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On a measure of axiality for triangular domains

Abstract. A measure of axial symmetry for triangles T in E_2 is studied, with the function $f(T)=\max_{T^*}\{[T^*]/[T]: T^* \text{ is axial and } T^* \subset T\}$ chosen as a measure of axiality, where $[T]$ and $[T^*]$ denote the area of the triangle and of the axially symmetric oval, respectively. The greatest lower bound, $g_\Delta=\inf_T\{f(T): T \text{ is a triangle in } E_2\}=2(\sqrt{2}-1)$, is approached as a limit. The least axial triangle is a triangle whose altitude h is arbitrarily close to zero and whose sides are in the ratio $(\sqrt{2}-1):1:\sqrt{2}$ in the limit of $h=0$.

1. Introduction

Ovals in the euclidean plane E_2 are compact convex sets with interior points. An oval can be symmetric with respect to a point (centrally symmetric or *centric*) or a line (axially symmetric or *axial*). Measures of centrality for convex sets have been critically reviewed by Grünbaum [7]. Measures of axiality for ovals have been investigated by Nohl [9], Krakowski [8], Chakerian and Stein [2], and de Valcourt [3–5]. In this paper we describe a measure of axiality for triangles – the simplexes in E_2.

Let K' denote the mirror image (*enantiomorph*) of an oval K obtained by reflection about a line k through an interior point. Let $K^*=K \cap K'$, a convex set, and $P=K \cup K'$. Then $K^* \subset K$ and $P \supset K$ are both necessarily axial, whereas K and K' are axial if and only if there exists a k (symmetry axis or mirror line) for which $K^*=P=K=K'$. In what follows, the chosen measure of axiality is the continuous real-valued function $f(K)$ defined on the class $\mathbf{K}_2$ of all ovals K in E_2 by

$$f(K)=\max_{K^*}\{[K^*]/[K]: K^* \text{ is axial and } K^* \subset K\},$$

where $[K]$ and $[K^*]$ denote the areas of the corresponding ovals [3, 4].

This function has the following properties:

(1) $0 \leq f(K) \leq 1$ for every oval $K \in \mathbf{K}_2$;
(2) $f(K)=1$ if and only if K is axial;
(3) $f(K)$ is similarity-invariant.

2. Maximal overlap ratios for enantiomorphous triangles

Let T denote a triangular oval, T' its enantiomorph, and T^* the intersection $T \cap T'$ of the enantiomorphous triangles. When T' is generated by reflection about a line k through an interior point of T, then T^* is an axial polygon inscribed in T and T', and k is its symmetry axis.
Alternatively, T^* may be generated simply by overlapping T and T'; in that case T^* is not necessarily axially symmetric. However, Giering [6] has shown that maximal overlap of enantiomorphous triangles obtains only if T^* is axial *and* the sides of T^* are segments of all six sides of the two overlapped triangles. Triangular intersections are ipso facto excluded for non-axial triangles, and it remains to discuss quadrilateral, pentagonal, and hexagonal intersections that satisfy Giering's conditions.

(a) *Quadrilateral intersections.* If the intersection of $T(ABC)$ and $T'(A'B'C')$ is quadrilateral, maximal overlap under Giering's conditions requires k to be the bisector of the shared angle. We choose the shared angle α opposite side a (Figure 1), and b, c as the two sides of T whose ratio is closest to unity, with $b \le c$. It is then easily seen that

$$f_1(T) = \max_{T^*} \{[T^*]/[T]\colon T^* \text{ is an axial quadrilateral and } T^* \subset T\}$$

$$= \frac{2}{1+\frac{c}{b}}, \quad \text{with} \quad 1 \le \frac{c}{b}. \tag{2.1}$$

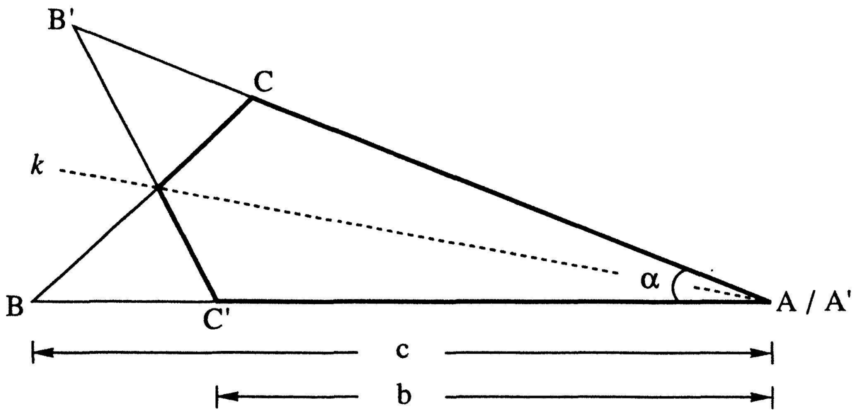

Figure 1.

(b) *Pentagonal intersections.* For a pentagonal intersection of T and T', maximal overlap under Giering's conditions requires k to be perpendicular to the shared side, c (Figure 2). With reference to Figure 2, for $1 \le \frac{2b\cos\alpha}{c} \le 2$ (acute angle at B) P is the projection of vertex C onto side c, and t is a segment of c, with $0 \le \frac{t}{c} \le \frac{1}{2}$. For $2 < \frac{2b\cos\alpha}{c}$ (obtuse angle at B), P is the projection of vertex C onto an extension of c, and t is negative. Let x, with $0 \le x \le c - 2t$, denote the segment of c that is not coextensive with c', the side opposite C'. The intersection T^* is the pentagon $BDED'B'$, and the overlap ratio is

$$\frac{[T^*]}{[T]}=\frac{(c+x)^2}{2c(c-t)}-\frac{2x^2}{c(c-2t)}.$$

This ratio is maximal for $x=\dfrac{c(c-2t)}{3c-2t}$, and it follows that

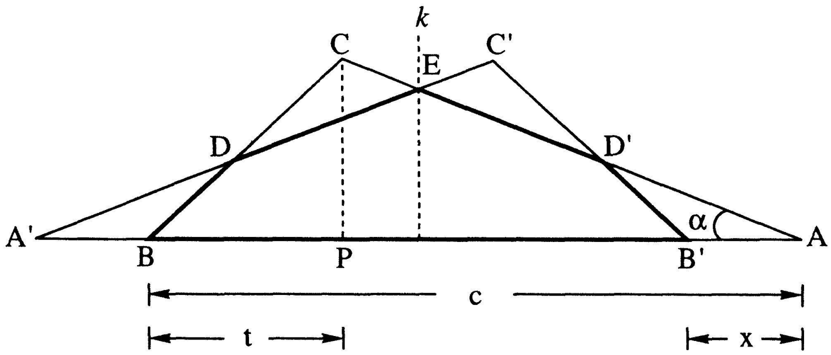

Figure 2.

$$f_2(T)=\max_{T^*}\{[T^*]/[T]\colon T^* \text{ is an axial pentagon and } T^* \subset T\}$$

$$=\frac{2c}{3c-2t}, \quad \text{with} \quad \frac{t}{c}\leq\frac{1}{2}, \tag{2.2}$$

with (2.2) equivalently expressed as

$$=\frac{2}{1+\dfrac{2b\cos\alpha}{c}}, \quad \text{with} \quad 1\leq\frac{2b\cos\alpha}{c}, \tag{2.3}$$

where $b\equiv AC$ and $\alpha\equiv CAB$ as shown in Figure 2.

It is obvious that $f_2(T)$ is achieved when c is chosen as the side for which $\frac{t}{c}$ is closest to $\frac{1}{2}$ (or $\frac{2b\cos\alpha}{c}$ is closest to 1).

(c) *Hexagonal intersections.* Two alternatives are distinguished [6] if T^* is hexagonal: the six sides of T^* belong alternately to T and T', i.e., no two adjacent sides of T^* belong to one triangle (alternant hexagonal intersection), or one pair of adjacent sides in T^* belongs to T and another pair belongs to T' (nonalternant hexagonal intersection). We consider the latter case first.

In Figure 3, T_h^* is the hexagon $CEFGC'D$, and k is a bisector of the inscribed square $EFGD$ [6]. Without loss of generality, let us assign unit dimensions to the square. Then $c=1+x+y$, where c is the side of T that is collinear with a side of the square, and x and

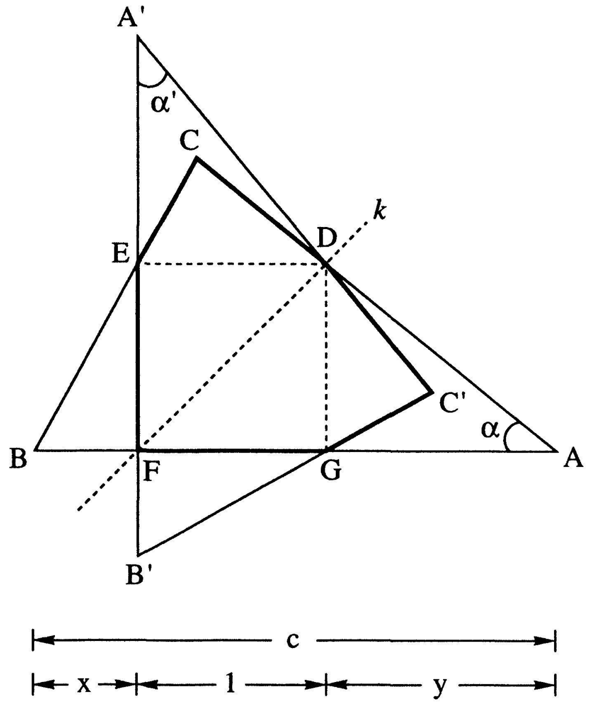

Figure 3.

y are variables. In terms of these variables the overlap ratio is given by

$$\frac{[T_h^*]}{[T]} = \frac{2}{1+x+y}, \quad \text{with} \quad 1<y \text{ and } 0<x\leq y\,.$$

With reference to Figure 3, $BC \equiv a < c$ for every $1<y$ and $0<x\leq y$. Hence

$$x+y > \frac{x+y}{\sqrt{1+x^2}} = \frac{c}{a}, \quad \text{and} \quad \frac{2}{1+x+y} < \frac{2}{1+\frac{c}{a}} \leq \frac{2}{1+\frac{c''}{b''}},$$

where c'' and b'' are sides of the triangle such that $\frac{c''}{b''}$ is closest to 1 and $b'' \leq c''$.
By comparison with (2.1) it follows that nonalternant hexagonal intersections are incapable of yielding $f(T)$.
We next consider the case of alternant hexagonal intersections. To achieve maximal overlap, two additional conditions have to be satisfied [6]: (i) two enantiomorphous triangles abc and $a'b'c'$ in the interior of T^* (Figure 4), which are similar to and share one side with T and T', respectively, have a vertex in common that lies on k; (ii) the three lines m_a, m_b, and m_c, which are perpendicular to and pass through the midpoints of alternating sides of T^*, meet at a common point in the interior of T^*.
We define

$$f_3(T) = \max_{T^*} \{[T^*]/[T]\colon\ T^* \text{ is an axial alternant hexagon and } T^* \subset T\}\,.$$

It can be shown that $f_3(T)=2/\mu$, where μ is the ratio of magnification (or scaling factor) that relates similar triangles: $AB=\mu c$, $AC=\mu b$, $BC=\mu a$. In terms of the quantities in Figure 4:

$$\mu=1+\frac{a\sin\theta}{c\sin\beta}+\frac{b\sin\phi}{c\sin\alpha}. \tag{2.4}$$

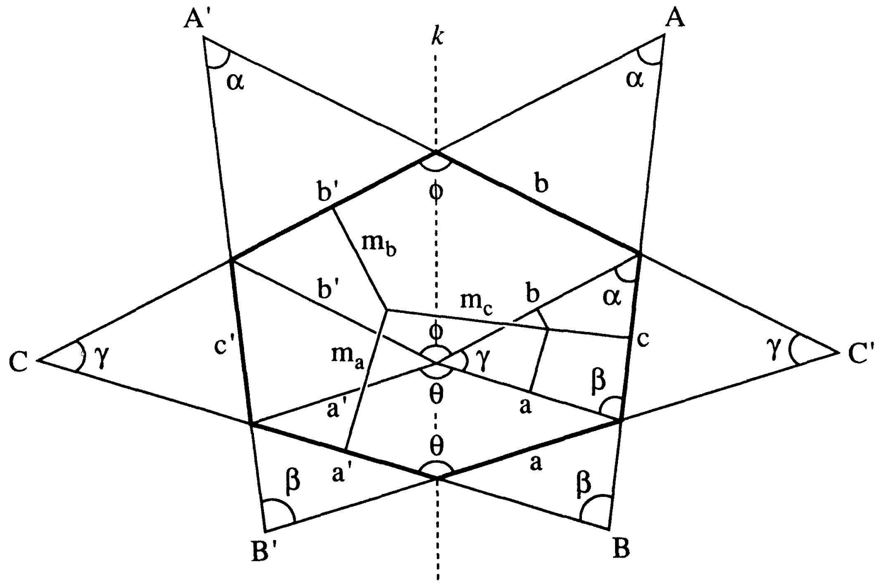

Figure 4.

For axial (isosceles or equilateral) triangles, T^* is axial *and* centric, $\mu=3$, and $f_3(T)=2/3$. This result accords with a finding by Besicovitch [1] that the largest value of the area of a centric oval inscribed in a triangle of area a is $2a/3$. However, for non-axial triangles T^* cannot be centric.

We shall now prove that for all triangles $f_3(T)\leq 2/3$, i.e., that $\mu\geq 3$. After appropriate substitutions in (2.4) this assertion takes the form

$$a^2\sin\theta+b^2\sin\phi\geq 2ab\sin\left(\frac{\theta+\phi}{2}\right). \tag{2.5}$$

It is obvious that if $\gamma=\frac{\pi}{2}$, m_a, m_b, and m_c meet at the midpoint of c and $\theta=\phi=\frac{\pi}{2}$. Therefore (2.5) is always satisfied. The condition that the three lines m_a, m_b, m_c meet at a common point in the interior of T^* is fulfilled if and only if

$$\frac{a^2}{b^2}=\frac{\cos\phi}{\cos\theta}. \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$\frac{\cos\phi\sin\theta}{\cos\theta}+\sin\phi-2\left(\frac{\cos\phi}{\cos\theta}\right)^{\frac{1}{2}}\cdot\sin\left(\frac{\theta+\phi}{2}\right)\geq 0.$$

That this relationship is true for any θ and ϕ in the domains $\frac{\pi}{2}<\theta<\pi$ and $\frac{\pi}{2}<\phi<\pi$ defined by the constraints of the problem can be shown by elementary means. This completes the proof of the assertion. Accordingly

$$f_3(T) \leq \tfrac{2}{3}$$

for all alternant hexagonal intersections of enantiomorphous triangles. Furthermore, since, for every $T, f_1(T)$ and $f_2(T)$ are both equal to or greater than 2/3 for quadrilateral and pentagonal intersections, respectively, it follows that alternant hexagonal intersections are also incapable of yielding $f(T)$.

3. Greatest lower bound of $f(T)$ for triangles

We have seen that only quadrilateral and pentagonal intersections, with shared angles and shared sides, respectively, need be considered as candidates for $f(T)$. Moreover, in order to achieve the greatest lower bound of $f(T)$ we seek the condition under which $f_1(T)=f_2(T)$.

Let us take c and b as the sides of a triangle, with $b \leq c$, such that $\frac{c}{b}$ is closest to 1. Then, according to (2a), only one quadrilateral intersection has to be considered:

$$f_1(T)=\frac{2}{1+\frac{c}{b}}.$$

However, three alternatives need to be considered for pentagonal intersections (2b):

$$f_2(T)=\frac{2}{1+\frac{2b'' \cos \alpha''}{c''}}, \quad \text{with} \quad 1 \leq \frac{2b'' \cos \alpha''}{c''} \leq 2,$$

where c'' is the shared side and α'' is the angle subtended by b'' and c''.

(a) *Pentagonal intersection with shared side a.* For this intersection, $f_1(T)=f_2(T)$ takes the form

$$\frac{2}{1+\frac{c}{b}}=\frac{2}{1+\frac{2c \cos \beta}{a}}, \quad \text{with} \quad 1 \leq \frac{2c \cos \beta}{a} \leq 2.$$

This equation is satisfied for either an isosceles triangle ($b=c$) or for a scalene triangle with

$$b=\frac{c}{4 \cos^2 \beta-1}, \qquad a=\frac{2c \cos \beta}{4 \cos^2 \beta-1}, \qquad \frac{\sqrt{2}}{2} \leq \cos \beta \leq \frac{\sqrt{3}}{2}.$$

Further, since $\frac{c}{b}$ is closest to 1, it follows that

$$\frac{c}{b} \leq \sqrt{2 \cos \beta}$$

and, in addition, that

$$\frac{\sqrt{2}}{2} \leq \cos \beta < \frac{\sqrt{5}+1}{4}.$$

Therefore

$$f(T) = \frac{2}{1+\frac{c}{b}} \geq \frac{2}{1+\sqrt{2 \cos \beta}} > \frac{2}{1+\sqrt{\frac{\sqrt{5}+1}{2}}}.$$

(b) *Pentagonal intersection with shared side b.* For this intersection, $f_1(T) = f_2(T)$ takes the form

$$\frac{2}{1+\frac{c}{b}} = \frac{2}{1+\frac{2a \cos \gamma}{b}}, \quad \text{with} \quad 1 \leq \frac{2a \cos \gamma}{b} \leq 2.$$

This equation is satisfied for any triangle with $c = 2a \cos \gamma$ and $\cos \gamma \leq \frac{1}{2}$. However, no such triangle, with the exception of the equilateral triangle, satisfies the condition that $\frac{c}{b}$ is closest to 1.
Therefore $f(T) = 1$.

(c) *Pentagonal intersection with shared side c.* For this intersection, $f_1(T) = f_2(T)$ takes the form

$$\frac{2}{1+\frac{c}{b}} = \frac{2}{1+\frac{2b \cos \alpha}{c}}, \quad \text{with} \quad 1 \leq \frac{2b \cos \alpha}{c} \leq 2,$$

which can be expressed as

$$\frac{c}{b} = \sqrt{2 \cos \alpha}, \quad \text{with} \quad \tfrac{1}{2} \leq \cos \alpha < 1. \tag{3.1}$$

Therefore

$$f(T) = \frac{2}{1+\frac{c}{b}} = \frac{2}{1+\sqrt{2 \cos \alpha}} > \frac{2}{1+\sqrt{2}} = 2(\sqrt{2}-1). \tag{3.2}$$

Since

$$2(\sqrt{2}-1) < \frac{2}{1+\sqrt{\frac{\sqrt{5}+1}{2}}} < 1 ,$$

it follows that the greatest lower bound of $f(T)$ is $2(\sqrt{2}-1)$.

Equation (3.1) gives the unique geometry of a triangle T, with a given $\frac{c}{b}$ or a given α, that corresponds to the lowest value of $f(T)$ for all possible intersections of T and its enantiomorph under conditions of maximal overlap.

Let g_Δ denote the greatest lower bound of $f(T)$. Then

$$g_\Delta = \inf_T \{f(T)\colon T \text{ is a triangle in } E_2\} .$$

As seen from (3.2), for the general triangle the infimum of $f(T)$ corresponds to $\alpha=0$. Therefore the greatest lower bound is approached as a limit. Indeed, the similarity-invariance of $f(T)$ implies [7] that $\mathbf{K}_2$ is not a compact space and that an extremal triangle for which $f(T)$ assumes a minimal value may not exist. Accordingly, for the general triangle we have

$$f(T) > 2(\sqrt{2}-1) \approx 0.828 .$$

The least axial triangle is therefore a triangle whose altitude ($h \equiv CP$, Figure 2) is arbitrarily close to zero and for which $\frac{c}{b}=\sqrt{2}$ and $\frac{a}{b}=\sqrt{2}-1$ in the limit of $h=0$. The same lower bound was found by Nohl [9] for centric ovals K_c, with equality for a special class of parallelograms:

$$f(K_c) \geq 2(\sqrt{2}-1) .$$

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Klassische Beleuchtungsgeometrie im E^d $(d \geqq 2)$

I. Bekannte Kurvenklassen in der Beleuchtungsgeometrie des E^d $(d \geqq 2)$

Untersuchungen zur Beleuchtung von Flächen sind naturgemäss mit der Physik und Geometrie des dreidimensionalen euklidischen Raumes $\boldsymbol{E}^3$ verknüpft. Gerade in jüngster Zeit hat die *Beleuchtungsgeometrie* eine merkliche Wiederbelebung erfahren, so dass auch Betrachtungen, die über eigentliche Anregungen hinausgehen, nahegelegt werden. Dazu gehört eine Übertragung klassischer Ergebnisse auf den beliebigdimensionalen Raum $\boldsymbol{E}^d$ $(d \geqq 2)$.

1. Begriffswelt mit *d*-dimensionalem Abstandsgesetz

Grundbegriffe der auf den $\boldsymbol{E}^3$ bezogenen Beleuchtungstechnik und -geometrie werden sinngemäss aus [8] bzw. [3] übernommen. Der geometrische Raum $\boldsymbol{E}^d$ $(d \geqq 2)$ sei bezüglich eines kartesischen Normalkoordinatensystems durch den Raum der Koordinatenvektoren $\boldsymbol{R}^d$ beschrieben, wobei Punkte durch ihre Koordinatenvektoren bezeichnet sind (z. B. $\boldsymbol{x}$). Weiterhin steht $\langle \cdot, \cdot \rangle$ für das *innere Produkt*, $\| \cdot \|$ für die *euklidische Norm*, $S^{d-1} := \{\boldsymbol{u} \in \boldsymbol{R}^d \mid \langle \boldsymbol{u}, \boldsymbol{u} \rangle = 1\}$ für die *Einheitssphäre* und $\boldsymbol{o}$ für den *Koordinatennullpunkt* des $\boldsymbol{E}^d$.

Ein *orientiertes Flächenelement* sei mit $(\boldsymbol{x}, \boldsymbol{u})$ bezeichnet, wobei $\boldsymbol{x} \in \boldsymbol{R}^d$ den Träger und $\boldsymbol{u} \in S^{d-1}$ den Stellung und Orientierung des Elements angebenden Normaleneinheitsvektor bedeuten. Ist $(\boldsymbol{x}, \boldsymbol{u})$ von t Parametern $v_1, \ldots, v_t$ abhängig, dann liegt (im Anschluss an [6], S. 528 ff. und S. 33 ff., sowie [4], S. 102 ff.) eine *Element-t-Schar* vor.

Für $\{\boldsymbol{x}(v_1, \ldots, v_t), \boldsymbol{u}(v_1, \ldots, v_t)\}$ seien alle wünschenswerten analytischen Eigenschaften vorausgesetzt und uninteressante Ausartungen ausgeschlossen. Eine Elementschar ist ein *Element-t-Verein,* wenn in jedem Punkt der Trägermannigfaltigkeit $\{\boldsymbol{x}(v_1, \ldots, v_t)\}$ die durch $\boldsymbol{u}$ beschriebenen $(d-1)$-Ebenen den Tangentialraum enthalten. Insbesondere sind hier jene Elementvereine $\{\boldsymbol{x}(v_1), \boldsymbol{u}(v_1)\}$ interessant, die im differentialgeometrischen Sinne *Streifen* bilden (Streifenbedingung: $\langle \dot{\boldsymbol{x}}, \boldsymbol{u} \rangle = 0$). Eine *geometrische Zentralbeleuchtung* des $\boldsymbol{E}^d$ wird durch das Paar $(\boldsymbol{q}, I(\boldsymbol{n}))$ beschrieben, wobei $\boldsymbol{q} \in \boldsymbol{R}^d$ die punktförmige Lichtquelle