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In [3], Grünbaum and Mycielski proposed the following model of plane geometry. The points of this model are the points of the Euclidean plane $\mathbb{R}^2$. There are four types of lines for this model: a vertical Euclidean line; a horizontal Euclidean line; a translate of the hyperbola $L = \{(x,y) : x > 0, y = 1/x\}$; and a translate of the hyperbola

Modelle der ebenen Geometrie, die mit Ausnahme des Parallelenaxioms alle anderen erfüllen, haben für die Entwicklung der Mathematik eine ausserordentlich wichtige Rolle gespielt. Ihre Entdeckung durch Beltrami, Klein und Poincaré zeigte ja nicht nur die logische Unabhängigkeit des Parallelenaxioms, sondern sie machte auch klar, dass die Mathematik Axiome weitgehend unabhängig von irgendwelchen Bezügen zur Wirklichkeit setzen kann. Ausser den berühmten Modellen von Klein und Poincaré werden jedem Studierenden der Mathematik auch andere vorgeführt, die nur einen Teil der Axiome der ebenen Geometrie erfüllen; die endlichen affinen und projektiven Ebenen sind für diese Zwecke besonders beliebt. Unendliche Modelle sind weniger bekannt. Der vorliegende Beitrag von Powers, Riedel und Sahoo beschäftigt sich mit einer ganzen Klasse von einfach zu beschreibenden unendlichen Modellen dieser Art. Die Autoren gehen dabei auch der Frage nach, unter welchen Bedingungen zwei Modelle ihrer Klasse zueinander isomorph sind. *ust*

$L^* = \{(x,y) : x < 0, y = -1/x\}$. We will follow [3] and label this model G2. Notice that G2 does not satisfy the Euclidean, hyperbolic, or elliptic parallel postulates. Thus G2 makes a nice example for students who take a geometry-for-teachers course.

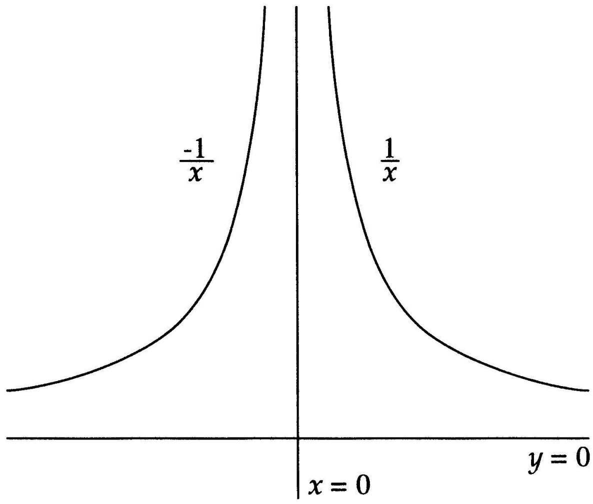

Fig. 1 The lines of G2

There is no reason why one needs to use the hyperbolas $1/x$ and $-1/x$ (plus translates) as lines of G2. Indeed, if we replace $1/x$ with $1/x^2$ and $-1/x$ with $-1/x^2$, then we generate yet another model of plane geometry that does not satisfy any of the standard parallel postulates. In [3], the authors proposed the curves e^{-x} and e^x as yet another version of model G2. One might think that all these versions of G2 are essentially the same. As our theorem below demonstrates, these versions are not isomorphic as models of incidence geometry. This result parallels the one given in [3] and further developed in [2] and [4].

Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ be two functions where I and J are subintervals of $\mathbb{R}$. We propose the following generalization of G2 using the functions f and g. As before, the points of this generalization of G2 are the points of $\mathbb{R}^2$. There are four types of lines: a vertical Euclidean line; a horizontal Euclidean line; a translate of the graph $\{(x,y) : x \in I, y = f(x)\}$; and a translate of the graph $\{(x,y) : x \in J, y = g(x)\}$. We denote this interpretation of incidence geometry by $M_{(f,g)}$.

In order for $M_{(f,g)}$ to be a model of incidence geometry we need f and g to be one-to-one. In particular, if these functions are continuous then one needs to be strictly decreasing and the other needs to be strictly increasing. Thus, we will require $f(x)$ to be strictly decreasing on I and $g(x)$ to be strictly increasing on J. If $y = f(x)$ is bounded above by $y = m$ and below by the line $y = n$, then there is no line in $M_{(f,g)}$ that passes through the points $(0, m+1)$ and $(1, n-1)$. Therefore, we will require $f(x)$ and $g(x)$ to be unbounded from either above or below. It is possible that f and g are unbounded in both directions (e.g., $f(x) = \log(-x)$ for $x < 0$ and $g(x) = \log(x)$ for $x > 0$) or in different directions (e.g., $f(x) = 1/x$ for $x > 0$ and $g(x) = -1/x$ for $x > 0$). Our goal, however, is to determine those functions $f(x)$ and $g(x)$ such that $M_{(f,g)}$ and G2 are isomorphic as models of incidence geometry. Toward this end, we will require $f(x)$ and $g(x)$ to be continuous, unbounded from above and bounded from below, and bounded on one side by vertical asymptotes.

Lemma 1 *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a bijection that induces a map from $M_{(f,g)}$ onto $M_{(f,g)}$. Then*

$$F(x,y) = (f_1(x), f_2(y)) \tag{1}$$

for some bijections f_1 and f_2 of $\mathbb{R}$.

Proof The following geometric fact is clear in $M_{(f,g)}$. If L is a line and p is a point not on L such that there is exactly one line through p parallel to L, then L is a horizontal line. It follows from this fact that the family of horizontal lines is mapped to itself. Since a line in $M_{(f,g)}$ is vertical if and only if it intersects every horizontal line it follows that F maps the family of vertical lines to the family of vertical lines. It now follows that $F(x,y)$ has the form described by equation (1). $\square$

We point out that G2 = $M_{(1/x,-1/x)}$ satisfies Lemma 1. In fact, we can prove more.

Lemma 2 $F : \mathbb{R}^2 \to \mathbb{R}^2$ *is a bijection that induces a map from* G2 *onto* G2 *if and only if*

$$F(x,y) = \begin{pmatrix} \lambda & 0 \\ 0 & 1/|\lambda| \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2}$$

for some nonzero $\lambda \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$.

Proof By Lemma 1, we know that $F(x,y) = (f_1(x), f_2(y))$ for some bijections f_1 and f_2 of $\mathbb{R}$. Since F maps G2 onto G2 it follows that $\{F(x,y) | x > a, y = 1/(x-a) + b\}$ is a line of G2. Since $\{F(x,y) | x > a, y = 1/(x-a) + b\}$ is neither a horizontal nor vertical line it follows that it is a translate of either $\{(x,y) | x > 0, y = 1/x\}$ or $\{(x,y) | x < 0, y = -1/x\}$. Thus, there exist $a', b' \in \mathbb{R}$ such that

$$(f_1(x) - a')(f_2(y) - b') = 1 \text{ or } -1 \tag{3}$$

for $x > a$ and $y = 1/(x-a) + b$. If the above product is 1, then

$$(f_1(x) - a')(f_2(y) - b') = (x-a)(y-b),$$

and so

$$\frac{f_1(x) - a'}{x - a} = \frac{y - b}{f_2(y) - b'}.$$

Therefore, there exists a nonzero constant λ such that

$$\frac{f_1(x) - a'}{x - a} = \lambda = \frac{y - b}{f_2(y) - b'}.$$

Thus, $f_1(x) = \lambda x + \alpha$ and $f_2(y) = \frac{1}{\lambda} y + \beta$ for some constants α and β and for all $x > a$ and $y = 1/(x-a) + b$. Since the line $\{(f_1(x), f_2(y)) | x > a, y = 1/(x-a) + b\}$ is bounded below it follows that $\lambda > 0$. If the product in equation (3) is -1, then a similar argument will show that there exists a constant $\lambda < 0$ such that $f_1(x) = \lambda x + \alpha$ and $f_2(y) = \frac{1}{(-\lambda)}y + \beta$ for some constants α and β, for all $x > a$, and $y = 1/(x-a) + b$. Since a and b can be chosen arbitrarily it follows that $f_1(x) = \lambda x + \alpha$ and $f_2(y) = \frac{1}{(-\lambda)}y + \beta$ for all x and y. Hence $F(x,y)$ satisfies (2).

It is straightforward to verify that if $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a bijection of the form described by equation (2), then F induces a map from G2 onto G2. □

We can now prove our main result.

Theorem 3 *If there exists a continuous bijection ϕ of $\mathbb{R}^2$ onto itself which induces a map from G2 onto $M_{(f,g)}$, then, up to translations, $f(x) = a/x$ for $x > 0$ and $g(x) = -a/x$ for $x < 0$ for some positive constant a.*

Proof The argument used to establish equation (1) in Lemma 1 can be used to show that, for all $(x,y) \in \mathbb{R}^2$,

$$\phi(x,y) = (\phi_1(x), \phi_2(y)) \tag{4}$$

for some bijections ϕ_1 and ϕ_2 of $\mathbb{R}$. Now consider the family of automorphisms of $M_{(f,g)}$ given by $\tau_t(x,y) = (x+t, y)$ where $t \in \mathbb{R}$. Then, for each t, $\tau_t' = \phi^{-1}\tau_t\phi$ is an automorphism of G2. It follows from Lemma 2 that

$$\phi_1^{-1}(\phi_1(x) + t) = \lambda x + a \tag{5}$$

for some nonzero constant λ. If $\lambda \neq 1$, then the equation $x = \lambda x + a$ has $x = \frac{a}{1-\lambda}$ as a solution. If $x = \frac{a}{1-\lambda}$ then equation (5) becomes

$$\phi_1(x) + t = \phi_1(x),$$

and so, $t = 0$. So for $t \neq 0$, it follows that $\lambda = 1$. Hence

$$\tau_t' = (x + h(t), a(t)y + b(t))$$

where h, a, b are functions and where h satisfies the Cauchy Functional Equation $h(r+s) = h(r) + h(s)$ for all $r, s \in \mathbb{R}$. Since $\phi \circ \tau_t' = \tau_t \circ \phi$ it follows that $\phi_1(x + h(t)) = \phi_1(x) + t$. The last equation holds if we replace $\phi_1(x)$ by $\phi_1(x) - \phi_1(0)$, thus we can assume that $\phi_1(0) = 0$. If we set $x = 0$, then we get $\phi_1(h(t)) = t$ for all t. Hence $\phi_1 = h^{-1}$ where h is continuous. It follows from Cauchy's Functional Equation (see [1]) that $h(t) = kt$ for some nonzero constant k. Hence $\phi_1(x) = \frac{1}{k}x$ for all x. A similar argument will show that, for all y, $\phi_2(y) = k'y$ for some nonzero constant k'. It now follows that equation (4) has, up to composition of a translation, the form

$$\phi(x,y) = (k_1 x, k_2 y)$$

for some nonzero constants k_1 and k_2.

The line $L = \{(x,y)|x > 0, y = 1/x\}$ in G2 is mapped by ϕ to the line $\{(k_1x, k_2y)|\ x > 0, y = 1/x\}$ in $M_{(f,g)}$. The latter must be a translate of either the graph of f or the graph of g. If it is a translate of the graph of f, then there exist α and β such that

$$k_2y = f(k_1x + \alpha) + \beta$$

for all $x > 0$ and $y = 1/x$. Since $f(x)$ is assumed to be bounded below and y ranges over the interval $(0, \infty)$ it follows that $k_2 > 0$. This in turn forces $k_1 > 0$, since $y = 1/x$ (for $x > 0$) and f are strictly decreasing functions. Replacing x by $\frac{x-\alpha}{k_1}$ in the last equation leads to

$$f(x) = \frac{k_1k_2}{x-\alpha} - \beta \tag{6}$$

for all $x > \alpha$.

Note that ϕ maps a translate of $L = \{(x,y)|x > 0, y = 1/x\}$ to some translate of the graph of f. Since ϕ maps G2 onto $M_{(f,g)}$, it follows that ϕ maps

$$L^* = \{(x,y)|x < 0, y = -1/x\}$$

to a translate of the graph of g. Thus

$$g(x) = \frac{-k_1k_2}{x-\alpha'} - \beta' \tag{7}$$

for all $x < \alpha'$. In fact, it is easy to see that $\alpha = \alpha'$ and $\beta = \beta'$.

If the image of L under ϕ is a translate of the graph of g then, using the same type of argument as above, we get $k_1 < 0$, $k_2 > 0$, $f(x) = \frac{-k_1k_2}{x-\alpha} - \beta$ for all $x > \alpha$, and $g(x) = \frac{k_1k_2}{x-\alpha'} - \beta'$ for all $x < \alpha'$.

So $f(x)$ and $g(x)$ are just scaled translations of $1/x$. Given that $M_{(f,g)}$ is closed under translations, we can take $f(x) = a/x$ for $x > 0$ and $g(x) = -a/x$ for $x < 0$ where a is some positive constant. □

In conclusion, we note that the converse of Theorem 3 is true since, for $a > 0$, $\phi(x,y) = (\sqrt{a}x, \sqrt{a}y)$ is a bijection of $\mathbb{R}^2$ which induces a map from *G2* onto $M_{(a/x,-a/x)}$.

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