Inequalities involving convex sequences

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$\bar{\Pi}$ ist von 24 kongruenten Fünfecken begrenzt; jedes besitzt eine Symmetrieachse. Eines ist in Figur 2 mit I II III IV V bezeichnet. Man findet

$$\overline{\mathrm{I,II}}=\overline{\mathrm{I,V}}=(\sqrt{p}+\sqrt{p^{-1}})/3, \qquad \overline{\mathrm{II,III}}=\overline{\mathrm{III,IV}}=\overline{\mathrm{IV,V}}=2\sqrt{p}/3\,,$$
$$\overline{\mathrm{II,V}}=2\sqrt{p^{-1}}/3\,. \tag{30}$$

5.2. *Inzidenzen im Kantensystem von $\bar{\Pi}$.* Die Polarität an $\varkappa$ führt die Verbindungsebene $[\mu\nu]$ komplanarer Kanten von Π in den Schnittpunkt der entsprechenden Kanten $\bar{k}_\mu, \bar{k}_\nu$ von $\bar{\Pi}$ über; dieser sei mit $\langle\mu\nu\rangle$ bezeichnet. In Figur 2 tragen $\bar{k}_\mu$ und $\bar{k}_\nu$ einen gegen $\langle\mu\nu\rangle$ gerichteten Pfeil. Aus der Ebene eines ebenen Vierkantenzuges von Π und aus der Ebene $[34]^{\mathrm{I}}$, gegeben durch (21) und (22), erhält man (beim gewählten Radius von $\varkappa$)

$$\langle 23\rangle^{\mathrm{I}}=\langle 33\rangle=\langle 24\rangle=\left(\frac{p^{-1}}{3},\frac{1}{3},\frac{p^{-2}}{3}\right), \qquad \langle 34\rangle^{\mathrm{I}}=\left(-\frac{p^{-2}}{3},-\frac{p^{-1}}{3},\frac{p^{-4}}{3}\right). \tag{31}$$

Aus den Ebenen $[23]^{\mathrm{II}}$ und $[34]^{\mathrm{II}}$, gegeben durch (23) und (24), folgt

$$\langle 23\rangle^{\mathrm{II}}=\left(\frac{p^{-1}}{3},\frac{p^{-3}}{3},\frac{p^{-3}}{3}\right), \qquad \langle 34\rangle^{\mathrm{II}}=\left(-\frac{p}{3},-\frac{p^{-1}}{3},\frac{p^{-3}}{3}\right). \tag{32}$$

In Figur 2 sind jene Punkte $\langle\mu\nu\rangle$ bezeichnet, die zu den in Figur 1 mit $(\mu\nu)$ bezeichneten Punkten gehören.

Die Punkte $\langle 23\rangle^{\mathrm{I}}=\ldots$ bilden den abgestumpften Würfel, der aus Π durch die Streckung aus M mit dem Faktor $p^{-2}/3$ hervorgeht. Wie in 3.3 findet man: *Die Punkte $\langle 23\rangle^{\mathrm{II}}$ bilden ein symmetrisches Polyeder, hingegen $\langle 34\rangle^{\mathrm{I}}$ und $\langle 34\rangle^{\mathrm{II}}$ je ein unsymmetrisches Polyeder von Γ.* Die Koordinaten der Ecken verhalten sich, abgesehen von Reihenfolge und Vorzeichen, bei $\Pi\langle 23\rangle^{\mathrm{I}}$ wie $1:p:p^2$, bei $\Pi\langle 34\rangle^{\mathrm{I}}$ wie $1:p^2:p^3$, bei $\Pi\langle 23\rangle^{\mathrm{II}}$ wie $1:1:p^2$, bei $\Pi\langle 34\rangle^{\mathrm{II}}$ wie $1:p^2:p^4$. Daraus folgt übrigens: $\Pi\,(34)^{\mathrm{II}}$ und $\Pi\,\langle 34\rangle^{\mathrm{I}}$ sind ähnliche Polyeder. Fritz Hohenberg, Graz

Inequalities involving convex sequences

To Ernst Trost on his 70th anniversary

1. Introduction

As an extension of the familiar estimate

$$a+\frac{1}{a}>2 \qquad (a>0, a\neq 1),$$

Wilson [12] showed that

$$\frac{1+a^2+a^4+\cdots+a^{2n}}{a+a^3+a^5+\cdots+a^{2n-1}}>\frac{n+1}{n}\qquad(n\geq 1, a>0, a\neq 1);$$

several other proofs were found at the same time ([8], p. 205).
Then Nanson [9] proved that

$$\frac{c_1+c_3+\cdots+c_{2n+1}}{n+1}\geq\frac{c_2+c_4+\cdots+c_{2n}}{n}$$

for any convex sequence $\{c_1, c_2, \ldots, c_{2n+1}\}$ with an odd number of terms; equality holds if and only if the sequence is an arithmetic progression. The particular case in which it is a positive geometric progression, say $c_k=a^{k-1}$ with $a>0$, is Wilson's inequality.
Nanson's proof consists in adding the inequalities

$$k(n-k)\ (c_{2k}-2c_{2k+1}+c_{2k+2})\geq 0,\qquad k(n-k+1)\,(c_{2k-1}-2c_{2k}+c_{2k+1})\geq 0$$

for $k=1,\ldots,n-1$.
Another proof, indicated in ([3], Theorem 130), uses the 'majorization theorem' due (in various forms) to Schur [10], Hardy, Littlewood and Pólya ([2] and [3], Theorem 108), and Karamata [4].
In section 2 we prove an inequality for convex sequences with an odd number of terms which is equivalent to Nanson's. Then we discuss in section 3 an inequality for convex functions which follows from the result in section 2, and its relation to a result of Szegö's. Finally, several estimates similar to Wilson's are deduced from the inequality in section 2.

2. An inequality for convex sequences

The following inequality contains Nanson's:
If $\{c_1, c_2, \ldots, c_{2n+1}\}$ is a convex sequence with an odd number of terms, then

$$c_1-c_2+c_3-c_4+\cdots+c_{2n+1}\geq\frac{c_1+c_3+\cdots+c_{2n+1}}{n+1}\geq\frac{c_2+c_4+\cdots+c_{2n}}{n},\tag{1}$$

with strict inequality throughout unless the sequence is an arithmetic progression, when there is equality throughout.

Proof: We begin by observing that for *any* real sequence,

$$\frac{c_1+c_3+\cdots+c_{2n+1}}{n+1}\geq\frac{c_2+c_4+\cdots+c_{2n}}{n}\tag{2}$$

is equivalent to

$$c_1-c_2+c_3-c_4+\cdots+c_{2n+1}\geq\frac{c_1+c_3+\cdots+c_{2n+1}}{n+1}, \tag{3}$$

and also to

$$c_1-c_2+c_3-c_4+\cdots+c_{2n+1}\geq\frac{c_2+c_4+\cdots+c_{2n}}{n}, \tag{4}$$

all three being strict if one of them is $\left(\text{subtract } \frac{1}{n}(c_1+c_3+\cdots+c_{2n+1})\right.$ from both sides of (2) to get (3), and $\frac{1}{n+1}(c_2+c_4+\cdots+c_{2n})$ to get (4); similarly, each of (3) and (4) implies $\left.(2)\right)$.

For $n=1$, (2) is simply the condition of convexity for a sequence of three terms: $c_1+c_3\geq 2c_2$.

We proceed by induction: assume (2) is true for all convex sequences with $2n-1$ terms (some $n\geq 2$). Let $\{c_1,c_2,\ldots,c_{2n+1}\}$ be a convex sequence with $2n+1$ terms:

$$c_{\nu-1}-c_\nu\geq c_\nu-c_{\nu+1}\qquad(\nu=2,3,\ldots,2n). \tag{5}$$

Set $\nu=2,4,\ldots,2n$ in (5), and add up:

$$\sum_{k=1}^{n}(c_{2k-1}-c_{2k})\geq\sum_{k=1}^{n}(c_{2k}-c_{2k+1}).$$

Add c_{2n+1} to each side:

$$\sum_{k=1}^{2n+1}(-1)^{k-1}c_k\geq\sum_{k=2}^{2n}(-1)^k c_k. \tag{6}$$

Now $\{c_2,c_3,\ldots,c_{2n}\}$ is a convex sequence of $2n-1$ terms. By the induction hypothesis, it satisfies (3):

$$\sum_{k=2}^{2n}(-1)^k c_k\geq\frac{1}{n}\sum_{k=1}^{n}c_{2k}. \tag{7}$$

Finally, (6) and (7) imply

$$\sum_{k=1}^{2n+1}(-1)^{k-1}c_k\geq\frac{1}{n}\sum_{k=1}^{n}c_{2k}, \tag{8}$$

which is (4) for the sequence $\{c_1,c_2,\ldots,c_{2n+1}\}$.

It is easily verified that equality holds throughout (1) if the sequence is an arithmetic progression. That (1) is strict throughout for all other convex sequences can be shown by induction. It is clear for $n=1$. For $n\geq 2$, equality in (8) requires equality in (7), and also in (5) for $\nu=2$ and $\nu=2n$. Hence $\{c_1,c_2,c_3\}$, $\{c_2,c_3,\dots,c_{2n}\}$ and $\{c_{2n-1},c_{2n},c_{2n+1}\}$ are arithmetic progressions.

3. Szegö's inequality

Szegö ([11]; [1], section 49) has proved that:
If $a_1\geq a_2\geq\cdots\geq a_{2n+1}\geq 0$, and if $f\colon[0,a_1]\to\mathbf{R}$ is convex, then

$$f(a_1)-f(a_2)+f(a_3)-f(a_4)+\cdots+f(a_{2n+1})\geq f(a_1-a_2+a_3-a_4+\cdots+a_{2n+1}). \quad (9)$$

(The same conclusion holds if $0\leq a_1\leq a_2\leq\cdots\leq a_{2n+1}$, and $[0,a_1]$ is replaced by $[0,a_{2n+1}]$: replace a_k by a_{2n+2-k}, $k=1,\dots,2n+1$.)
Wright [13] has given an interesting proof of (9), and discussed the type of convexity required of f; he shows that (9) holds if

$$f(x_1+\delta)-f(x_1)\geq f(x_2+\delta)-f(x_2) \quad (\mathrm{C}_1)$$

for all x_1,x_2,δ with $0\leq x_2\leq x_1<x_1+\delta\leq a_1$.
Under different hypotheses, a different lower bound for the left side of (9) can be obtained from (1):
Let $f\colon I\to\mathbf{R}$ be convex on an interval I containing the sequence $A=\{a_1,a_2,\dots,a_{2n+1}\}$. If f is non-decreasing and A is convex, or if f is non-increasing and A is concave, then

$$f(a_1)-f(a_2)+f(a_3)-f(a_4)+\cdots+f(a_{2n+1})\geq\frac{f(a_1)+f(a_3)+\cdots+f(a_{2n+1})}{n+1}. \quad (10)$$

Indeed, under these hypotheses $\{f(a_1),f(a_2),\dots,f(a_{2n+1})\}$ is a convex sequence.
In (10), mid-point convexity is all we require of f:

$$2f\left(\frac{x+y}{2}\right)\leq f(x)+f(y) \quad \text{for all} \quad x,y. \quad (\mathrm{C}_2)$$

For all f, (C_1) implies (C_2). There are f for which (C_2) is true but not (C_1) ([6], [7]). If f is continuous, (C_2) implies (C_1) ([3], Theorem 110).
In (10), f is bounded above on any finite subinterval of I and satisfies (C_2); hence f is continuous on the interior of I ([3], Theorem 111), and satisfies (C_1) there. And if I contains some of its endpoints, and f is monotonic, (C_2) implies (C_1) even if x_2 or $x_1+\delta$ is an endpoint of I (for instance, if f increases, its only possible discontinuity is at the right endpoint of I).
Suppose now that all the conditions for (9) and (10) are met: let $\{a_1,a_2,\dots,a_{2n+1}\}$ be non-negative, monotonic and convex (respectively, concave), and let f be non-decreasing (respectively, non-increasing) and convex (C_2) on a suitable interval. One may then ask whether one of (9), (10) is stronger than the other; but simple examples show that in general no such comparison is possible.

For instance, take $a_1 > a_2 = a_3 = \cdots = a_{2n+1} \geq 0$ and f strictly increasing; then the right side of (9) is strictly larger than that of (10).
But if A is an arithmetic progression, the right side of (9) is equal to

$$f\left(\frac{a_1 + a_3 + \cdots + a_{2n+1}}{n+1}\right),$$

and this is strictly less than the right side of (10) if f is strictly convex.

4. Wilson's and other inequalities

We return briefly to our observation at the beginning of the proof in section 2, and remark that for *any* real sequence $\{c_1, c_2, \ldots, c_{2n+1}\}$, (2) and

$$c_1 - c_2 + c_3 - c_4 + \cdots + c_{2n+1} \geq \frac{c_1 + c_2 + c_3 + \cdots + c_{2n+1}}{2n+1} \tag{11}$$

are equivalent. Indeed, the right side of (11) lies between the two terms of (2), of which it is a convex combination. Hence (3) implies (11), and (11) implies (4). We conclude that for any real sequence (convex or not), the chain

$$\sum_{k=1}^{2n+1} (-1)^{k+1} c_k \geq \frac{1}{n+1} \sum_{k=0}^{n} c_{2k+1} \geq \frac{1}{2n+1} \sum_{k=1}^{2n+1} c_k \geq \frac{1}{n} \sum_{k=1}^{n} c_{2k} \tag{12}$$

is equivalent to any one of the six inequalities it contains. Equality either holds throughout, or not at all. And if the sequence is convex, the only case of equality is for an arithmetic progression.
Now apply (12) to the convex sequence $c_k = a^{k-1}$ $(a > 0)$: for $n \geq 1, a > 0, a \neq 1$,

$$\sum_{k=0}^{2n} (-1)^k a^k > \frac{1}{n+1} \sum_{k=0}^{n} a^{2k} > \frac{1}{2n+1} \sum_{k=0}^{2n} a^k > \frac{1}{n} \sum_{k=0}^{n-1} a^{2k+1}. \tag{13}$$

On dividing a term in (13) by another to its right we get six inequalities, one of them Wilson's: for $n \geq 1, a > 0, a \neq 1$,

$$\frac{1 + a^2 + a^4 + \cdots + a^{2n}}{a + a^3 + a^5 + \cdots + a^{2n-1}} > \frac{n+1}{n}, \tag{14}$$

$$\frac{1 - a + a^2 - a^3 + \cdots + a^{2n}}{1 + a^2 + a^4 + \cdots + a^{2n}} > \frac{1}{n+1}, \tag{15}$$

$$\frac{1 - a + a^2 - a^3 + \cdots + a^{2n}}{1 + a + a^2 + a^3 + \cdots + a^{2n}} > \frac{1}{2n+1}, \tag{16}$$

$$\frac{1-a+a^2-a^3+\cdots+a^{2n}}{a+a^3+a^5+\cdots+a^{2n-1}}>\frac{1}{n}, \tag{17}$$

$$\frac{1+a^2+a^4+\cdots+a^{2n}}{1+a+a^2+a^3+\cdots+a^{2n}}>\frac{n+1}{2n+1}, \tag{18}$$

$$\frac{1+a+a^2+a^3+\cdots+a^{2n}}{a+a^3+a^5+\cdots+a^{2n-1}}>\frac{2n+1}{n}. \tag{19}$$

Any one of these implies the other five. Each is of the form $f_n(a)>f_n(1)$, where $f_n(a)$ denotes the quotient on the left side.
The case $a<0$ is easy to dispose of; (13) holds for $a<0$ (consider the effect on each sum of replacing an $a>0$ by $-a$). For $a<0$, the lower bounds in (15), (16) and (18) may be replaced by 1, but the quotients in (14), (17) and (19) are unbounded from below, as $a\to-\infty$.
Finally, we remark that (14) through (19) imply

$$f_n(a)<f_n(-1) \quad \text{for} \quad a<0, a\neq -1: \tag{20}$$

consider the effect on $f_n(a)$ of replacing a by $-a$.
As a last example, take

$$\frac{1+a+a^2+\cdots+a^n}{a+a^2+a^3+\cdots+a^{n-1}}>\frac{n+1}{n-1} \qquad (n\geq 2, a>0, a\neq 1) \tag{21}$$

([5], p. 111). If $f_n(a)$ denotes the left side of (14), then

$$f_n(a)f_{n-1}(a)>f_n(1)f_{n-1}(1)=\frac{n+1}{n-1} \qquad (n\geq 2, a>0, a\neq 1)$$

whence

$$f_n(\sqrt{a})f_{n-1}(\sqrt{a})>\frac{n+1}{n-1} \qquad (n\geq 2, a>0, a\neq 1),$$

and this is (21).
Another proof of (21) starts from

$$\frac{c_1+c_2+\cdots+c_{n+1}}{n+1}\geq\frac{c_2+c_3+\cdots+c_n}{n-1}, \tag{22}$$

valid for any convex sequence $\{c_1, c_2, \ldots, c_{n+1}\}$, which we leave to the reader.

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Aufgaben

Aufgabe 841. Sechs in einer Ebene liegende Kreise, die einen gegebenen Grundkreis k rechtwinklig schneiden, seien so angeordnet, dass im Inneren von k ein rechtwinkliges Kreisbogensechseck s entsteht. Zu je zwei Gegenseiten von s existiert dann genau ein Mittellot, d.i. ein Kreis, der diese Seiten und den Grundkreis rechtwinklig schneidet. Man zeige, dass sich die drei Mittellote innerhalb s in einem Punkt schneiden. Der Beweis ist ohne Hilfsmittel der hyperbolischen Geometrie zu führen.

P. Buser, Bonn, BRD

Lösung: Im folgenden wird bezeichnet: Der Mittelpunkt von k mit M; die Kreise, auf denen die Seiten des Kreisbogensechsecks liegen, mit $k_1,\dots,k_6$, ihre Mittelpunkte mit $M_1,\dots,M_6$; die Lotkreise der Seitenpaare (k_i,k_{i+3}) mit l_i und ihre Mittelpunkte mit L_i, $i=1,2,3$.
Aus den Voraussetzungen folgt zunächst, dass die l_i paarweise reelle Schnittpunkte haben. Zum Nachweis, dass sie genau zwei Punkte S_1, S_2 gemeinsam haben, also demselben elliptischen Kreisbüschel angehören, genügt es zu zeigen, dass ihre Mittelpunkte L_i kollinear sind.
Das Dreieck $M_1M_3M_5$ liegt polar zum Dreieck $M_2M_4M_6$ in bezug auf den Grundkreis, denn die Schnittpunktsehne von k_1 und dem Grundkreis ist einerseits die Polare von M_1 bezüglich k, da k_1 den Grundkreis rechtwinkelig schneidet, und andererseits müssen die Punkte M_2 und M_6 in ihrer Eigenschaft als Potenzpunkte von k_1,k_3 und k bzw. von k_5,k_1 und k auf dieser Schnittpunktsehne in ihrer Eigenschaft als Potenzlinie von k_1 und k liegen; k_2 und k_6 schneiden ja k_1 und k