sort of precision which we find in Euclid. I do not mean that we should use his famous *Elements* as a textbook, nor that here and there a certain compression in his mode of exposition is not advisable.

All this is mere detail. What I do mean is that the sort of logical transition which he made explicit, we should make explicit, and that the sort of transition which he omits, we should omit.

I doubt however whether it is desirable to plunge the student into the full rigour of Euclidean Geometry without some mitigation. It is for this reason that the modern habit, at least in England, of laying great stress, in the initial stages, on training the pupil in simple constructions from numerical data is to be praised. It means that after a slight amount of reasoning on the Euclidean basis of accuracy, the mind of the learner is relieved by doing the things in various special cases, and noting by rough measurements that the desired results are actually attained. It is important however that the measurements be not mistaken for the proofs. Their object is to make the beginner apprehend what the abstract ideas really mean.

Again in algebra, the notation and the practical use of the symbols should be acquired in the simplest cases, and the more theoretical treatment of the symbolism reserved to a suitable later stage. In fact my rule would be initially to learn the meaning of the ideas by a crude practice in simple ways, and to refine the logical procedure in preparation for an advance to greater generality. In fact the thesis of my paper can be put in another way thus, the object of a mathematical education is to acquire the powers of analysis, of generalization, and of reasoning. The two processes of analysis and generalization were in my previous statement put together as the power of grasping abstract ideas.

But in order to analyse and to generalize, we must commence with the crude material of ideas which are to be analysed and generalized. Accordingly it is a positive error in education to start with the ultimate products of this process, namely the ideas in their refined analysed and generalized forms. We are thereby skipping an important part of the training, which is to take the ideas as they actually exist in the child's mind, and to exercise the child in the difficult art of civilizing them and clothing them.

The schoolmaster is in fact a missionary, the savages are the ideas in the child's mind; and the missionary shirks his main task if he refuse to risk his body among the cannibals.

At this point I should like to turn your attention to those pupils forming the mathematical section. There is an idea, widely prevalent, that it is possible to teach mathematics of a relatively advanced type — such as differential calculus, for instance — in a way useful to physicists and engineers without any attention to its logic or its theory.

This seems to me to be a profound mistake. It implies that a merely mechanical knowledge without understanding of ways of arriving at mathematical results is useful in applied science. It seems to me to be of no use whatever. The results themselves can all be found stated in the appropriate pocket-books and in other elementary works of reference. No one when applying a result need bother himself as to why it is true. He accepts it and applies it. What is of supreme importance in physics and in engineering is a mathematically trained mind, and such a mind can only be acquired by a proper mathematical discipline.

I fully admit that the proper way to start such a subject as the Differential Calculus is to plunge quickly into the use of the notation in a few absurdly simple cases, with a crude explanation of the idea of rates of increase. The notation as thus known can then be used by the lecturers in the Physical and Engineering Laboratories. But the mathematical training of the applied scientists consists in making these ideas precise and the proofs accurate.

I hope that the thesis of this paper respecting the position of logical precision in the teaching of mathematics has been rendered plain. The habit of logical precision with its necessary concentration of thought upon abstract ideas is not wholly possible in the initial stages of learning. It is the ideal at which the teacher should aim. Also logical precision, in the sense of logical explicitness, is not an absolute thing: it is an affair of more or less. Accordingly the quantity of explicitness to be introduced at each stage of progress must depend upon the practical judgment of the teacher. Lastly, in a sense, the instructed mind is less explicit; for it travels more quickly over a well-remembered path, and may save the trouble of putting into words trains of thought which are very obvious to it. But on the other hand it atones for this rapidity by a concentration on every subtle point where a fallacy can lurk. The habit of logical precision is the instinct for the subtle difficulty.

Résumé. — Le rapport ci-dessus traite de la place à donner aux recherches mathématiques modernes dans les écoles secondaires anglaises et principalement pour l'ensemble des jeunes gens qui désirent réduire leur éducation mathématique au minimum, section non-mathématique, par opposition à la section mathématique, qui comprendrait ceux qui recherchent des connaissances ou un développement mathématique plus complet.

M. Whitehead prend comme point de départ les questions suivantes: Pourquoi, à part l'arithmétique la plus élémentaire, enseignerait-on des mathématiques à la section non-mathématique? Quelles sont les qualités de l'esprit qu'une éducation mathématique est destinée à former lorsqu'elle est considérée comme un élément dans une éducation générale?

Pour y répondre, M. Whitehead considère les deux résultats vers lesquels doit tendre l'éducation mathématique. Premièrement développer la *faculté d'abstraction,* ce qui n'est possible qu'en l'appliquant à des groupes d'idées qui s'y prêtent, tels que: en première ligne les principes fondamentaux relatifs à la géométrie, aux proportions, aux notions de quantité, de nombre. Ceci au début dans des cas particuliers très simples et pour des idées générales évidentes pour tous.

La deuxième faculté mentale à développer est la *faculté de raisonnement logique* (logique déductive). Enseigner les mathématiques doit être enseigner la précision logique. Cette précision est non seulement un but pour elle-même, mais aussi l'instrument qui permet d'atteindre à la faculté d'abstraction.

Quant au degré de précision logique à rechercher, M. Whitehead estime qu'elle est approximativement celle des mathémati-

ciens grecs, mais qu'il ne faut pas oublier qu'elle doit être obtenue par approximations successives et qu'elle est le but et non le point de départ de l'enseignement.

Au sujet de la section mathématique et principalement des futurs physiciens et ingénieurs, il estime que l'éducation mathématique doit donner des notions précises avec leurs démonstrations exactes et ne pas se contenter de procédés appliqués mécaniquement.

(La Rédaction.)

SUR LA CLASSIFICATION ET LA CONSTRUCTION DES COURBES TRANSCENDANTES

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1. Au sujet de mon article *Courbes transcendantes et interscendantes*, paru dans l'*Enseignement Mathématique* (mai 1912, pp. 209-214), M. Gino Loria a fait, dans le numéro suivant (pp. 291-293), quelques remarques dont l'importance n'échappera à aucune des personnes qui s'occupent des courbes particulières.

Jusqu'à présent, en effet, les courbes interscendantes n'avaient été citées qu'en passant par quelques auteurs, et leur topologie, ainsi que l'écrit M. G. Loria est toute à faire. Ce que l'on en avait dit de plus intéressant peut se résumer dans ces brèves et précises considérations d'Euler :

«.... De là naît la première espèce et comme la plus simple des « courbes transcendantes ; ce sont celles dont l'équation renferme « des exposants irrationnels. Comme il n'entre dans leur expression « ni logarithmes, ni arcs de cercles et qu'elles proviennent de « la seule considération des nombres irrationnels, elles paraissent « en quelque sorte appartenir à la Géométrie ordinaire; et c'est « pour cette raison que Leibniz les a appelées *interscendantes*, « comme si elles tenaient un certain milieu entre les courbes algé- « briques et les courbes transcendantes.

« On aura donc une courbe interscendante dans celle qui est « exprimée par l'équation $y = x^{\sqrt{2}}$... Si nous nous contentons « de prendre seulement une valeur approchée de $\sqrt{2}$ en mettant « à sa place quelques-unes des fractions $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$ qui ex- « priment à peu près la valeur de $\sqrt{2}$, nous aurons bien à la vé-