On the upward motion of a double cone on an inclined plane

Autor(en): **Bottema, O. / Groenman, J.T.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **36 (1981)**

Heft 5

PDF erstellt am: **24.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-35551

ANMERKUNGEN

1 In § 36 (S.312) schreibt er: «Solutio haec eo maiorem attentionem meretur, quod ad eam nulla certa methodo, sed potius quasi divinando sum perductus; et quoniam ea adeo octo numeros arbitrarios implicat, qui quidem facta reductione ad unitatem ad septem rediguntur, vix dubitare licet, quin ista solutio sit universalis et omnes prorsus solutiones possibiles in se complectatur. Si quis ergo viam directam ad hanc solutionem manuducentem investigaverit, insignia certe subsidia Analysi attulisse erit censendus. Utrum autem similes solutiones pro amplioribus quadratis, quae numeris 25, 36 et maioribus constant, expectare liceat, vix affirmare ausim. Non solum autem hinc Algebra communis, sed etiam Methodus Diophantea maxima incrementa adeptura videtur.» Siehe dazu auch O. Volk: Miscellanea from the History of Celestial Mechanics *14,* (1976), 365-382, Anmerkung 7, S.378-379.
2 Perturbation Theory of Kepler Motion based on Spinor Regularization. Journal für die reine und angewandte Mathematik *218,* (1965), 204-219.
3 Der Briefverkehr findet grösstenteils in lateinischer Sprache statt. Beide, Goldbach wie auch Euler, verfügten über einen guten lateinischen Stil. (Dies trifft nur auf *die etwa 30 ersten Briefe* zu. Der Hauptteil der ganzen Korrespondenz ist deutsch geschrieben [E.A.F.].)
4 Der Satz von der «Quadratzerlegung» findet sich zum ersten Male bei C.G. Bachet de Méziriac (1581-1638) im Kommentar zur Ausgabe «Diophanti Alexandrini Arithmeticorum libri sex et de numeris multangulis liber unus», S.241-242. Paris 1621. Siehe dazu die Anmerkung 6 des «Briefwechsels», S.32.

On the upward motion of a double cone on an inclined plane

1. In some older 'physical cabinets' the following apparatus may be found. Its main part are two rails, s_1 and s_2, both starting at a point O, going upward and having the same inclination with the horizontal plane through O (fig. 1; where the cone K_1 and the line s_1 are given by their horizontal and vertical orthogonal projection). (As a rule these 'rails' are the upper rims of two vertical triangular wooden partitions.) The space between the rails is open.
The bisector plane U of the angle between s_1 and s_2 is a plane of symmetry of the apparatus. The rails determine an inclined plane V with inclination a. A rigid homogeneous body B consists of two congruent right circular cones, K_1, K_2, with the same basic circle C (centre M, radius R) and height h; the vertex of K_i is T_i, $i = 1,2$.
This double cone is placed in a symmetric way (that is with C in the plane U) on the rails, K_1 on s_1, K_2 on s_2, the tangent points being A_1 and A_2, respectively. The surprising result is that B, under gravity, *moves upward along the rails.* The explanation is of course that if A_1 and A_2 move upward more material of B is pushed into the space between the rails with the effect that M, the mass-centre of B, moves downward as it should be.
This device, a simple and ingenious example of *physique amusante,* seems almost forgotten. It is mentioned for instance by Sutton [1] and discussed at some length in such books as Violle [2], Auerbach [3] and Müller-Pouillet [4], but these restrict themselves to a description and some general remarks; no further references are given. The apparatus is determined by three essential parameters;

for instance the slope α of the plane V, β equal to half the angle between the projections s_1', s_2' of s_1, s_2 on the horizontal plane and γ, the angle between the generators of a cone and its axis. [The length R or h determines only the scale.] It is reasonable to expect that certain inequalities between them must be satisfied for the apparatus to be operational. It was not easy to trace some 'theory' on the apparatus but by good fortune we came across a paper of 1854 about the subject [5]. It makes uses of calculus and spherical trigonometry and restricts itself moreover to the case of B being in equilibrium on the rails. The text implies that the apparatus was well-known at its time, but the origin remains a mystery. We try to derive the said conditions by elementary means, restricting ourselves to the geometry of the phenomenon and not dealing with dynamical considerations.

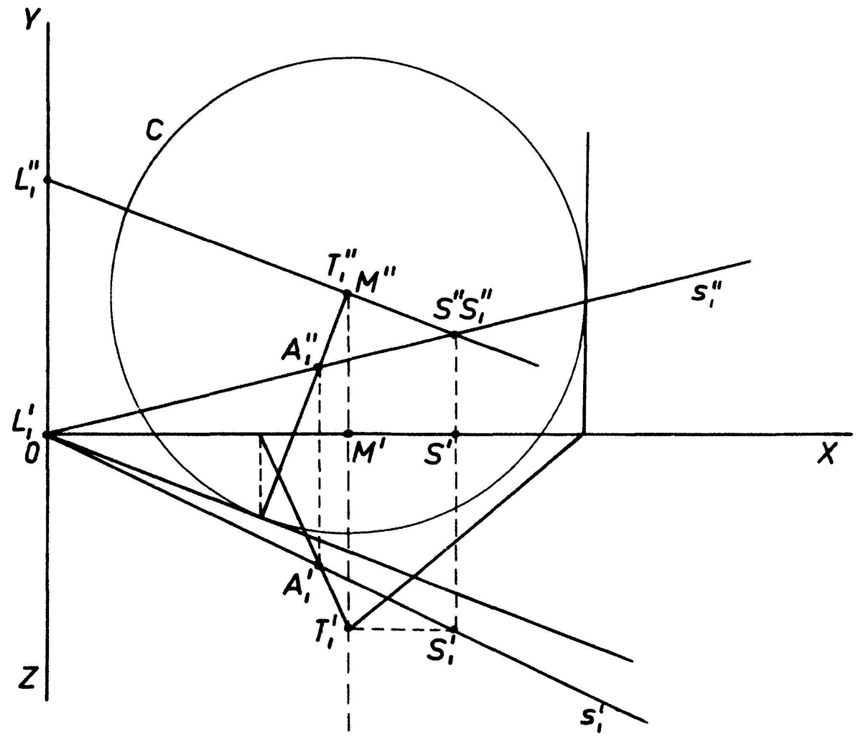

Figure 1

2. We introduce the following cartesian frame: O is the origin, OY is vertical upward, OXZ is the horizontal plane, OX is in U. We put $\tan\alpha = a$, $\tan\beta = b$, $\tan\gamma = R/h = c$; a, b, c are positive numbers.
The circle C is in the plane $z=0$; let M be the point $(x_0, y_0, 0)$.
We try to determine the locus of M for the various positions of B on the rails. Obviously we may restrict ourselves to one half of the figure, $z \geq 0$ say, with s_1 and T_1. The equation of K_1 is

$$(x-x_0)^2+(y-y_0)^2-c^2(z-h)^2=0. \tag{2.1}$$

Equations for s_1 are $y=ax$, $z=bx$; its intersections with K_1 follow from

$$(a^2+1-b^2c^2)x^2+2(-x_0-ay_0+bc^2h)x+(x_0^2+y_0^2-c^2h^2)=0. \tag{2.2}$$

If s_1 is tangent to K_1 this equation must have two equal roots. We find the following condition for (x_0, y_0):

$$(b^2c^2-a^2)x_0^2+2ax_0y_0+(b^2c^2-1)y_0^2-2bc^2hx_0-2abc^2hy_0 \\ +(a^2+1)c^2h^2=0, \tag{2.3}$$

which represents therefore the locus of M.
(2.3) is the equation of a conic but after some algebra (calculating the associated determinant) it is seen to be degenerate: it consists of two straight lines l_1, l_2 through the point $S=(h/b, ah/b)$. This point has a physical meaning: if M is at S, then T_1 has the coordinates $(h/b,\ ah/b,\ h)$ which implies that T_1 is on s_1. Hence at this moment the double cone is entirely between the rails, A_1 and A_2 have arrived at their highest point and the phenomenon has reached its natural finish.
Although S is a real point (2.3) could be the equation of two conjugate imaginary lines. To verify this we intersect (2.3) with the axis OY. For $x_0=0$ (2.3) gives us the equation

$$(b^2c^2-1)y^2-2abc^2hy+(a^2+1)c^2h^2=0 \tag{2.4}$$

which has real and distinct roots if and only if

$$D=a^2+1-b^2c^2>0 \tag{2.5}$$

(we can neglect the case $D=0$, for the two equal roots of (2.2) would be infinite). (2.5) is therefore a necessary condition for the configuration to be possible. If it is satisfied the intersections L_1, L_2 of l_1, l_2 and OY are $(0, y_{1,2}, 0)$ with

$$y_{1,2}=\frac{(-abc^2+c\sqrt{D})\,h}{1-b^2c^2}. \tag{2.6}$$

3. At first sight it is surprising that the locus of M consists of *two* lines. But we must keep in mind that our analytical condition for s_1, to be tangent to K_1 covers two possibilities: s_1 can be *below* K_1 (that is what we want) or *above* it (which is meaningless for the problem). We have only to consider those line-segments SL_i $(i=1,2)$ *above* SO.
There is, however, another consequence of the analytical method we must give attention to. The equation (2.1) of K_1 is not only valid for the part of the cone between T_1 and the circle C, but also for those points on its generators which lie on the other side of T_1. In other words: the z-coordinate of A_1 must be less than h, or what is the same thing its x-coordinate must be less than that of S. This x-coordinate is the root of (2.2) if this equation has two equal roots. Hence

$$x(A_1)=\frac{x_0+ay_0-bc^2h}{a^2+1-b^2c^2}<\frac{h}{b}. \tag{3.1}$$

As $D>0$ this implies

$$bx_0+aby_0-(a^2+1)h<0, \tag{3.2}$$

which means that M must lie below the line g with the equation

$$bx+aby-(a^2+1)h=0. \tag{3.3}$$

This line g also passes through S and it intersects OY at

$$G=(0,y_3) \quad \text{with}$$

$$y_3=\frac{a^2+1}{ab}h. \tag{3.4}$$

4. To discuss the conditions to be satisfied by SL so far we distinguish three cases: 1. $1-b^2c^2>0$, 2. $1-b^2c^2=0$ (both implying $D>0$) and 3. $1-b^2c^2<0$ (from which it follows that $1-b^2c^2>-a^2$).

Ad. 1. If $1-b^2c^2>0$ the two roots of 2.4 have a negative product; one is positive and the other negative. From (2.6) it follows that the +-sign corresponds to the positive root, y_1 say. Then l_1 is above SO and l_2 is below SO and must therefore be rejected (if M is on it s_1 is tangent to K_1 on the wrong side). Furthermore $y_1<y_3$ must hold, or in view of (2.6) and (3.4):

$$ab(-abc^2+c\sqrt{D})<(1-b^2c^2)(a^2+1),$$

which is equivalent with $(a^2+1)(1-b^2c^2)>0$ which is indeed true.
Hence l_1 is the locus of M to be considered. The situation is given in figure 2a.

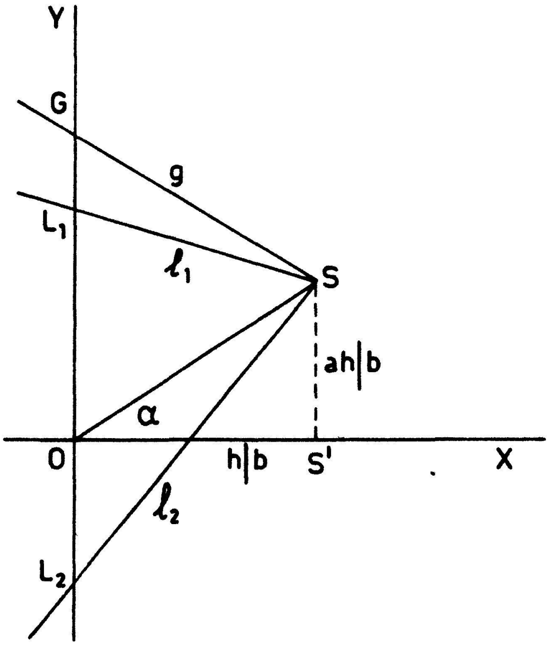

Figure 2a

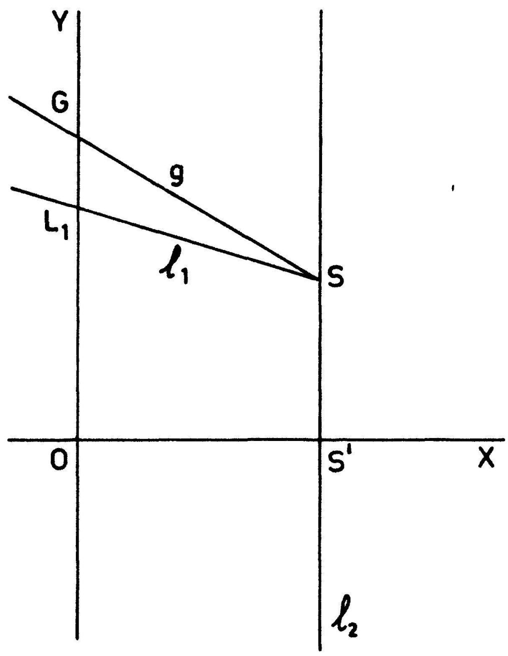

Figure 2b

Ad. 2. If $1-b^2c^2=0$ one of the roots of 2.4 y_2 say, is infinite, l_2 is the vertical line through S. The other root y_1 is seen to be $(a^2+1)h/2ab$ which is less than y_3 as it should be (fig. 2b).

Ad. 3. If $1-b^2c^2<0$ both roots of (2.4) are positive. From (2.6) it follows $y_1<y_2$. Furthermore it is easy to check that $y_1<y_3$ and $y_2>y_3$ (fig. 2c). Summing up we have: *the locus of M is the line-segment SL_1*, where $S=(h/b, ah/b)$ and $L_1=(0,y_1)$ with

$$y_1=\frac{-abc^2+c\sqrt{D}}{1-b^2c^2}h, \qquad \text{if } \ 1-b^2c^2\neq 0, \tag{4.1}$$

$$y_1=\frac{a^2+1}{2ab}h, \qquad \text{if } \ 1-b^2c^2=0. \tag{4.2}$$

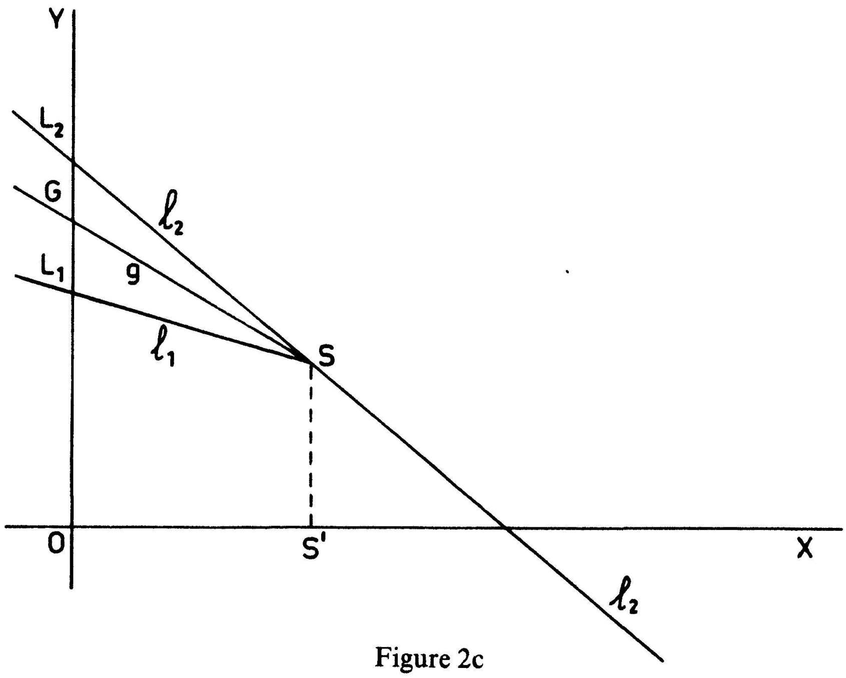

Figure 2c

5. Now we have located M we can introduce the further condition: the path of M must be a descending line, L_1 must be higher than S, $y_1>ah/b$. For (4.1) this gives us, for $1-b^2c^2>0$ and for $1-b^2c^2<0$: $a^2<b^2c^2$ and for (4.2): $a^2<1=b^2c^2$. Keeping in mind that $D>0$ we have: *the necessary and sufficient conditions for B going upward on the rails are:*

$$a^2<b^2c^2<a^2+1, \tag{5.1}$$

or

$$\tan\alpha<\tan\beta\,\tan\gamma<1/\cos\alpha. \tag{5.2}$$

6. If we restrict ourselves to the case that B is in equilibrium at any position on the rails, as Fleury did, the necessary and sufficient condition is obviously

$\tan\alpha = \tan\beta \tan\gamma$, in accordance with his result. The second inequality of (5.2) could be new, together with the fact that the path of M is a straight line.

O. Bottema and J.T. Groenman, Delft

REFERENCES

1 R.M. Sutton: Demonstration experiments in physics, p.26 (1938). "A double cone will appear to roll up an incline made of two gradually separating rails if the slope is such that the center of gravity of the body is slowly lowered as it rolls."

2 J. Violle: Cours de physique, tome I, p.164 (1883). «... le double cône homogène que l'on place sur deux règles inclinées réunies à leur sommet par une charnière permettant de les écarter plus ou moins; si l'angle de ces deux lames a une valeur convenable, le double cône roule vers la partie la plus haute des lames inclinées et paraît remonter, tandis qu'en réalité son centre de gravité ne fait que descendre.»

3 F. Auerbach and W. Hort: Handbuch der Physikalischen und Technischen Mechanik, Band II. Technische und Physikalische Mechanik starrer Systeme, p.527–528 (1927).

4 Müller-Pouillet: Lehrbuch der Physik, 1.Band. 1.Teil: Mechanik punktförmiger Massen und starrer Körper, p.632 (1929). «Wir erwähnen alle diese Apparätchen, weil sie nun einmal zum Inventar der physikalischen Kabinette gehören und deshalb dem Physiker bekannt sein müssen.»

5 H. Fleury: Condition d'équilibre du double cône sur deux droites concourantes et également inclinées sur le plan horizontal. Nouvelles Annales de Mathématiques, tome 13, p.211–219 (1854).

Elementarmathematik und Didaktik

Berlekamp-Algorithmus und programmierbarer Taschenrechner

Zahlentheoretische Probleme werden am Gymnasium fast ausschliesslich in der Orientierungsstufe behandelt. Zu diesem Themenbereich gehört auch die Bestimmung des grössten gemeinsamen Teilers (ggT) zweier natürlicher Zahlen, die im Schulunterricht meist mit Hilfe der Primzahlzerlegung erfolgt. Die Einbeziehung von programmierbaren Taschenrechnern in den Unterricht ermöglicht es, derartige Problemstellungen in höheren Klassenstufen nochmals anzusprechen. Hierbei kommt der wohlbekannte euklidische Algorithmus, der in der Orientierungsstufe m.E. zu wenig beachtet wird, wieder voll zur Geltung. Da sich zudem der ggT zweier natürlicher Zahlen a und b als Linearkombination von a und b mit ganzzahligen Koeffizienten darstellen lässt, möchte man diese Koeffizienten ebenfalls mit dem Rechner ermitteln.
Eine Möglichkeit besteht darin, die Zwischenergebnisse, die man im Verlauf der Rechnung erhält, in die letzte Gleichung des euklidischen Algorithmus sukzessive einzusetzen und auf diese Art rückwärts die gewünschte Linearkombination zu berechnen. Dazu müssen aber vorher alle Zwischenergebnisse gespeichert werden, und dies kann, insbesondere wenn a und b ziemlich gross sind, erheblichen Speicherbedarf erfordern.
Man ist folglich an einem Algorithmus interessiert, der mit weniger Speicherplatz auskommt. Um dies zu erreichen, muss die Speicherung der Zwischenergebnisse