The forcibly-tree and forcibly-unicyclic degree sequences

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Dedicated to Frank Boesch

1. Introduction

We follow the notation and terminology of [3]. Thus the degree sequence π of a graph G is written $d_1 \geq d_2 \geq \cdots \geq d_p$. A given nonincreasing sequence σ of positive integers is called *graphical* if σ is the degree sequence of some graph. Criteria for σ to be graphical are well known; see [3], p. 58, or Hakimi and Schmeikel [2] which presents a recent survey of the literature on graphical degree sequences.

Let P be a property of graphs, such as hamiltonian, tree, unicyclic, etc. A graphical sequence π is called *potentially-P* if there exists a graph G which realizes π having property P. Similarly π is *forcibly-P* if it is not only potentially-P but every graph which realizes π satisfies property P. The latter concept was introduced by Nash-Williams [4] in his discussion of forcibly-hamiltonian degree sequences.

It is easy to see [3], p. 62, that a degree sequence $(d_1, d_2, \ldots, d_p)$ is potentially-tree if and only if $2q = \sum d_i$, every d_i is positive and $q = p - 1$.

Potentially-unicyclic degree sequences were characterized by Boesch and Harary [1] as precisely those graphical sequences $d_1, \ldots, d_p$ such that $d_3 > 1, d_p > 0$ and $\sum d_i = 2p$.

Our present object is to characterize forcibly-tree and forcibly-unicyclic degree sequences.

Theorem 1. *A graphical degree sequence $d_1 \geq d_2 \geq \cdots d_p \geq 1$ is forcibly-tree if and only if either $p = 2$ and $d_1 = d_2 = 1$ or $d_3 = 1$ and $d_1 + d_2 = p$.*

Proof: If a tree T contains a path of length 4, let $v_1v_2v_3v_4v_5$ be its points. Let $G(T)$ be the graph with point set $V(T)$, whose line set is $E(G) = E(T) \cup \{v_1v_5, v_2v_4\} - \{v_1v_2, v_4v_5\}$. Then $G(T)$ is not a tree but has the same degree sequence as T. Hence no forcibly-tree degree sequence can be realized by a tree containing P_5 and so is only realized by K_2, a star or a double star. The latter two families of trees satisfy $d_3 = 1$ and $d_1 + d_2 = p$.

For the converse it is clear that a degree sequence of the given type can only be the degree sequence of a double star, a star or K_2. These are precisely the trees with diameter at most three. □

Theorem 2. *A graphical degree sequence $d_1 \geq d_2 \geq \cdots \geq d_p \geq 1$ is forcibly-unicyclic if and only if it is either one of the sequences $2^3, 2^4, 2^5, d_2^4 d_1^{d-2}, d_2^3 d_1^{d-2}$, where $d_1 = d \geq 3$, or satisfies $d_3 \geq 2, d_4 = 1$ and $d_1 + d_2 + d_3 = p + 3$.*

Proof: The first step is to list a family of unicyclic graphs with the property that any unicyclic supergraph of one of these does not have a forcibly-unicyclic degree sequence. These minimal unicyclic graphs are the cycles $C_n, n \geq 6$, and the graphs in figure 1.

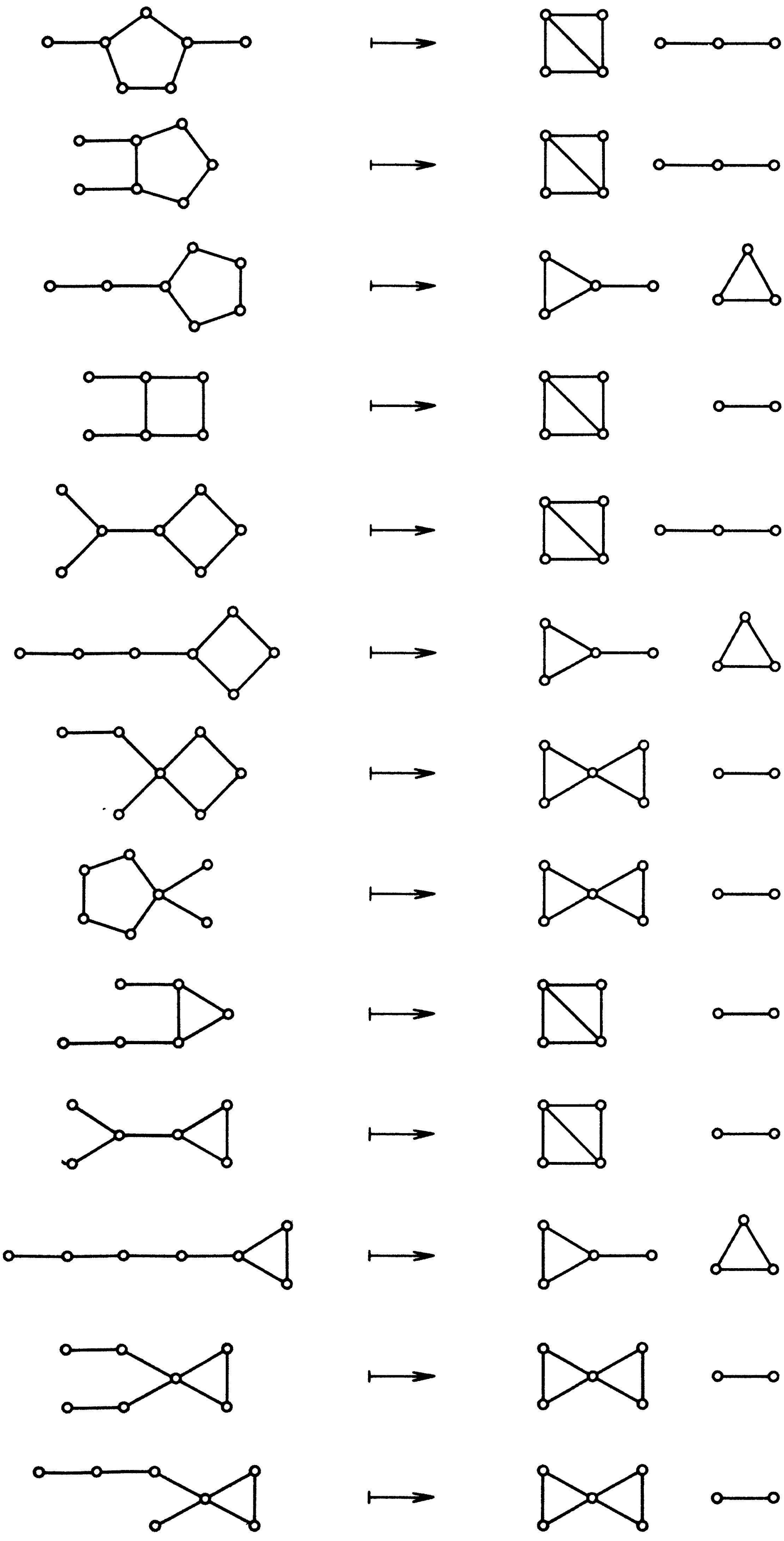

Figure 1. Minimal unicyclic graphs with nonforcible degree sequences.

We also show in figure 1 how to transform each minimal graph into a nonunicyclic graph with the same degree sequence. In each case it is immediate that any unicyclic graph which contains the given graph as a subgraph can be similarly transformed. Of course each cycle $C_n, n \geq 6$, is sequentially equivalent to $C_{n-3} \cup K_3$.

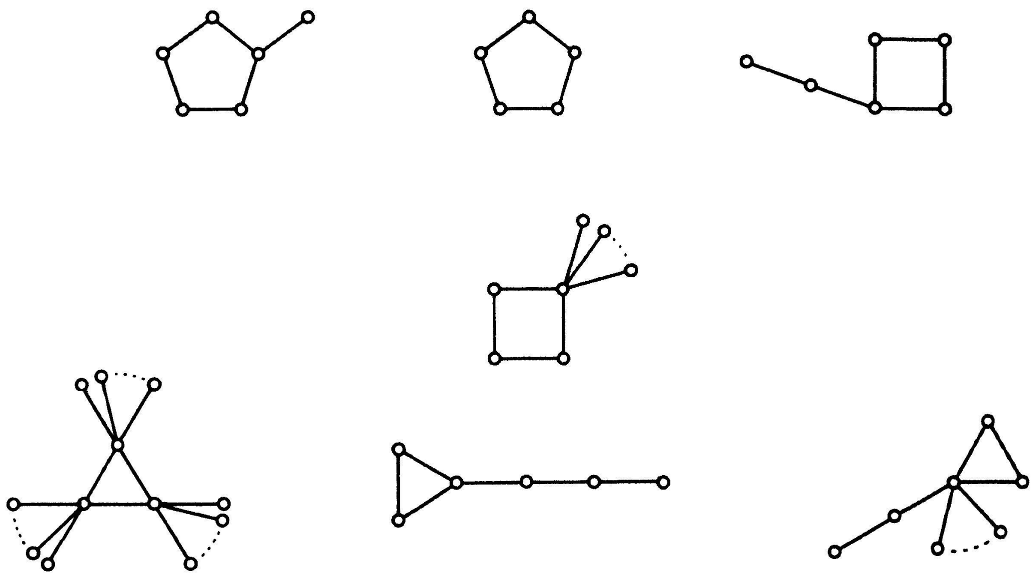

Figure 2. The graphs with forcibly-unicyclic degree sequences.

Next we determine and display in figure 2 the unicyclic graphs which contain no subgraph from the forbidden family, i.e., no long cycle and none of the graphs in figure 1.

Evidently the degree sequences of the graphs in figure 2 are precisely those given in the statement of theorem 2. It is easy to verify that each of these is forcibly-unicyclic.

In later communications we plan to study forcibly-*P* degree sequences for other properties *P*.

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