On two characteristic properties of Euclidean norms

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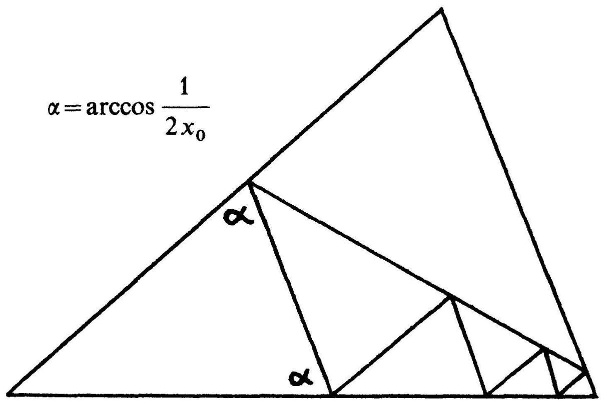

Fig. 6

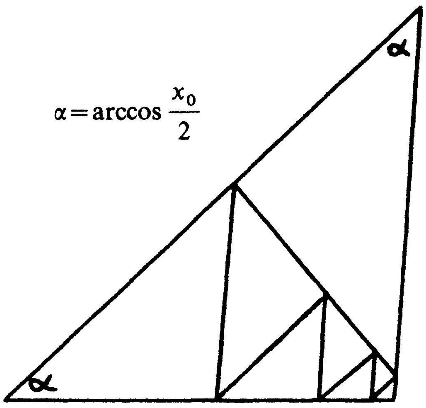

Fig. 7

Anmerkung. Über eine Reihe von Fallunterscheidungen werden in [5] mit einfachen kombinatorisch-topologischen Schlüssen die folgenden Minimalitätsaussagen bewiesen.

(i) Die beiden Dreiecke aus Satz 2 lassen sich nicht in weniger als acht Teile perfekt zerlegen.
(ii) Alle übrigen ungleichseitigen Dreiecke, die nicht rechtwinklig-nichtgleichseitig sind, lassen sich nicht in weniger als sechs Teile perfekt zerlegen.

Aus (i) und (ii) folgt, dass die Zerlegungen der Figuren 4, 6 und 7 jeweils die minimale Anzahl von Teilen realisieren, in die die dargestellten Dreiecke perfekt zerlegt werden können.

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On two characteristic properties of Euclidean norms

0. Introduction

In the analysis of functions of several real variables it is useful to consider different norms on $\mathbb{R}^n$, depending on the problem at hand. This is not a dirty trick, because it can be shown, that on a finite dimensional real vector space all norms are topologically equivalent. Nevertheless, the (standard) euclidean norm on $\mathbb{R}^n$ is in some sense «nicer» and «more natural» than any other norm. If we compare, for instance, the unit spheres for the euclidean and the maximum norm on $\mathbb{R}^2$ or $\mathbb{R}^3$, we must admit that the euclidean unit sphere is «more homogeneous» than the maximum unit sphere with its corners! This shall be made more precise in the first part of the paper, while the second part deals with the trigonometry in a normed vector space: There is more about it than only the triangle inequality, because we shall see that Euclid's definition of a right angle – if correctly interpreted – makes sense in a normed vector space and that the existence of orthogonal complements characterizes euclidean norms.

1. Linear Isometries

From now an, E will always denote a finite-dimensional real vector space, endowed with a norm $\| \ \|$. $S(E, \| \ \|)$ denotes the unit sphere, and $Iso(E, \| \ \|)$ the group of linear isometries $T: E \to E$. This is a subgroup of $GL(E)$, the group of all regular linear transformations of E.

Let us call the norm $\| \ \|$ *euclidean* if it is the norm induced by an inner product $\langle , \rangle$ on E:

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

The following theorem [1] says that the unit sphere for a euclidean norm is more homogeneous than for any other norm:

Theorem 1. *The norm $\| \ \|$ is euclidean if and only if the group $Iso(E, \| \ \|)$ acts transitively on the unit sphere S.*

Proof: Let $\| \ \|$ be a euclidean norm, induced by the scalar product $\langle , \rangle$. The group of isometries is then the corresponding orthogonal group which acts transitively on the unit sphere.

Now let the norm $\| \ \|$ be such that the isometry group acts transitively on the unit sphere. $\| \ \|$ induces an operator norm on the real vector space $L(E, E)$ of all linear transformations of E, and the group $Iso(E, \| \ \|)$ is bounded with respect to this norm, because it is contained in the unit sphere.

But $Iso(E, \| \ \|)$ is also closed in $L(E, E)$: Let $(T_\nu)_{\nu \in \mathbb{N}}$ be a sequence of isometries $T_\nu \in Iso(E, \| \ \|)$ converging to the linear map $T: E \to E$. The real vector space $L(E, E)$ being finite-dimensional, «convergence» means convergence with respect to any norm on $L(E, E)$, and taking the operator norm induced by the given norm on E we obtain for any $x \in E$,

$$| \|T(x)\| - \|T_\nu(x)\| | \le \|T(x) - T_\nu(x)\| \le \|T - T_\nu\| \cdot \|x\|$$

and hence

$$\|T(x)\| = \lim_{\nu \to \infty} \|T_\nu(x)\| = \|x\|,$$

which means that T is also an isometry.

[1] This theorem is not new; the referee informed me, that another proof can be found in [2, p. 250 ff.].

Thus, the subgroup $Iso(E, \| \ \|)$ of the Lie group $GL(E)$ is compact, and therefore there exists an inner product $\langle \, , \rangle$ on E which is invariant under the action of $Iso(E, \| \ \|)$. (See for instance [1, page 54, Prop. XVI].) Without loss of generality we may suppose $\sqrt{\langle x, x\rangle} = 1 = \| x \|$ for some fixed vector $x \in S(E, \| \ \|)$. Now let $y \in S(E, \| \ \|)$ be any other unit vector. By hypothesis, there is a transformation $T \in Iso(E, \| \ \|)$ with $T(x) = y$, and as the inner product $\langle \, , \rangle$ is invariant under T, we have also

$$\| y \| = 1 = \sqrt{\langle x, x\rangle} = \sqrt{\langle T(x), T(x)\rangle} = \sqrt{\langle y, y\rangle}\,.$$

2. Orthogonality

Recall Euclid's definition of a right angle, as it is stated in the first book of the Elements: A right angle is an angle which is congruent to its supplementary angle. This makes sense in a normed vector space if we define the congruence of angles by the congruence of triangles, the latter being defined by comparing corresponding sides.
But one has to be careful in applying these definitions of congruence: Let $x, y \in E$ be two non-zero vectors. We want to compare the angle formed by the rays defined by x and y to the angle formed by the rays defined by x and $-y$. In order to do so, it is not sufficient to compare simply the triangle with sides x, y and $x-y$ to that with sides, x, $-y$ and $x+y$, as we can see from the following example: Let $E = \mathbb{R}^3$, endowed with the maximum norm, and consider the vectors $x = (1, 0.5)$ and $y = (0, 0.5)$: The two triangles formed by x, $\pm y$ and $x \mp y$ are congruent, because $\| x+y \| = \| x-y \|$. If we replace y by $2y$, we are still considering the same angles formed by the rays defined by x and $\pm y$, but the triangles formed by x, $\pm 2y$ and $x \mp 2y$ are no longer congruent, because $\| x+2y \| = 1.5$, whereas $\| x-2y \| = 1$. This leads to the following definition:

Definition. *Two linear subspaces $F_1, F_2 \subset E$ are said to be orthogonal if $\| x+y \| = \| x-y \|$ for $\forall x \in F_1$ and $\forall y \in F_2$.*
Two vectors $x, y \in E$ are orthogonal if the linear subspaces $\mathbb{R}x$ and $\mathbb{R}y$ are orthogonal.

Thus, in the above example of $\mathbb{R}^3$ endowed with the maximum norm, $x = (1, 0)$ and $y = (0, 1)$ are orthogonal, but the reader may prove as an exercise that there is no non-zero vector orthogonal to $(1, 0.5)$. (Do this before reading Lemma 3 below!)
If the norm $\| \ \|$ is euclidean, this notion of orthogonality is the usual one, defined by the inner product, as follows easily from Pythagoras' theorem. So one direction of the following theorem is obvious:

Theorem 2. *The norm $\| \ \|$ is euclidean if and only if every linear subspace $F \subset E$ has an orthogonal complement in E.*

We shall first show that orthogonal complements are unique.

Lemma 1. *If the linear subspaces $F_1, F_2 \subset E$ are orthogonal, then $F_1 \cap F_2 = 0$.*

Proof: For $x \in F_1 \cap F_2$ we have $\| x+x \| = \| x-x \| = 0$, and thus $x = 0$.

Lemma 2. *Let $x, y \in E$ be non-zero vectors with $x \perp y$. Then $x+y \not\perp x$.*

Proof: Suppose $x+y \perp x$. As we have also $x \perp y$, we obtain the following chain of equations:

$$\begin{aligned} \|y\| &= \|x+y-x\| = \|x+y+x\| \\ &= \|y+2x\| = \|y-2x\| = \|x+y-3x\| = \|x+y+3x\| \\ &= \|y+4x\| = \ldots = \|y+2nx\| \quad \forall n \in \mathbb{N}. \end{aligned}$$

But this is impossible unless $x=0$.

As a consequence of this Lemma, a linear subspace $F \subset E$ has at most one orthogonal complement $F^{\perp} \subset E$, and in this case the *orthogonal reflection* ϱ_F in F is well-defined:

$$\varrho_F(x+y) := x-y \quad \text{for} \quad x \in F \quad \text{and} \quad y \in F^{\perp}.$$

By our definition of orthogonality, ϱ_F is an isometry with respect to the norm $\| \; \|$. We want to show that these reflections act transitively on the unit sphere if there exists an orthogonal complement to every one-dimensional linear subspace, and we need the following Lemma:

Lemma 3. *Let E be two-dimensional and let x, y be an orthogonal basis for E with $\|x+y\| = \|x\|$. Then there exists no orthogonal complement to the subspace $\mathbb{R}(x+\alpha y)$ for $0<\alpha<1$.*

Proof: If we restrict the norm $\| \; \|$ to the straight line $x+\mathbb{R}y$, we obtain a convex function which by hypothesis takes the same value at the three points $x-y$, x and $x+y$. Thus the norm must be constant on the segment between $x-y$ and $x+y$ and, in particular, $\|x+\alpha y\| = \|x\|$ for $0 \leq \alpha \leq 1$.
Let $z = x+\alpha y$ with $0<\alpha<1$. By Lemma 2, neither x nor y is orthogonal to z. So we have to prove that $\lambda x+\mu y$ is not orthogonal to z for $\lambda \neq 0$ and $\mu \neq 0$. For $\varepsilon>0$ small enough,

$$x+\alpha y \pm \varepsilon(\lambda x+\mu y) = (1 \pm \varepsilon\lambda)\left(x + \frac{\alpha \pm \varepsilon\mu}{1 \pm \varepsilon\lambda} y\right),$$

and thus

$$\|x+\alpha y \pm \varepsilon(\lambda x+\mu y)\| = |1 \pm \varepsilon\lambda| \, \|x\|$$

for ε sufficiently small, because $\lim\limits_{\varepsilon \to 0} \dfrac{\alpha \pm \varepsilon\mu}{1 \pm \varepsilon\lambda} = \alpha \in \,]0,1[$.
Thus, $\|x+\alpha y+\varepsilon(\lambda x+\mu y)\| \neq \|x+\alpha y-\varepsilon(\lambda x+\mu y)\|$, and that means that $\lambda x+\mu y$ is not orthogonal to $x+\alpha y$.

This Lemma shows in particular that in $\mathbb{R}^2$ endowed with the maximum norm, there is no orthogonal complement to the line $\mathbb{R}(1, 0.5)$.

Lemma 4. *Let the norm* $\| \ \|$ *on* E *be such that every one-dimensional linear subspace has an orthogonal complement. Then, for* $x, y \in S(E, \| \ \|)$ *with* $x \neq \pm y$ *the reflection* $\varrho = \varrho_F$ *in the orthogonal complement* $F = \mathbb{R}(x-y)^{\perp}$ *maps* x *to* y.

Proof: Let $z := x - y$ and consider the decompositions

$$x = a + \lambda z \quad \text{and} \quad y = b + \mu z \text{ with } a, b \in F \text{ and } \lambda, \mu \in \mathbb{R}.$$

The uniqueness of the orthogonal decomposition implies $a = b \neq 0$ and $\lambda - \mu = 1$. Moreover, a and z must be linearly independent.
Now $\varrho(x) = a - \lambda z$, and in order to show $\varrho(x) = y$ we have to show $\mu = -\lambda$ or simply $|\mu| = |\lambda|$. So let us suppose $|\lambda| \neq |\mu|$. Then there are at least three different points among $a \pm \lambda z$ and $a \pm \mu z$, and all these points lie on the same line and have the same norm, namely $\| a \pm \lambda z \| = \| a \pm \mu z \| = 1$. By the convexity of the norm, we have also $\| a \| = 1$, and by Lemma 3, for $0 < \alpha < \max\{|\lambda|, |\mu|\}$, the linear subspace $\mathbb{R}(a + \alpha z)$ has no orthogonal complement in the plane spanned by a and z. This is a contradiction to the hypothesis, because an orthogonal complement in E would intersect this plane in a line orthogonal to $\mathbb{R}(a + \alpha z)$.
Now we can achieve the proof of Theorem 2:

Proof of Theorem 2, second part: In order to apply Theorem 1, we shall prove that the group of isometries $Iso(E, \| \ \|)$ acts transitively on the unit sphere $S(E, \| \ \|)$. So let $x, y \in S(E, \| \ \|)$. If $x = \pm y$, there is no problem to construct an isometry mapping x to y; else Lemma 4 tells us that the reflection in the orthogonal complement of the line $\mathbb{R}(x-y)$ is such an isometry.

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Aufgaben

Aufgabe 1034. Unter einem perfekten Quader versteht man einen Quader mit ganzzahligen Seiten s_i, ganzzahligen Flächendiagonalen f_i $(i = 1, 2, 3)$ und ganzzahliger Raumdiagonale r. Man zeige: Für perfekte Quader gilt

$$s_1 s_2 s_3 f_1 f_2 f_3 r \equiv 0 (\bmod 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 37).$$

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