A very elementary proof of a probabilistic limit relation

Das exakte Differential

$$(\sqrt{x^2 - y} - x)\,dy + (\sqrt{x^2 - y} - x)^2\,dx$$

besitzt die Stammfunktion

$$\frac{2}{3}x^3 - \frac{2}{3}(x^2 - y)^{\frac{3}{2}} - x\,y\,.$$

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LITERATUR

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Let

$$a_n = e^{-n} \sum_{k=0}^{n-1} \frac{n^k}{k!}\,. \tag{1}$$

It is well known that

$$\lim a_n = \tfrac{1}{2}\,. \tag{2}$$

This limit relation has a definite probabilistic flavor. In «wise» terms, a_n is the probability that the sum of n independent, equally distributed Poisson random variables with parameter $\lambda = 1$ is smaller than mean value. Relation (2) hence follows immediately as a very particular case of the Central Limit Theorem.
(I first met quantities (1) when dealing with certain problems concerning probability measures in $\mathbb{R}^n$: confronting Gaussian distribution versus discrete measures concentrated on vertices of the n-cube.)
One inevitably encounters (1) and (2) in quite simple probabilistic considerations concerning interrelation between Poisson, normal and binomial distributions. A glance at the first few chapters of W. Feller's book [1] (its examples and exercises) will suffice to ascertain this.

Relation (2) however deserves an interest of its own. D. Newman has included it in his charming Problem Seminar [2] (Problem 96). There are several ways of proving (2); as a rule, they start from the integral representation of Taylor's remainder and then use some more or less advanced calculus methods (Euler's gamma function, Lebesgue Dominated Convergence Theorem, Poisson integral, and so on; Stirling's formula is the cheapest). It might be therefore of some interest to see an entirely elementary proof of (2).
Even deprived of its natural probabilistic context, relation (2) retains some grace. The sum in (1) is the initial segment of the power series representation of e^x, taken at $x = n$; summation is carried as long as the terms increase and stops when they begin to decrease. Thus, asymptotically, (2) asserts that the «growing part» of the series $\sum n^k/k!$ carries about a half of its entire mass. With this comment (and with a proof which avoids the use of nontrivial methods), formula (2) can be taught at the very beginning of a course of calculus. (It is then recommended to return to it several times during, say, two years to exhibit new possibilities given by more advanced techniques.)
Proof of Limit Relation (2). Split e^n into three summands,

$$e^n = A_n + B_n + C_n, \tag{3}$$

where

$$A_n = \sum_{k=0}^{n-1} \frac{n^k}{k!}, \quad B_n = \sum_{k=n}^{2n-1} \frac{n^k}{k!}, \quad C_n = \sum_{k=2n}^{\infty} \frac{n^k}{k!}.$$

The idea is to show that A_n and B_n are approximately equal, whereas the «tail» C_n is negligible, as compared with e^n.
We have

$$A_n = \frac{n^{n-1}}{(n-1)!}\left(1 + \sum_{k=1}^{n-1} n^{-k} \prod_{j=1}^{k} (n-j)\right),$$

$$B_n = \frac{n^n}{n!}\left(1 + \sum_{k=1}^{n-1} n^k \prod_{j=1}^{k} (n+j)^{-1}\right),$$

and since $n^{n-1}/(n-1)! = n^n/n!$, this yields

$$B_n - A_n = \frac{n^n}{n!} \sum_{k=1}^{n-1} F(n,k), \tag{4}$$

where

$$F(n,k) = \frac{n^k}{(n+1)\dots(n+k)} - \frac{(n-1)\dots(n-k)}{n^k} = \frac{N(n,k)}{D(n,k)}, \tag{5}$$

with numerator

$$N(n,k) = n^{2k} - (n^2-1)\dots(n^2-k^2)$$

and denominator

$$D(n,k) = n^k (n+1) \dots (n+k).$$

These are positive numbers, so that

$$F(n,k) > 0. \tag{6}$$

Consider the k-th degree polynomial

$$P(x) = \prod_{j=1}^{k} (n^2 - j^2 x).$$

It has k roots in $(1, +\infty)$, hence it is convex in $(-\infty, 1]$, hence is supported by the straight line $y = L(x)$,

$$L(x) = n^{2k} - n^{2k-2}\left(\sum_{j=1}^{k} j^2\right)x.$$

Setting $x = 1$ in the inequality $L(x) \leqq P(x)$ we obtain for the numerator of (5) the estimate

$$N(n,k) \leqq n^{2k-2} \sum_{j=1}^{k} j^2 = n^{2k-2} \cdot \tfrac{1}{6} k(k+1)(2k+1) \leqq n^{2k-2} k^3. \tag{7}$$

As to the denominator, note that if $k \geqq 2$ then

$$(n+1)\dots(n+k) > \sum_{\substack{i,j=1\\(i<j)}}^{k} ij \cdot n^{k-2}$$

$$= \tfrac{1}{2} n^{k-2}\left(\left(\sum_{j=1}^{k} j\right)^2 - \left(\sum_{j=1}^{k} j^2\right)\right) = \frac{1}{24} n^{k-2} k(k+1)(3k^2 - k - 2) \geqq \tfrac{1}{8} n^{k-2} k^4,$$

this being true also for $k = 1$. Thus

$$D(n,k) > \tfrac{1}{8} n^{2k-2} k^4$$

and so, by (5), (6) and (7),

$$0 < F(n,k) < \frac{8}{k}.$$

Hence, in view of (4),

$$0 < (B_n - A_n)e^{-n} < 8 \cdot \frac{n^n}{n!\, e^n} \sum_{k=1}^{n-1} \frac{1}{k}. \tag{8}$$

Stirling would now lead a quick way to (11). In our desire to be as elementary as possible we may also argue like this: the inequalities

$$\left(1+\frac{1}{n}\right)^{n+\frac{1}{3}} < e, \tag{9}$$

$$n!\,e^n > n^{n+\frac{1}{3}} \tag{10}$$

hold for $n = 1, 2, 3, \ldots$; (9) is proved by standard calculus method (examine $\varphi(x) = \log(x+1) - \log x - (x+\frac{1}{3})^{-1}$) and then (10) is proved by induction, with the aid of (9). Since the sum on the right hand of (8) is $o(n^{1/3})$, we get from (8) and (10)

$$\lim (B_n - A_n)\, e^{-n} = 0. \tag{11}$$

We are left with the «tail» term C_n (see (3)). But this is immediate:

$$C_n = \frac{n^{2n}}{(2n)!}\left(1 + \sum_{k=1}^{\infty} n^k \prod_{j=1}^{k} (2n+j)^{-1}\right) < \frac{n^{2n}}{(2n)!} \sum_{k=0}^{\infty} 2^{-k} = \frac{2n^{2n}}{(2n)!}.$$

In view of (10), $(2n)! > (2n)^{2n} e^{-2n}$, and therefore $C_n < 2^{1-2n} e^{2n}$, whence

$$\lim C_n e^{-n} = 0. \tag{12}$$

Now, relations (3), (11) and (12) yield the desired conclusion

$$\lim a_n = \lim A_n e^{-n} = \lim \tfrac{1}{2}((A_n + B_n)\, e^{-n} + (A_n - B_n)\, e^{-n}) = \tfrac{1}{2}.$$

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