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A *complex normed algebra A with unit 1* is a normed vector space over $\mathbb{C}$ where a multiplication $A \times A \to A$ is defined such that

$$\| x \cdot y \| \leqq \| x \| \, \| y \| , \qquad \text{for all } x, y \in A ,$$

where $\cdot$ denotes the multiplication and $\| \ \|$ the norm on A. With respect to this multiplication the unit 1 of A has of course the property $1 \cdot x = x \cdot 1 = x$ for every $x \in A$.
The normed algebras we usually meet in the applications and in introductory functional analysis courses are the so-called *Banach algebras.* These are normed algebras such that the norm induces a *complete* topology (i.e. one such that Cauchy sequences converge).
Now, every second year mathematics student knows that the set of all invertible elements of a Banach algebra A is *open* ($x \in A$ is *invertible* if there is a $y \in A$ such that $x \cdot y = y \cdot x = 1$. In such case we write $y =: x^{-1}$. The set of invertibles of A is denoted by $\mathrm{Inv}(A)$). The point is that the converse of this statement is false: just take a look at the algebra $R(D)$ of all complex rational functions defined on the closed unit disk of $\mathbb{C}$, endowed with the norm $\| q \| := \sup_{z \in \mathbb{C}} | q(z) |$. $q \in R(D)$ is invertible if and only if it has no zeros in D, and thus $\mathrm{Inv}(R(D))$ is open in $R(D)$, by the Maximum Principle. On the other hand, $R(D)$ is clearly no Banach algebra since there are analytic functions on D which are not rational (e.g. $\sin(z)$!).
Since the condition that $\mathrm{Inv}(A)$ be open has a well-mixed topological and algebraic nature, it seems interesting to define *Q-algebras* (or *open algebras,* as they are sometimes called) as those algebras (with unit!) which satisfy it. Of course, all Banach algebras are Q-algebras.
Our purpose is to show that almost all fundamental properties of Banach algebras are shared by the larger class of normed Q-algebras. Quite surprisingly, it turns out that some of these properties do actually characterize the normed Q-algebras among the normed algebras with unit.
In the following A will be a complex normed algebra with unit 1 and norm $\| \cdot \|$.
Fuster and Marquina [3] have proved the equivalence of the statements

(Q) A is a Q-algebra

(Q_{FM1}) $\exists\, \delta \in (0,1] : x \in A$ and $\| 1 - x \| < \delta$ imply $x \in \mathrm{Inv}(A)$

(Q_{FM2}) $\exists\, \delta \in (0,1] : x \in A$ and $\| x \| < \delta$ imply that $\sum x^n$ converges in A.

In an unpublished paper [4] Th. W. Palmer has given a further characterization:

(Q_{P}) A is inverse-closed in its completion, that is, if A^* is the completion of A, then $\mathrm{Inv}(A^*) \cap A \subset \mathrm{Inv}(A)$.

Let's now state our theorem:

Theorem. *Let A be a complex normed algebra with unit 1. Then the following conditions are equivalent:*

(Q) *A is a Q-algebra*

(Q_1) *If* $x \in A$ *and* $\| 1 - x \| < 1$ *then* $x \in \mathrm{Inv}(A)$

(Q_2) *If* $x \in A$ *and* $\| x \| < 1$ *then* $\sum x^n$ *converges in A*

(Q_3) $r(x) = \lim_n \| x^n \|^{1/n} = \inf_n \| x^n \|^{1/n}$ *for all* $x \in A$

(Q_4) $\sup_{\| x \| = 1} r(x) < \infty$

(Q_4') $r(x) \leqq \| x \|$ *for all* $x \in A$

(Q_5) $\partial \mathrm{Inv}(A) \subset \mathrm{TDZ}(A)$

(Q_6) $\mathrm{Rad}(A)$ *is closed and* $A/\mathrm{Rad}(A)$ *is a Q-algebra*

(Q_7) $\sigma: A \to P(\mathbb{C})$, $x \mapsto \mathrm{Sp}(x)$ *is upper semicontinuous*

(Q_7') σ *is upper semicontinuous at* $0 \in A$

(Q_8) D: $x \mapsto \mathrm{diam}(\mathrm{Sp}(x))$ *is upper semicontinuous*

(Q_8') *D is continuous at* $0 \in A$.

Remarks on notation: 1. $\mathrm{TDZ}(A)$ in (Q_5) is the set of *topological divisors of zero* in A. Recall that x is in $\mathrm{TDZ}(A)$ if there is a sequence (w_n) in A with $\| w_n \| = 1$ for all n, and such that $\lim_n w_n x = 0 = \lim_n x w_n$ (see [2], p. 12).

2. $\mathrm{Rad}(A)$ is defined as the intersection of all maximal left ideals in A (see [2], p. 124). Rad stands for *radical.*

3. If $x \in A$, then $\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} : \lambda 1 - x \notin \mathrm{Inv}(A)\}$ is the *spectrum* of x (in A). $r(x) := \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$ is the *spectral radius* of x.

4. If K is a subset of A, ∂K is the *boundary* of K in A, i.e. $\partial K = \bar{K} \setminus \mathring{K}$, where $\bar{K}$ is the closure of K and $\mathring{K}$ is the set of its inner points.

5. *Upper semicontinuous* in (Q_7) means that, for each $x \in A$ and each open subset U of $\mathbb{C}$ such that $\mathrm{Sp}(x) \subset U$, there is a $\delta > 0$ with $\| y - x \| < \delta \Rightarrow \mathrm{Sp}(y) \subset U$.

Proof:
$(Q) \Rightarrow (Q_{\mathrm{FMI}})$: This is trivial since if $\mathrm{Inv}(A)$ is open then 1 is an inner point of $\mathrm{Inv}(A)$, that is, there exists a $\delta > 0$ with $\{y : \| 1 - y \| < \delta\} \subset \mathrm{Inv}(A)$. Since $0 \notin \mathrm{Inv}(A)$, clearly $\delta \leqq 1$.

$(Q_{\mathrm{FMI}}) \Rightarrow (Q_4)$: Let $\delta \in (0, 1]$ be as in (Q_{FMI}). We have $r(x) \leqq \dfrac{\| x \|}{\delta}$ for all $x \in A$, hence $\sup_{\| x \| = 1} r(x) \leqq 1/\delta$.

$(Q_4) \Rightarrow (Q_3)$: The formulas $r(a) \geqq \lim_n \| a^n \|^{1/n} = \inf_n \| a^n \|^{1/n}$ and $\mathrm{Sp}(a) \neq \emptyset$ are true in all complex normed algebras (see [2], Prop. 2.8 and Th. 5.7). It remains to prove that $r(x) \leqq \lim_n \| x^n \|^{1/n}$ for all $x \in A$. Since $\mathrm{Sp}(q(x)) = q(\mathrm{Sp}(x))$ for all nonconstant polynomials q ([2], Prop. 5.5), we have, for $n \in \mathbb{N}$,

$$r(x)^n = r(x^n) \leqq M \cdot \| x^n \| ,$$

where $0 < M := \sup_{\|y\|=1} r(y) < \infty$. Now it follows immediately that

$$r(x) \leqq \lim_n M^{1/n} \| x^n \|^{1/n} = \lim_n \| x^n \|^{1/n} .$$

$(Q_3) \Rightarrow (Q_4')$: $r(x) = \inf_n \| x^n \|^{1/n} \leqq \| x \|$, for all $x \in A$.

$(Q_4') \Rightarrow (Q_1)$: Let $x \in A$ and $\| 1 - x \| < 1$. Then $r(1-x) < 1$, that is, $1 \notin \mathrm{Sp}(1-x)$, hence $x = 1 - (1-x) \in \mathrm{Inv}(A)$.

$(Q_1) \Rightarrow (Q_2)$: (see [3]). Let (Q_1) hold and let $\| x \| < 1$. Define $s_N := \sum_{n=0}^{N} x^n$ for all $N \geqq 0$ ($x^0 := 1$). By (Q_1), $1 - x$ is invertible. Let $y := (1-x)^{-1}$. We have then

$$\begin{aligned} \| s_N - y \| &= \| y(1-x) s_N - y \| \leqq \| y \| \; \| (1-x) s_N - 1 \| \\ &= \| y \| \; \| x^{N+1} \| \leqq \| y \| \; \| x \|^{N+1} . \end{aligned}$$

Since $\| x \| < 1$, we get $\lim_N s_N = y$, that is, $\sum x^n$ converges.

$(Q_2) \Rightarrow (Q_1)$: Let $x \in A$ and $\| 1 - x \| < 1$. It follows from (Q_2) that $\sum (1-x)^n$ converges to some $y \in A$. Now, since

$$\begin{aligned} \left\| 1 - x \sum_{n=0}^{N} (1-x)^n \right\| &= \left\| 1 + (1-x) \sum_{n=0}^{N} (1-x)^n - \sum_{n=0}^{N} (1-x)^n \right\| \\ &= \| (1-x)^{N+1} \| \leqq \| 1 - x \|^{N+1} , \end{aligned}$$

we get $x y = 1$. Similarly, $y x = 1$ and thus x is invertible.

$(Q_1) \Rightarrow (Q)$: Let x be invertible, and let $y \in A$ with $\| x - y \| < 1 / \| x^{-1} \|$. This implies

$$\| 1 - x^{-1} y \| = \| x^{-1} (x - y) \| < 1 ,$$

that is, $x^{-1} y$ is invertible, by (Q_1). Let $w := (x^{-1} y)^{-1}$. It is clear that $(w x^{-1}) y = 1$, and thus y is left invertible. Analogously we prove that $y x^{-1}$ is invertible and, with $z := (y x^{-1})^{-1}$, we have $y (x^{-1} z) = 1$. Since y is left and right invertible, y must be invertible. This proves that $\mathrm{Inv}(A)$ is open.

For completeness' sake we prove

$(Q) \Rightarrow (Q_P)$: Let $x \in A \cap \mathrm{Inv}(A^*)$, $x^{-1} \in A^* \backslash A$, where A^* is the completion of A. If we had $x \in \partial \mathrm{Inv}(A)$, there would exist $(x_n) \in \mathrm{Inv}(A)^{\mathbb{N}}$ such $\lim_n x_n = x$. Since $\lim_n x_n^{-1} = x^{-1}$ in A^*, we would have in particular that $M := \sup_n \| x_n^{-1} \| < \infty$. Taking n sufficiently

large, we would get $M \cdot \| x - x_n \| < 1$ and $x = x_n(1 + x_n^{-1}(x - x_n)) \in \mathrm{Inv}(A)$, by $(Q) \Leftrightarrow (Q_1)$: a contradiction. Hence $x \notin \partial\mathrm{Inv}(A)$.
Since $\mathrm{Inv}(A^*)$ is open in A^*, we get thus a neighbourhood U of x in A^* such that $U \subset \mathrm{Inv}(A^*)$ and $U \cap \mathrm{Inv}(A) = \emptyset$. Since $a \mapsto a^{-1}$ is a homeomorphism in A^* ([2], Prop. 2.6), we have that U^{-1} is open in A^* and $U^{-1} \cap A = \emptyset$, which contradicts the denseness of A in A^*.

$(Q) \Rightarrow (Q_5)$: Let $x \in A \cap \partial\mathrm{Inv}(A)$, $(x_n) \in \mathrm{Inv}(A)^{\mathbf{N}}$ such that $\lim\limits_n x_n = x$. We claim that $\sup\limits_n \| x_n^{-1} \| = \infty$. In fact, if we assume that $\| x_n^{-1} \| \leqq N < \infty$ for all n, we have

$$\| x_m^{-1} - x_n^{-1} \| = \| x_m^{-1}(x_n - x_m) x_n^{-1} \| \leqq N^2 \cdot \| x_n - x_m \| .$$

It follows that (x_n^{-1}) is a Cauchy sequence in A (say, with limit $y \in A^*$, A^* the completion of A). Then $xy = yx = 1$ by continuity of multiplication and thus $x \in \mathrm{Inv}(A^*)$. By $(Q_P) \Leftarrow (Q)$, $x \in \mathrm{Inv}(A)$, which is a contradiction.
Without loss of generality let also $\| x_n^{-1} \| \geqq n$, for all n, and define $w_n := x_n^{-1}/\| x_n^{-1} \|$. It is now easy to see that $\lim\limits_n x\, w_n = \lim\limits_n w_n x = 0$, that is, $x \in \mathrm{TDZ}(A)$.

$(Q_5) \Rightarrow (Q)$: If A were not a Q-algebra, there would exist $x \in \mathrm{Inv}(A) \cap \partial\mathrm{Inv}(A)$. Since $x \in \mathrm{Inv}(A)$, x cannot be in $\mathrm{TDZ}(A)$, contradicting (Q_5).

$(Q) \Rightarrow (Q_6)$: If A is a Q-algebra, then maximal left ideals are closed. This is an easy consequence of $J \subset A \backslash \mathrm{Inv}(A)$ for every proper left ideal J. It is also an easy task to prove that A/I is a Q-algebra for *every* ideal I.

$(Q_6) \Rightarrow (Q_3)$: We have that

$$\mathrm{Rad}(A) = \{x\colon 1 - xy \in \mathrm{Inv}(A) \text{ and } 1 - yx \in \mathrm{Inv}(A) \text{ for all } y \in A\}$$

([2], Prop. 24.16, Cor. 24.17). Using this result, we may follow Aupetit ([1], Lemme I,1.2) to obtain

$$\mathrm{Sp}(x) = \mathrm{Sp}(\hat{x})$$

for all $x \in A$, where $\hat{x}$ denotes the class of x in $A/\mathrm{Rad}(A)$. Let $x \in A$. Since $A/\mathrm{Rad}(A)$ is a Q-algebra, $(Q) \Leftrightarrow (Q_3)$ gives

$$r(\hat{x}) = \lim_n \| \hat{x}^n \|^{1/n}$$

and thus

$$\lim_n \| x^n \|^{1/n} \leqq r(x) = \lim_n \| \hat{x}^n \|^{1/n} \leqq \lim_n \| x^n \|^{1/n},$$

which was to be proved (the first inequality follows from the general theorem already quoted in the proof of $(Q_4) \Rightarrow (Q_3)$).

$(Q) \Rightarrow (Q_7)$: Let σ be not upper continuous at $x \in A$. Choose U open in $\mathbb{C}$ such that $\mathrm{Sp}(x) \subset U$ and $(x_n) \in A^{\mathbb{N}}$, $(\alpha_n) \in \mathbb{C}^{\mathbb{N}}$ such that $\lim_n x_n = x$, $\alpha_n \in \mathrm{Sp}(x_n) \setminus U$. Since (by $(Q) \Leftrightarrow (Q_4')$)

$$\sup_n |\alpha_n| \leqq \sup_n r(x_n) \leqq \sup_n \| x_n \|,$$

we may assume that the x_n are chosen in such a way that (α_n) converges. Let $\alpha := \lim_n \alpha_n$. Then $\alpha \notin U$ and since $\alpha_n 1 - x_n \notin \mathrm{Inv}(A)$ for all n, and since $\alpha 1 - x = \lim_n \alpha_n 1 - x_n$, we have that $\alpha \notin \mathrm{Sp}(x)$ and $\alpha 1 - x \in \mathrm{Inv}(A) \cap \partial \mathrm{Inv}(A)$, a contradiction with (Q).

$(Q_7) \Rightarrow (Q_7') \Rightarrow (Q_8')$ and $(Q_7) \Rightarrow (Q_8) \Rightarrow (Q_8')$ are clear.

$(Q_8') \Rightarrow (Q)$: Choose $\delta > 0$ such that $\| x \| < \delta$ implies $\mathrm{Sp}(x) \subset U_{1/2}(0)$. It follows that $0 \notin \mathrm{Sp}(1-x) = 1 - \mathrm{Sp}(x)$, that is, $1 - x \in \mathrm{Inv}(A)$. $(Q) \Leftrightarrow (Q_1)$ now does the rest.

Remarks: 1. As regards Palmer's characterization (Q_P), the implication $(Q_P) \Rightarrow (Q)$ is very easy to prove: if $x \in A$ and $\| 1 - x \| < 1$, then $x \in \mathrm{Inv}(A^*)$, since A^* is a Banach algebra, but this implies $x \in \mathrm{Inv}(A)$ by (Q_P).

2. I believe that our Theorem may sufficiently increase the popularity of normed Q-algebras. It is now clear that lots of elementary results about Banach algebras are true for Q-algebras, too: it is unfortunate that they are usually confusingly proved under completeness assumptions (see, for instance, [5], Chapter 18).

3. Our Theorem clearly has many applications. One may use the standard Banach-algebra-proofs to obtain, for instance, the following "Gelfand-Theorems":

Theorem (*): *Commutative complex normed Q-algebras are exactly those A, for which there exist a compact space K and an isomorphism ϕ of $A/\mathrm{Rad}(A)$ onto a full subalgebra of $C(K)$, which is separating in $C(K)$ and contains 1_K.*

Theorem (**): *Commutative Q^*-algebras* (defined analogously to C^*-algebras) *are the full dense subalgebras of $C(K)$ which are separating and contain 1_K, for a certain compact space K.*

(Recall that a subalgebra B of A is *full* if B contains the unity of A and if, whenever $b \in B$ has an inverse b^{-1} in A, b^{-1} is in B. A subalgebra of $C(K)$ is *separating* if, given points p and q in K, there is an f in A with $f(p) \neq f(q)$.) It is a very useful exercise to prove these theorems!

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