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Autor(en): **Dimitrov, Nikolay**

Objektyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **68 (2013)**

PDF erstellt am: **18.04.2024**

Persistenter Link: <https://doi.org/10.5169/seals-515903>

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## On some results related to Napoleon configurations

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Nikolay Dimitrov

Nikolay Dimitrov completed his undergraduate studies at Sofia University “Saint Kliment Ohridski” in Bulgaria with a major in mathematics. In 2009 he received a doctoral degree in mathematics from Cornell University. Currently, he is a postdoctoral fellow at McGill University and Centre de Recherches Mathématiques in Montreal.

### 1 Introduction

In this short article we discuss some results from planar Euclidean geometry which have a close connection to Napoleon’s theorem. They are summarized in Theorem 1. The statement of Theorem 1 appears in [1] where the proof is based on coordinate descriptions and algebraic computations. Since both Theorem 1 and Napoleon’s theorem (see Theorem 2) are elementary geometric results, it makes sense to provide a proof that remains in the same simple geometric domain. For that reason, the arguments presented in the current paper are entirely in the spirit of synthetic Euclidean geometry and use only geometric methods with almost no algebraic computations. Thus, one gets a better feeling for the geometry and the properties of Napoleon configurations.

**Definition 1.** Let  $\triangle ABC$  be an arbitrary triangle. We say that the points  $A_1, B_1$  and  $C_1$  form a non-overlapping Napoleon configuration for the triangle  $\triangle ABC$  if all three triangles  $\triangle ABC_1$ ,  $\triangle AB_1C$  and  $\triangle A_1BC$  are equilateral and no one of them overlaps with  $\triangle ABC$  (see Figure 1). Alternatively, we say that the points  $A'_1, B'_1$  and  $C'_1$  form an overlapping Napoleon configuration for  $\triangle ABC$  if all three triangles  $\triangle ABC'_1$ ,  $\triangle AB'_1C$  and  $\triangle A'_1BC$  are equilateral and all of them overlap with  $\triangle ABC$ .

Um den Satz von Napoleon kreisen in der Euklidischen Geometrie zahlreiche Varianten. Bekannt sind etwa die Kiepert-Dreiecke und deren schöne Eigenschaften. Branko Grünbaum hat 2001 eine besonders ausführliche Version des Satzes von Napoleon formuliert, in der zahlreiche neue Eigenschaften der Konfiguration beschrieben werden. Grünbaum benutzt in seinem Beweis Methoden der analytischen Geometrie. Der Autor der vorliegenden Arbeit beweist nun Grünbaums Variante des Satzes mit elementaren Methoden der synthetischen Geometrie, die sich darüberhinaus als besonders anschaulich erweisen.

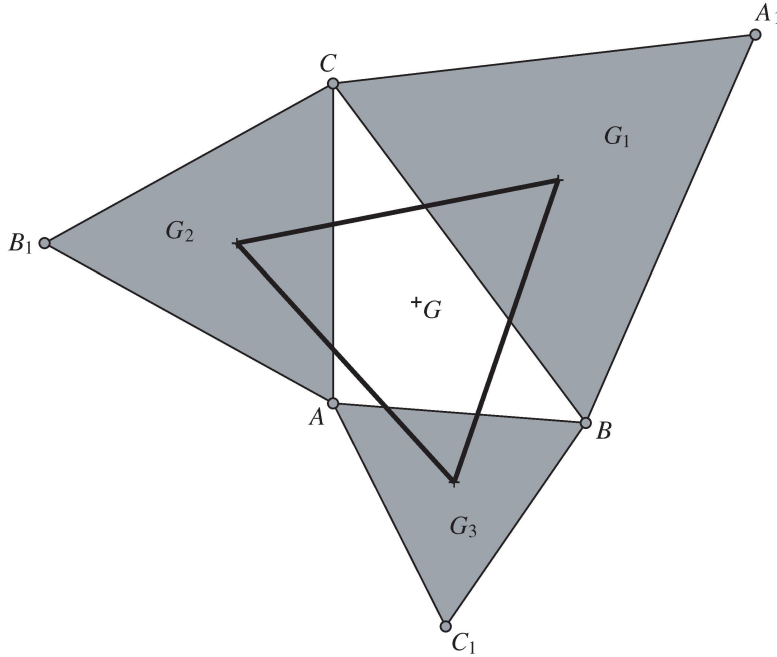


Fig. 1 A non-overlapping Napoleon configuration and the first part of Napoleon's theorem.

## 2 The Main Result

The main result of the current article is the following theorem.

**Theorem 1.** *Let us have an arbitrary triangle  $\triangle ABC$  and let  $A_1$ ,  $B_1$  and  $C_1$  form a non-overlapping Napoleon configuration for that triangle. Denote the midpoints of  $B_1C_1$ ,  $C_1A_1$  and  $A_1B_1$  by  $A_2$ ,  $B_2$  and  $C_2$  respectively. Also, denote the centroids of the triangles  $\triangle A_1BC$ ,  $\triangle AB_1C$  and  $\triangle ABC_1$  by  $G_1$ ,  $G_2$  and  $G_3$  respectively. Then the following statements are true:*

1. *The triangles  $\triangle A_2B_2C$ ,  $\triangle AB_2C_2$  and  $\triangle A_2BC_2$  are equilateral;*
2. *The centroids  $A^*$ ,  $B^*$ ,  $C^*$  of  $\triangle AB_2C_2$ ,  $\triangle A_2B_2C$ ,  $\triangle A_2BC_2$  respectively are vertices of an equilateral triangle, whose centroid coincides with the centroid  $G$  of  $\triangle ABC$ ;*

*Similarly, let  $A'_1$ ,  $B'_1$  and  $C'_1$  be an overlapping Napoleon configuration for  $\triangle ABC$ . Denote the midpoints of  $B'_1C'_1$ ,  $C'_1A'_1$  and  $A'_1B'_1$  by  $A'_2$ ,  $B'_2$  and  $C'_2$  respectively. Also, denote the centroids of triangles  $\triangle A'_1BC$ ,  $\triangle AB'_1C$  and  $\triangle ABC'_1$  by  $G'_1$ ,  $G'_2$  and  $G'_3$  respectively. Then*

3. *The triangles  $\triangle A'_2B'_2C$ ,  $\triangle AB'_2C'_2$  and  $\triangle A'_2BC'_2$  are equilateral;*
4. *The centroids  $A^{**}$ ,  $B^{**}$ ,  $C^{**}$  of  $\triangle AB'_2C'_2$ ,  $\triangle A'_2B'_2C$ ,  $\triangle A'_2BC'_2$  respectively are vertices of an equilateral triangle, whose centroid coincides with the centroid  $G$  of  $\triangle ABC$ ;*

5. Triangle  $\triangle A^*B^*C^*$  is homothetic to the triangle  $\triangle G'_1G'_2G'_3$  with homothetic center  $G$  and a coefficient of similarity  $-1/2$ ;
6. Triangle  $\triangle A^{**}B^{**}C^{**}$  is homothetic to the triangle  $\triangle G_1G_2G_3$  with homothetic center  $G$  and a coefficient of similarity  $-1/2$ .
7. The area of  $\triangle ABC$  equals four times the algebraic sum of the areas of  $\triangle A^*B^*C^*$  and  $\triangle A^{**}B^{**}C^{**}$ .

### 3 Napoleon's Theorem

Napoleon's theorem is a beautiful result from planar Euclidean geometry and there are various ways to prove it. In order to make this article more self-contained, we present one possible geometrically oriented proof. Before we state and prove Napoleon's theorem we are going to need the following lemma.

**Lemma 1.** *Given an arbitrary triangle  $\triangle ABC$ , let  $A_1$ ,  $B_1$  and  $C_1$  form a non-overlapping Napoleon configuration for that triangle. Then, the following properties are true:*

1. The segments  $AA_1$ ,  $BB_1$  and  $CC_1$  are of equal length. In other words,  $AA_1 = BB_1 = CC_1$ ;
2. They intersect at a common point, denoted by  $J$ ;
3.  $\angle AJB = \angle BJC = \angle CJA = 120^\circ$ ;
4. The circles  $K_1$ ,  $K_2$  and  $K_3$  circumscribed around the equilateral triangles  $\triangle A_1BC$ ,  $\triangle AB_1C$  and  $\triangle ABC_1$  respectively pass through the point  $J$  (see Figure 2).

*Proof.* Perform a  $60^\circ$  rotation  $R_A$  around the point  $A$  in counterclockwise direction. Since  $AC = AB_1$  and  $\angle CAB_1 = 60^\circ$ , the point  $C$  is mapped to the point  $B_1$ . Similarly,  $C_1$  is mapped to  $B$ . Therefore the segment  $CC_1$  maps to the segment  $B_1B$ . This implies that  $BB_1 = CC_1$  (see Figure 2). Moreover, if we denote by  $J$  the intersection point of  $BB_1$  and  $CC_1$ , then  $\angle CJB_1 = \angle C_1JB = 60^\circ$  and  $\angle BJC = 180^\circ - \angle CJB_1 = 180^\circ - 60^\circ = 120^\circ$ . We are going to show that the points  $A$ ,  $J$  and  $A_1$  lie on the same line.

Notice that  $\angle CJB_1 = \angle CAB_1 = 60^\circ$ . Therefore the quadrilateral  $CB_1AJ$  is inscribed in a circle  $K_2$ . Then,  $\angle CJA = 180^\circ - \angle AB_1C = 180^\circ - 60^\circ = 120^\circ$ . Since  $\angle BJC + \angle CA_1B = 120^\circ + 60^\circ = 180^\circ$ , the points  $B$ ,  $A_1$ ,  $C$  and  $J$  lie on a circle  $K_1$ . From here we can conclude that  $\angle A_1JC = \angle A_1BC = 60^\circ$ . Then,  $\angle A_1JA = \angle A_1JC + \angle CJA = 60^\circ + 120^\circ = 180^\circ$ . That means that  $J$  belongs to the straight line  $AA_1$ .

If we perform another  $60^\circ$  counterclockwise rotation  $R_B$ , this time around the point  $B$ , it will turn out that  $AA_1$  is mapped to  $C_1C$ . Therefore,  $AA_1 = CC_1$ . Also,  $\angle AJB = 360^\circ - \angle BJC - \angle CJA = 360^\circ - 120^\circ - 120^\circ = 120^\circ$ . Since  $\angle AJB + \angle BC_1A = 120^\circ + 60^\circ = 180^\circ$ , the points  $A$ ,  $C_1$ ,  $B$  and  $J$  lie on a circle  $K_3$ . We see that the circles  $K_1$ ,  $K_2$ ,  $K_3$  all pass through the same point  $J$ . This completes the proof of Lemma 1.  $\square$

**Remark.** The point  $J$  from Lemma 1 (see also Figure 2) is often called Fermat point or alternatively Torricelli point.



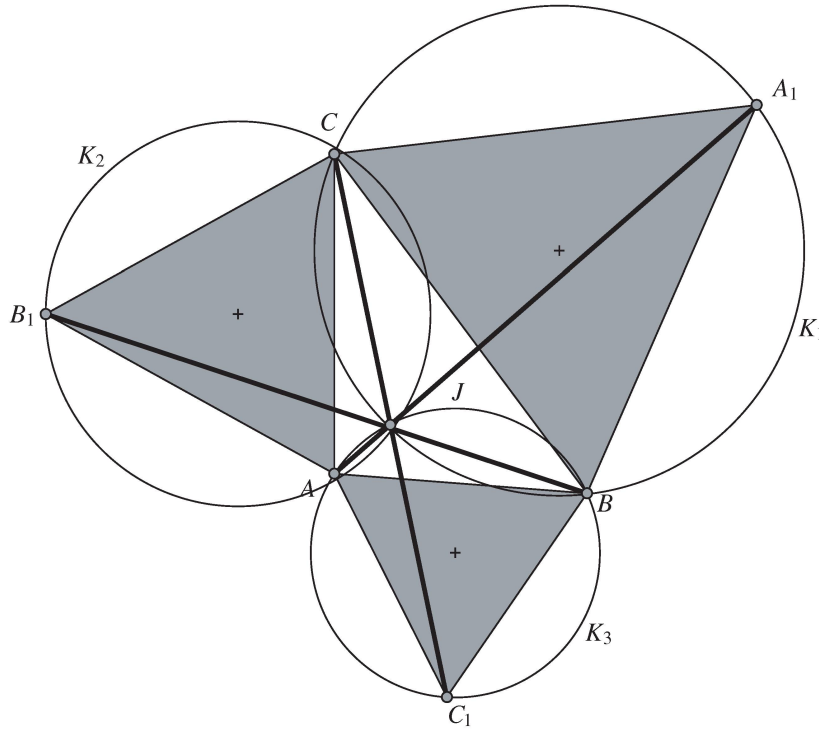


Fig. 2 Constructions in the proof of Lemma 1.

Next, we are ready to state and prove Napoleon's theorem.

**Theorem 2.** *Let  $\triangle ABC$  be an arbitrary triangle and let  $G$  be its centroid. Then, the following statements are true:*

1. *Assume  $A_1$ ,  $B_1$  and  $C_1$  form a non-overlapping Napoleon configuration for that triangle. Denote the centroids of the triangles  $\triangle A_1BC$ ,  $\triangle AB_1C$  and  $\triangle ABC_1$  by  $G_1$ ,  $G_2$  and  $G_3$  respectively. Then, the triangle  $\triangle G_1G_2G_3$  is equilateral with a centroid coinciding with the point  $G$ ;*
2. *Let  $A'_1$ ,  $B'_1$  and  $C'_1$  form an overlapping Napoleon configuration for that triangle. Denote the centroids of triangles  $\triangle A'_1BC$ ,  $\triangle AB'_1C$  and  $\triangle ABC'_1$  by  $G'_1$ ,  $G'_2$  and  $G'_3$  respectively. Then, the triangle  $\triangle G'_1G'_2G'_3$  is equilateral with a centroid coinciding with the point  $G$ ;*
3. *The area of  $\triangle ABC$  equals the algebraic sum of the areas of  $\triangle G_1G_2G_3$  and  $\triangle G'_1G'_2G'_3$ .*

*Proof.* We start with the first claim of the theorem (see also Figure 3). Let  $M_1$ ,  $M_2$  and  $M_3$  be the midpoints of the edges  $BC$ ,  $CA$  and  $AB$  respectively. Since  $G$  is the centroid

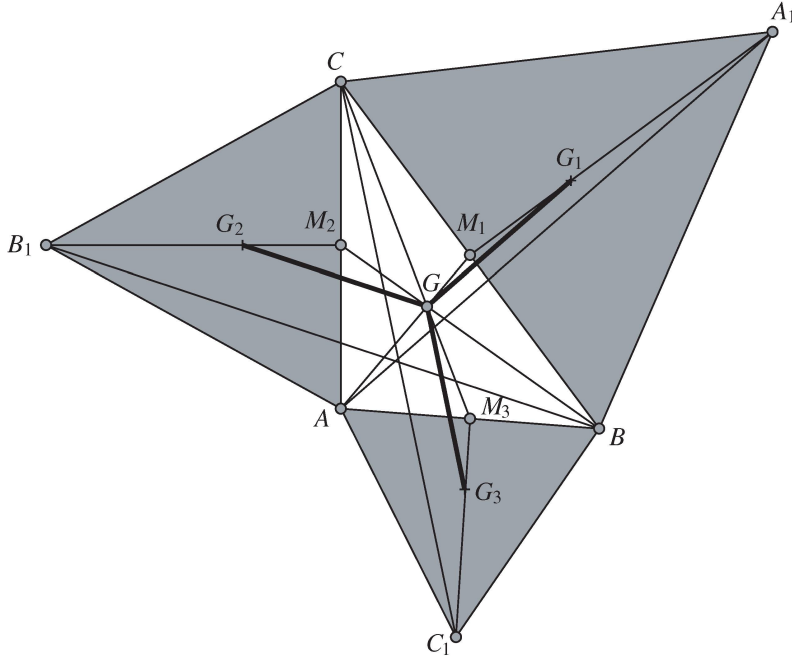


Fig. 3 Constructions in the proof of Napoleon's theorem.

of  $\triangle ABC$  and  $G_2$  is the centroid of  $\triangle AB_1C$ , we have the ratios  $M_2G : M_2B = M_2G_2 : M_2B_1 = 1 : 3$ . Therefore, by the intercept theorem  $GG_2 = \frac{1}{3}BB_1$  and  $GG_2$  is parallel to  $BB_1$ . Analogously,  $GG_1 = \frac{1}{3}AA_1$ ,  $GG_1$  is parallel to  $AA_1$ ,  $GG_3 = \frac{1}{3}CC_1$  and  $GG_3$  is parallel to  $CC_1$ . By part 1 of Lemma 1,  $AA_1 = BB_1 = CC_1$ , hence  $GG_1 = GG_2 = GG_3$ . By part 3 of Lemma 1,  $\angle A_1JB = \angle B_1JC = \angle C_1JA = 120^\circ$ , so  $\angle G_1GG_2 = \angle G_2GG_3 = \angle G_3GG_1 = 120^\circ$ .

We can conclude from here that  $\triangle G_1G_2G \cong \triangle G_2G_3G \cong \triangle G_3G_1G$  and hence  $G_1G_2 = G_2G_3 = G_3G_1$ , that is, the triangle  $\triangle G_1G_2G_3$  is equilateral.

The proof of claim 2 from Napoleon's theorem is analogous to the proof of claim 1. We just have to consider overlapping configurations and rename their notations appropriately.

In order to prove claim 3 from Theorem 2, we are going to show that  $\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2} \text{Area}(\triangle ABC) + \frac{1}{6}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$ . Let point  $P$  be the reflection image of the vertex  $C$  with respect to the line  $G_1G_2$ . In other words,  $P$  is chosen so that  $G_1G_2$  is the perpendicular bisector of  $CP$ . Hence,  $\triangle G_1G_2C \cong \triangle G_1G_2P$  and  $G_2P = G_2C = G_2A$ . If we denote  $\angle G_1G_2C = \alpha$  then  $\angle PG_2G_1 = \alpha$ . On the one hand,  $\angle AG_2P = \angle AG_2C - \angle PG_2C = 120^\circ - \angle PG_2C = 120^\circ - (\angle PG_2G_1 + \angle G_1G_2C) = 120^\circ - 2\alpha$ . On the other hand,  $\angle G_3G_2P = \angle G_3G_2G_1 - \angle PG_2G_1 = 60^\circ - \alpha$ . Therefore,  $\angle AG_2G_3 = \angle AG_2P - \angle G_3G_2P = 120^\circ - 2\alpha - (60^\circ - \alpha) = 60^\circ - \alpha$ . Since  $G_2P = G_2A$  and  $\angle AG_2G_3 = \angle G_3G_2P = 60^\circ - \alpha$ , the line  $G_2G_3$  is the bisector of  $\angle AG_2P$  in the isosceles triangle  $\triangle AG_2P$ , and hence it is the perpendicular

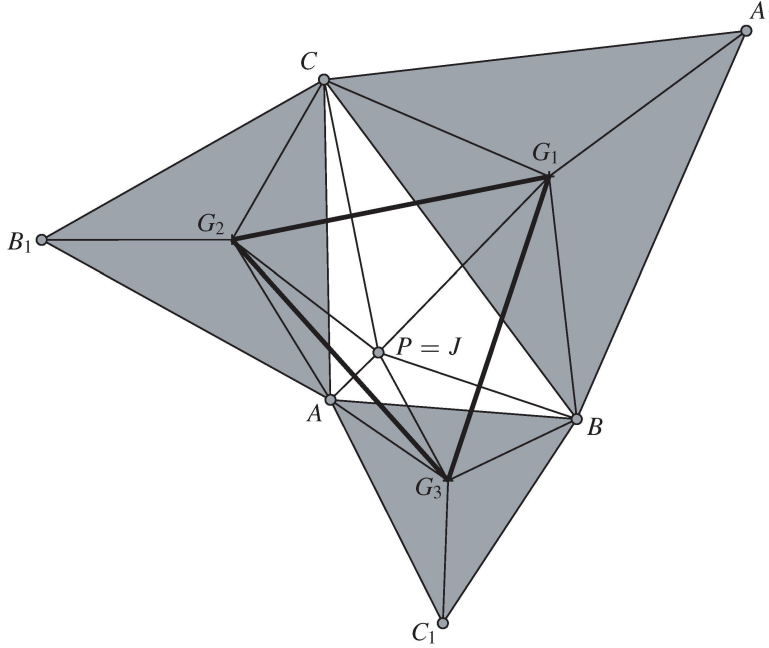


Fig. 4 Constructions in the proof of Napoleon's theorem.

bisector of the segment  $AP$ . Therefore,  $P$  is the reflection image of  $A$  with respect to  $G_2G_3$  and  $\triangle G_2G_3A \cong \triangle G_2G_3P$ . Analogously, we can show that the reflection of  $B$  with respect to  $G_3G_1$  is again  $P$  and  $\triangle G_3G_1B \cong \triangle G_3G_1P$ . All of the arguments above lead to the conclusion that  $\text{Area}(\triangle G_1G_2G_3) = \text{Area}(\triangle G_1G_2P) + \text{Area}(\triangle G_2G_3P) + \text{Area}(\triangle G_3G_1P) = \text{Area}(\triangle G_1G_2C) + \text{Area}(\triangle G_2G_3A) + \text{Area}(\triangle G_3G_1B)$ , so

$$\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2} \text{Area}(AG_3BG_1CG_2).$$

Notice that  $\text{Area}(AG_3BG_1CG_2) = \text{Area}(\triangle ABC) + \text{Area}(\triangle AG_3B) + \text{Area}(\triangle BG_1C) + \text{Area}(\triangle CB_2A) = \text{Area}(\triangle ABC) + \frac{1}{3}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$ . It follows from here that  $\text{Area}(\triangle G_1G_2G_3) = \frac{1}{2} \text{Area}(\triangle ABC) + \frac{1}{6}(\text{Area}(\triangle A_1BC) + \text{Area}(\triangle AB_1C) + \text{Area}(\triangle ABC_1))$ .

Using analogous arguments, one can show that  $\text{Area}(\triangle G'_1G'_2G'_3) = \frac{1}{2} \text{Area}(\triangle ABC) - \frac{1}{6}(\text{Area}(\triangle A'_1BC) + \text{Area}(\triangle AB'_1C) + \text{Area}(\triangle ABC'_1))$ . Now, we can deduce that

$$\text{Area}(\triangle G_1G_2G_3) + \text{Area}(\triangle G'_1G'_2G'_3) = \text{Area}(\triangle ABC).$$

An additional observation is that  $G_1P = G_1B = G_1C = G_1A_1$  and therefore  $P$  lies on the circle  $K_1$ , circumscribed around  $\triangle A_1BC$  (see Lemma 1 and Figure 2). Similarly,  $P$  lies on the circles  $K_2$  and  $K_3$  circumscribed around  $\triangle AB_1C$  and  $\triangle ABC_1$  respectively. That implies that  $P$  is the intersection point of  $K_1$ ,  $K_2$  and  $K_3$ , which was already denoted by  $J$ , i.e.,  $P \equiv J$ .  $\square$

#### 4 Proof of Theorem 1

This section contains the proof of the main result, namely Theorem 1. To prove this statement we are going to use several lemmas and corollaries which together will give us the desired result.

The next lemma is essentially the proof of fact 1 from Theorem 1.

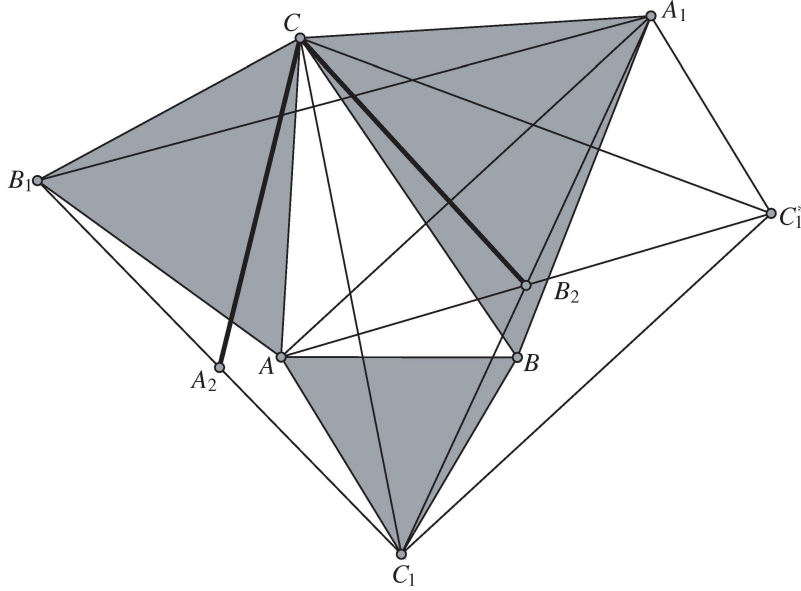


Fig. 5 Constructions in the proof of Lemma 2.

**Lemma 2.** *In the setting of Theorem 1, the points  $A_2$ ,  $B_2$  and  $C$  form an equilateral triangle (see Figure 5).*

*Proof.* Consider a  $60^\circ$  rotation  $R_C$  around the point  $C$  in counterclockwise direction. The point  $B_1$  maps to  $A$ . Denote by  $C_1^*$  the image of the point  $C_1$  (Figure 5). Then,  $B_1C_1$  maps to  $AC_1^*$ . We are going to show that the point  $B_2$  is the image of  $A_2$  under the rotation  $R_C$ . Since the midpoint  $A_2$  of  $B_1C_1$  maps to the midpoint of the image  $AC_1^*$ , we need to prove that  $B_2$  lies on  $AC_1^*$  and is the midpoint of that segment.

By the properties of the rotation  $R_C$ , we have that  $CC_1 = CC_1^*$  and  $\angle C_1CC_1^* = 60^\circ$ . Therefore triangle  $\triangle CC_1C_1^*$  is equilateral and so by Lemma 1 we can deduce that  $C_1C_1^* = CC_1 = AA_1$ .

Notice that the point  $A_1$  is the image of  $B$  under the rotation  $R_C$ . Since  $C_1$  maps to  $C_1^*$  we have that  $BC_1$  maps to  $A_1C_1^*$ . Thus,  $A_1C_1^* = BC_1 = AC_1$ .

The facts that  $CC_1 = AA_1$  and  $A_1C_1^* = AC_1$  imply that the quadrilateral  $AA_1C_1^*C_1$  is a parallelogram. For any parallelogram, the intersection point of the diagonals is the midpoint for both diagonals. That means that the midpoint  $B_2$  of the diagonal  $C_1A_1$  lies

on the diagonal  $AC_1^*$  and is the midpoint of  $AC_1^*$ . Therefore,  $B_2$  is the image of  $A_2$  under the rotation  $R_C$ . Hence,  $CA_2 = CB_2$  and  $\angle A_2CB_2 = 60^\circ$ , i.e., the triangle  $\triangle A_2B_2C$  is equilateral.  $\square$

We are going to need the following intermediate statement.

**Lemma 3.** *Consider the equilateral triangle  $\triangle ABC'_1$ , overlapping  $\triangle ABC$ . Then, the midpoint  $C_2$  of the segment  $A_1B_1$  is also the midpoint of  $CC'_1$  (see Figure 6).*

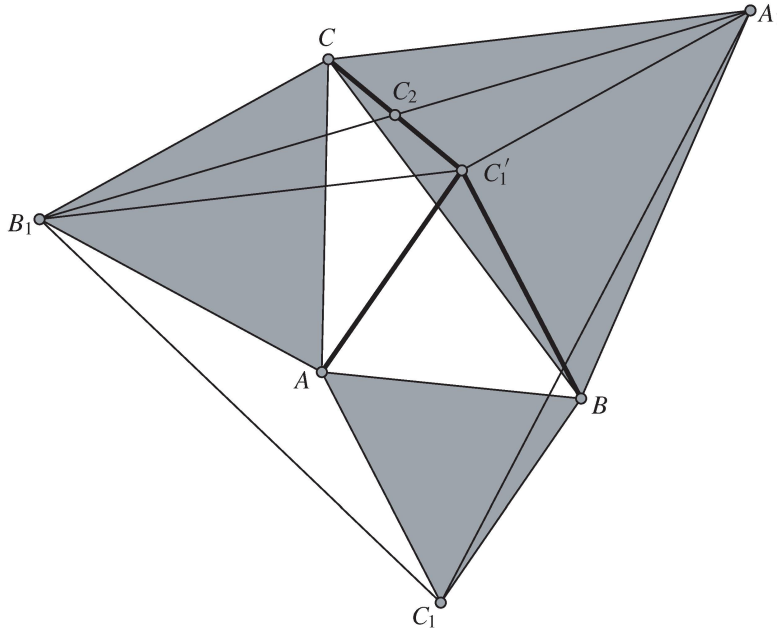


Fig. 6 Constructions in the proof of Lemma 3.

*Proof.* Consider a  $60^\circ$  degree clockwise rotation around the point  $A$ . Then  $B$  maps to  $C'_1$  and  $C$  maps to  $B_1$ . Therefore the segment  $BC$  maps to the segment  $C'_1B_1$ , so  $BC = C'_1B_1$ . Now consider a  $60^\circ$  degree counter-clockwise rotation around the point  $B$ . In this case  $A$  maps to  $C'_1$  and  $C$  maps to  $A_1$ . Thus, the segment  $AC$  maps to  $C'_1A_1$ , so  $AC = C'_1A_1$ . From the two identities  $BC = C'_1B_1$  and  $AC = C'_1A_1$  it can be concluded that the quadrilateral  $B_1C'_1A_1C$  is a parallelogram. Therefore, the midpoint  $C_2$  of the diagonal  $A_1B_1$  is also the midpoint of the diagonal  $CC'_1$ .  $\square$

Next, we are going to locate the centroids of  $\triangle A_1B_1C_1$  and  $\triangle A_2B_2C_2$ .

**Lemma 4.** *The centroids of  $\triangle A_1B_1C_1$  and  $\triangle A_2B_2C_2$  coincide with the centroid  $G$  of  $\triangle ABC$  (see Figure 7).*

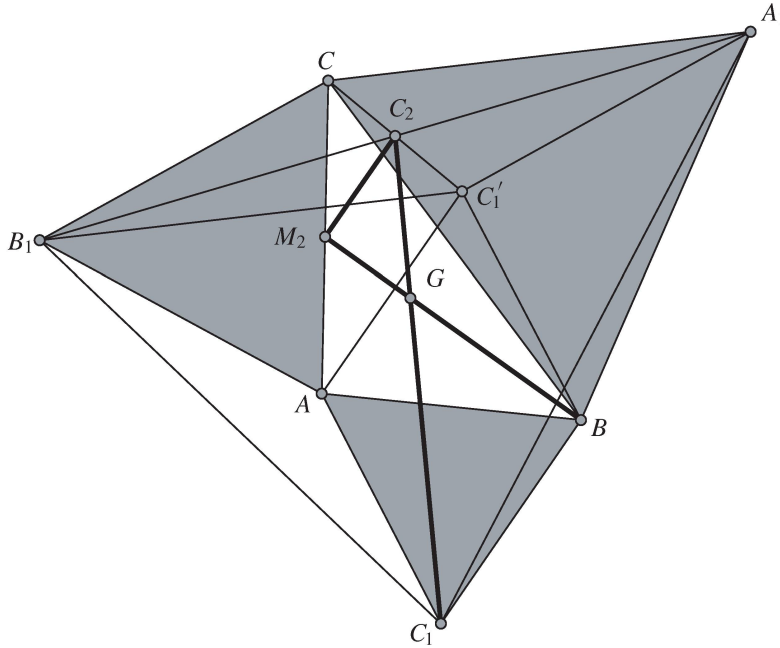


Fig. 7 Constructions in the proof of Lemma 4.

*Proof.* Let  $M_2$  be the midpoint of  $AC$ . Then  $M_2C_2$  is a mid-segment of the triangle  $\triangle AC_1' C$ . Therefore,  $M_2C_2$  is parallel to  $AC_1'$  and  $2M_2C_2 = AC_1'$ . Since triangles  $\triangle ABC_1$  and  $\triangle ABC_1'$  are equilateral, the quadrilateral  $AC_1BC_1'$  is a rhombus, so  $AC_1' = C_1B$  and  $AC_1'$  is parallel to  $C_1B$ . Hence  $M_2C_2$  is parallel to  $C_1B$  and  $2M_2C_2 = C_1B$ . Let  $G'$  be the intersection point of  $BM_2$  and  $C_1C_2$ . From here we can deduce that  $BG' : G'M_2 = C_1G' : G'C_2 = BC_1 : C_2M_2 = 2 : 1$ . But for the centroid  $G$  of  $\triangle ABC$  it is true that  $BG : GM_2 = 2 : 1$ , so  $G \equiv G'$  and  $G$  is the centroid of  $\triangle A_1B_1C_1$ . Since the triangles  $\triangle A_1B_1C_1$  and  $\triangle A_2B_2C_2$  have a common centroid, the statement is proved.  $\square$

The following corollary proves statements 1 and 2 from Theorem 1.

**Corollary 1.** *The points  $A, B$  and  $C$  form an overlapping Napoleon configuration for  $\triangle A_2B_2C_2$ . Moreover, the centroids  $A^*, B^*, C^*$  of the equilateral triangles  $\triangle AB_2C_2$ ,  $\triangle A_2BC_2$  and  $\triangle A_2B_2C$  respectively form an equilateral triangle, whose centroid coincides with the centroid  $G$  of  $\triangle ABC$ .*

*Proof.* By Lemma 2, first applied to the triple  $A, B_2, C_2$ , then to the triple  $A_2, B, C_2$ , and finally to the triple  $A_2, B_2, C$ , we obtain the first statement of Corollary 1. Thus, the points  $A, B$  and  $C$  form an overlapping Napoleon configuration for  $\triangle A_2B_2C_2$ . By the classical Napoleon's theorem for overlapping configurations, it follows that the centroids  $A^*, B^*, C^*$  of  $\triangle AB_2C_2$ ,  $\triangle A_2BC_2$  and  $\triangle A_2B_2C$  respectively form an equilateral triangle

whose centroid coincides with the centroid of  $\triangle A_2B_2C_2$ . By Lemma 4, the centroid of  $\triangle A_2B_2C_2$  coincides with the centroid  $G$  of  $\triangle ABC$ . The corollary is proved.  $\square$

Notice that the proof of the statements 3 and 4 from Theorem 1 is absolutely analogous to the proof of the statements 1 and 2. All we have to do is to follow more or less the same arguments, just changing the notation appropriately. What is left is the verification of the last three claims from Theorem 1. We proceed with the following lemma:

**Lemma 5.** *Consider the centroids  $C^*$  and  $G'_3$  of the equilateral triangles  $\triangle A_2B_2C$  and  $\triangle ABC'_1$  respectively. Then  $G'_3$  maps to  $C^*$  under a homothetic transformation of dilation factor  $-1/2$  with respect to the centroid  $G$  of  $\triangle ABC$  (see Figure 8).*

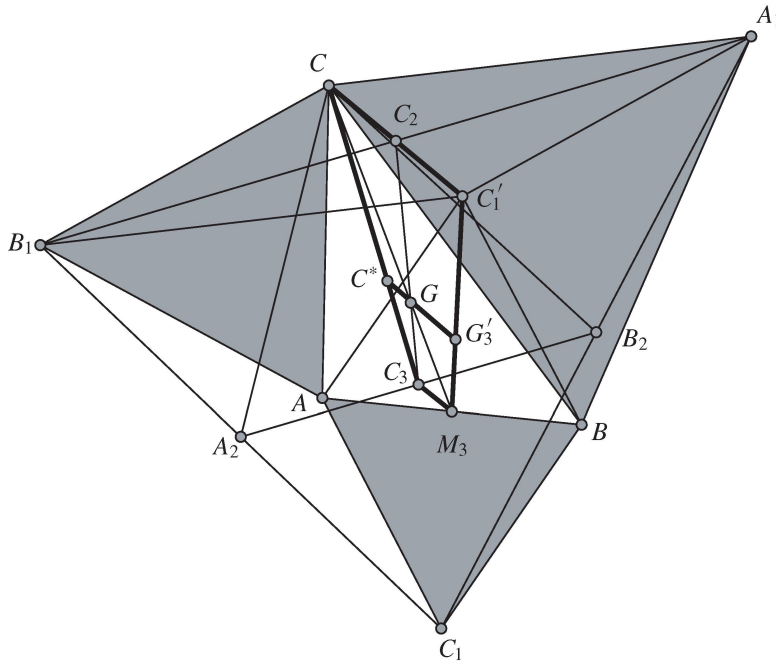


Fig. 8 Constructions in the proof of Lemma 5.

*Proof.* Perform a homothetic transformation of dilation factor  $-1/2$  with respect to the centroid  $G$  of  $\triangle ABC$ . By Lemma 4 the point  $G$  is also the centroid of  $\triangle A_1B_1C_1$ . Then  $A_1B_1$  maps to  $A_2B_2$  and so the midpoint  $C_2$  of  $A_1B_1$  maps to the midpoint  $C_3$  of  $A_2B_2$ . Also, the vertex  $C$  maps to the midpoint  $M_3$  of  $AB$  because  $G$  is the centroid of  $\triangle ABC$  (see Figure 8). From here we can conclude that  $C_3G : GC_2 = 1 : 2$  and  $M_3G : GC = 1 : 2$  which transforms into  $C_3G : C_3C_2 = 1 : 3$  and  $M_3G : M_3C = 1 : 3$ . As  $C^*$  is the centroid of  $\triangle A_2B_2C$ , we can see that  $C_3C^* : C_3C = C_3G : C_3C_2 = C^*G : CC_2 = 1 : 3$  and  $C^*G$  is parallel to  $CC_2$ . Similarly,  $G'_3$  is the centroid of  $\triangle ABC'_1$ , so  $M_3G'_3 : M_3C'_1 =$

$M_3G : M_3C = G'_3G : C'_1C = 1 : 3$  and  $G'_3G$  is parallel to  $C'_1C$ . By Lemma 3,  $C_2$  is the midpoint of  $CC'_1$  which means that both  $GC^*$  and  $GG'_3$  are parallel to the same line  $CC_2$ . Therefore  $G$  belongs to  $C^*G'_3$ . Moreover,  $C^*G = \frac{1}{3}CC_2 = \frac{1}{6}CC'_1$  and  $G'_3G = \frac{1}{3}CC'_1$ . Hence  $C^*G : GG'_3 = 1 : 2$ , so the point  $C^*$  is the image of the point  $G'_3$  under the homothetic transformation of factor  $-1/2$  with respect to  $G$ .  $\square$

After establishing the previous result, we are ready to confirm the validity of statements 5, 6 and 7 from Theorem 1.

**Corollary 2.** *In the setting of Theorem 1, triangle  $\triangle A^*B^*C^*$  is homothetic to the triangle  $\triangle G'_1G'_2G'_3$  with a homothetic center  $G$  and a coefficient of similarity  $-1/2$ . Similarly, triangle  $\triangle A^{**}B^{**}C^{**}$  is homothetic to the triangle  $\triangle G_1G_2G_3$  with a homothetic center  $G$  and a coefficient of similarity  $-1/2$ . Moreover, the area of  $\triangle ABC$  equals four times the algebraic sum of the areas of  $\triangle A^*B^*C^*$  and  $\triangle A^{**}B^{**}C^{**}$ .*

*Proof.* Applying Lemma 5 first to the pair of centroids  $C^*$  and  $G'_3$  of the equilateral triangles  $\triangle A_2B_2C$  and  $\triangle ABC'_1$ , then to the centroids  $A^*$  and  $G'_1$  of the equilateral triangles  $\triangle AB_2C_2$  and  $\triangle A'_1BC$ , and finally to the centroids  $B^*$  and  $G'_2$  of the equilateral triangles  $\triangle A_2BC_2$  and  $\triangle AB'_1C$ , we conclude that triangle  $\triangle A^*B^*C^*$  is homothetic to the triangle  $\triangle G'_1G'_2G'_3$  with respect to  $G$  and a dilation coefficient  $-1/2$ . Analogously, the same is true for the equilateral triangles  $\triangle A^{**}B^{**}C^{**}$  and  $\triangle G_1G_2G_3$ . Finally, due to the homothety, the area of  $\triangle A^*B^*C^*$  is  $1/4$  of the area of  $\triangle G'_1G'_2G'_3$  and the area of  $\triangle A^{**}B^{**}C^{**}$  is  $1/4$  of the area of  $\triangle G_1G_2G_3$ . Since by Napoleon's theorem the area of  $\triangle ABC$  equals the algebraic sum of the areas of  $\triangle G_1G_2G_3$  and  $\triangle G'_1G'_2G'_3$ , we conclude that the area of  $\triangle ABC$  equals four times the algebraic sum of the areas of  $\triangle A^*B^*C^*$  and  $\triangle A^{**}B^{**}C^{**}$ . This completes the proof of the corollary.  $\square$

## References

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