Geometric aspects of linear transformations of the plane

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mit den zusätzlich berechneten Spannungswerten in einigen der Stäbe angegeben. Ein positiver Spannungswert entspricht einer Zugbeanspruchung, ein negativer Wert einer Druckspannung. Die Beanspruchung auf Zug und Druck entsprechen der Anschauung.

Deformationen und Spannungen.

Verschiebungen der Knotenpunkte [cm]				Spannungen [N cm^{-2}]	
i	u_i	v_i	w_i	Stab	σ
1	0,3719	0,2093	−0,1150	P_6P_1	12866
2	0,4070	0,1501	−0,1242	P_1P_2	−5917
3	0,5375	0,1608	0,0234	P_2P_3	7246
4	0,4219	0,1182	0,0150	P_3P_4	−7222
5	0,4570	0,1682	0,0150	P_1P_4	5006
				$P_{10}P_4$	1548
				P_4P_5	5000

H.R. Schwarz, Universität Zürich

LITERATURVERZEICHNIS

1 J.P. Aubin: Approximation of elliptic boundary-value problems. Wiley, New York 1972.
2 Ph.G. Ciarlet: The finite element method for elliptic problems. North-Holland, Amsterdam 1978.
3 L. Collatz: Konvergenz des Differenzenverfahrens bei Eigenwertproblemen partieller Differentialgleichungen. Dt. Math. *3*, 200-212 (1938).
4 R. Courant, K. Friedrichs und H. Lewy: Über die partiellen Differenzengleichungen der mathematischen Physik. Math. Ann. *100*, 32-74 (1928).
5 R. Courant: Variational methods for the solution of problems of equilibrium and vibrations. Bull. Amer. Math. Soc. *49*, 1-23 (1943).
6 G. Pólya: Sur une interprétation de la méthode des différences finies qui peut fournir des bornes supérieures ou inférieures. C.R. Acad. Sci., Paris *235*, 995-997 (1952).
7 G. Pólya: Estimates for eigenvalues. Studies in Mathematics and Mechanics presented to R. von Mises, S.200-207. Academic Press, New York 1954.
8 P.M. Prenter: Splines and variational methods. Wiley, New York 1975.
9 H.R. Schwarz: Methode der finite Elemente. Teubner, Stuttgart 1980.
10 E. Stiefel: Einführung in die numerische Mathematik, 5. Aufl. Teubner, Stuttgart 1976.
11 G. Strang und G.J. Fix: An analysis of the finite element method. Prentice-Hall, Englewood Cliffs, N.J., 1973.

Geometric aspects of linear transformations of the plane

The purpose of this note is to present some interesting and useful connections between plane geometry and linear transformations of the plane into itself. Of course, such linear transformations are most often given by 2×2 matrices. While our development does not easily extend into higher dimensional space, the two-dimensional case is already one of considerable practical importance. For example, in engineering mechanics the planar stress and strain tensors and the inertia tensor of a solid with a plane of symmetry are each represented by a 2×2 matrix.

The following notational conventions will be adopted. A linear transformation A of the real plane $\mathbf{R}^2$ into itself will be represented by the matrix $[A]=\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$ relative to the standard ordered basis vectors (i,j). If $x=\xi i+\eta j\in\mathbf{R}^2$ then the image of the vector x under A will be written as Ax, which corresponds to the matrix multiplication $[A]\begin{bmatrix}\xi \\ \eta\end{bmatrix}$. The inner product on $\mathbf{R}^2$ will be denoted by $\langle,\rangle$, and the magnitude of x will be denoted by $\|x\|$; whence $\|x\|^2=\langle x,x\rangle$. A circle centered at $c\in\mathbf{R}^2$ of radius ρ will be denoted by $C(c;\rho)$. Equality by definition will be denoted by $\equiv$. I denotes the identity transformation.

In section 1, the eigenvalues and eigenvectors of the transformation A are constructed geometrically relative to a circle Γ which is associated with $[A]$. Section 2 applies these ideas to construct the zeros of a quadratic function; the method has interesting historical connections. The concluding section discusses a geometric approach to constructing the level curves of a general quadratic polynomial in two variables. Our methods have also been applied to the phase plane analysis of the differential equation $\dot{x}=Ax$ [1].

1. Eigenvalues and principal axes

Given the matrix $[A]=\begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}$, define the vectors

$$a\equiv\alpha i+\gamma j, \qquad b\equiv\delta i-\beta j.$$

By taking the line segment between a and b as a diameter, we define the (possibly degenerate) circle $\Gamma\equiv C(c;\|r\|)$, where

$$c\equiv\frac{1}{2}(a-b)=\frac{1}{2}(\alpha-\delta)i+\frac{1}{2}(\gamma-\beta)j,$$
$$r\equiv\frac{1}{2}(a-b)=\frac{1}{2}(\alpha-\delta)i+\frac{1}{2}(\gamma+\beta)j.$$

The trace $\alpha+\delta\equiv\tau$ of A is twice the abscissa of the center of Γ; that is, $\gamma_1=(1/2)\tau$ if $c=\gamma_1 i+\gamma_2 j$. The determinant $\alpha\delta-\beta\gamma\equiv\sigma$ of A can also be identified geometrically relative to the circle Γ as shown in figure 1, since

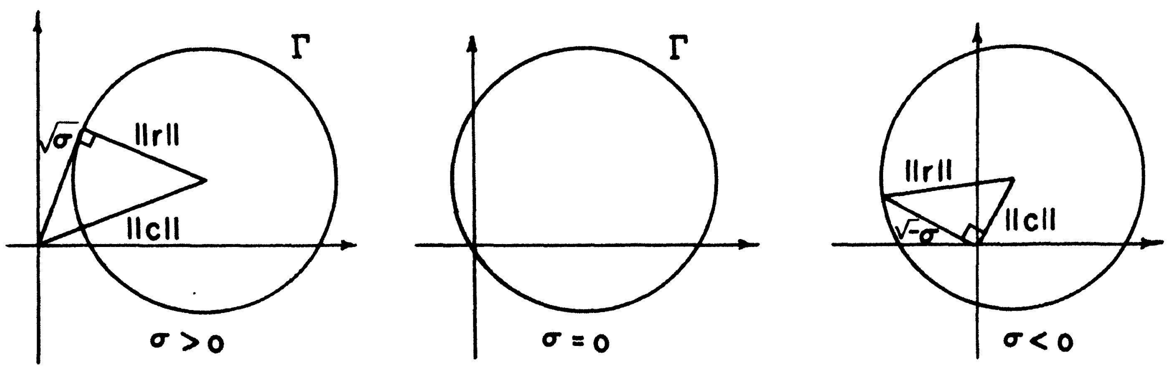

Figure 1

$$\sigma = \langle a,b\rangle = \langle c+r, c-r\rangle = \|c\|^2 - \|r\|^2 .$$

The eigenvalues of A are defined as the (possibly complex) roots of the characteristic equation $0 = \text{determinant } (A-\lambda I) = \lambda^2 - \tau\lambda + \sigma = \lambda^2 - 2\gamma_1\lambda + \|c\|^2 - \|r\|^2$. By the quadratic formula the eigenvalues are

$$\lambda_{1,2} = \gamma_1 \pm \sqrt{\gamma_1^2 - (\|c\|^2 - \|r\|^2)} = \gamma_1 \pm \frac{1}{2}\Delta^{1/2},$$

where Δ denotes the discriminant

$$\Delta \equiv \tau^2 - 4\sigma = 4\gamma_1^2 - 4(\|c\|^2 - \|r\|^2) = 4(\|r\|^2 - \gamma_2^2).$$

The invariants $\lambda_1, \lambda_2, \Delta$ can now be identified geometrically relative to the circle Γ as shown in figure 2. In the case $\Delta > 0$, the eigenvalues are real and distinct and are the abscissae of the points of intersection of Γ with the horizontal axis. When $\Delta = 0$, the circle Γ has a tangency with the horizontal axis at the value of the repeated eigenvalue $\lambda_1 = \lambda_2$. In the case $\Delta < 0$, the eigenvalues are complex and the real plane is identified with the complex plane. The circle centered at $(\gamma_1, 0)$ which is tangent to Γ is seen to have radius $(1/2)\sqrt{-\Delta}$, which means its two intersection points with the vertical line through the center of Γ geometrically determine the eigenvalues λ_1 and λ_2 in the complex plane. The circle $C(0; \sqrt{\sigma})$ also passes through λ_1 and $\lambda_2 = \bar{\lambda}_1$ (bar denotes complex conjugate), since $\sigma = \lambda_1 \lambda_2 = |\lambda_1|^2 = |\lambda_2|^2$.

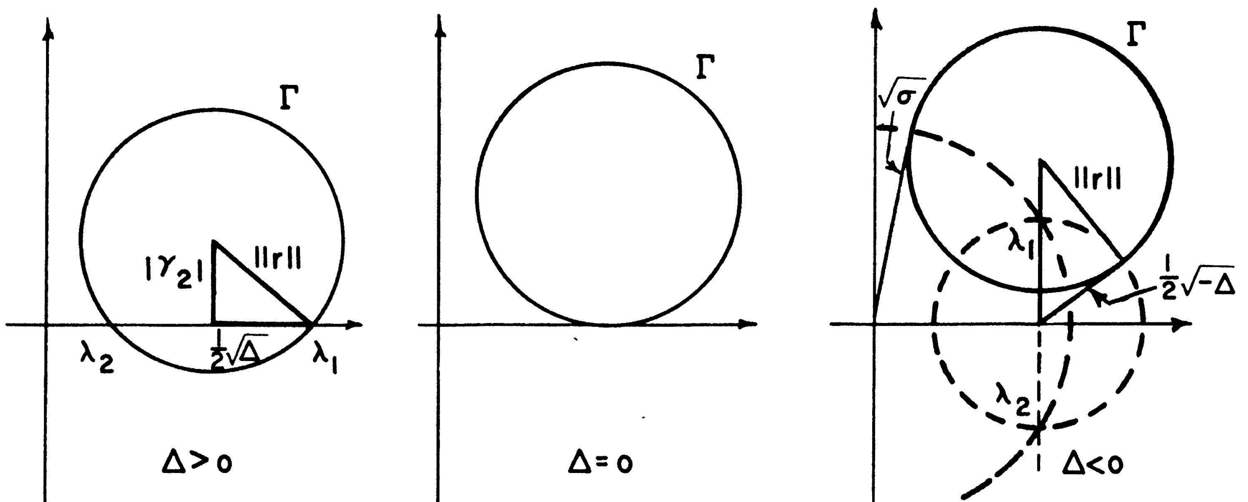

Figure 2

Although Γ depends on the matrix $[A]$ which, in the i,j-basis, represents the linear transformation A, the invariants $\tau, \sigma, \Delta, \lambda_1, \lambda_2$ associated geometrically with Γ are independent of the basis. Thus, any change of basis must result in a circle $\hat{\Gamma}$ which geometrically determines the same invariants. This is obvious for rotations of the plane, say counterclockwise through an angle θ: here c is unchanged, r is rotated clockwise an angle 2θ, and hence $\hat{\Gamma} = \Gamma$. Under a general change of basis in the case

$\Delta > 0, \hat{\Gamma}$ will be any of the circles which pass through the points λ_1 and λ_2 on the horizontal axis. In the case $\Delta < 0, \hat{\Gamma}$ is any of the circles orthogonal to the family of circles through λ_1 and $\lambda_2 = \bar{\lambda}_1$; that is, $\hat{\Gamma}$ is one of the Apollonian circles determined by the distinct complex eigenvalues, including possibly the point circles λ_1 and λ_2 themselves. The non-intersecting Apollonian circles, together with its orthogonal family of circles which intersect at two distinct common points, constitute the well-known Steiner circles. The case $\Delta = 0$ corresponds to a degenerate system of Steiner circles.

In the case $\Delta \geq 0$, it remains to geometrically determine the eigenspaces $\mathfrak{E}_1$ and $\mathfrak{E}_2$ which correspond respectively to the real eigenvalues λ_1 and λ_2. Let

$$u_1 \equiv a - \lambda_2 i = (a - \lambda_2)\, i + \gamma j$$
$$u_2 \equiv a - \lambda_1 i = (a - \lambda_1)\, i + \gamma j$$

define the vectors u_1 and u_2, as shown in figure 3.

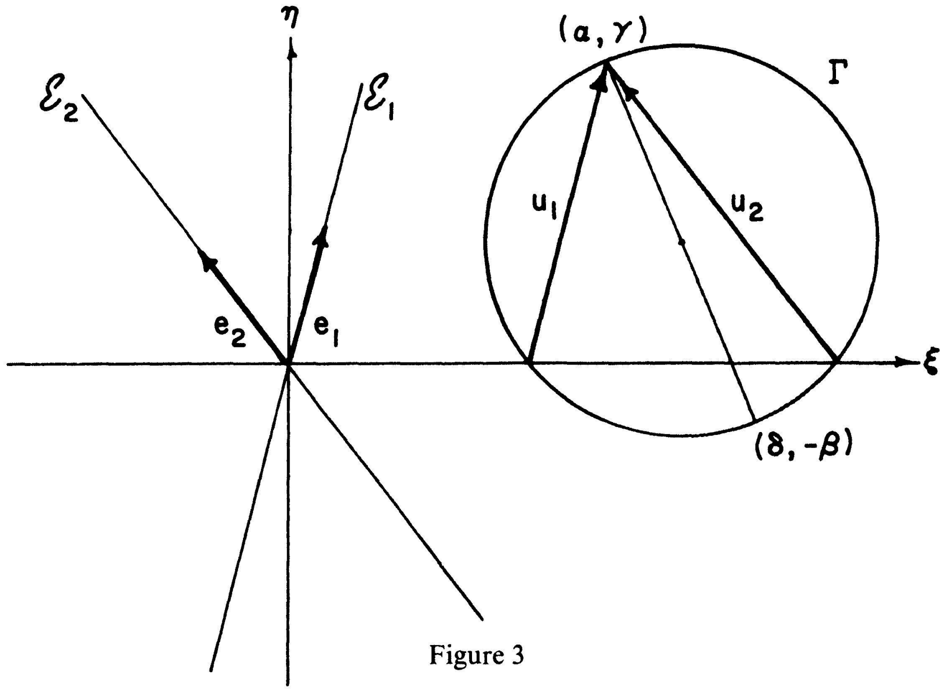

Figure 3

Then

$$(A - \lambda_1 I)\begin{bmatrix} a - \lambda_2 \\ \gamma \end{bmatrix} = \begin{bmatrix} a - \lambda_1 & \beta \\ \gamma & \delta - \lambda_1 \end{bmatrix}\begin{bmatrix} a - \lambda_2 \\ \gamma \end{bmatrix}$$
$$= \begin{bmatrix} a^2 - (\lambda_1 + \lambda_2)\, a + \lambda_1 \lambda_2 + \beta\gamma \\ \gamma a - (\lambda_1 + \lambda_2)\, \gamma + \delta\gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

since $\lambda_1 + \lambda_2 = a + \delta$ and $\lambda_1 \lambda_2 = a\delta - \beta\gamma$. If $u_1 \neq 0$, this shows that u_1, and hence also the unit vector $e_1 \equiv u_1 / \| u_1 \|$, is an eigenvector corresponding to λ_1. If $u_1 = 0$, e_1 is taken as a unit tangent vector to Γ at a. Defining e_2 analogously, it follows that in all cases, e_1 and e_2 are unit eigenvectors corresponding respectively to λ_1 and λ_2.

In the case of distinct eigenvalues ($\Delta > 0$), it is geometrically clear that e_1 and e_2 are linearly independent. Moreover, they are orthogonal if and only if Γ is centered on

the horizontal axis, and therefore corresponds to a matrix for which $\beta=\gamma$; that is, $[A]$ is a symmetric matrix.
In the case $\Delta=0$, $\lambda\equiv\lambda_1=\lambda_2$ is a repeated eigenvalue. There is either one eigenvector (when $r\neq 0$ and Γ is a proper circle), or else all directions are eigenvectors (when $r=0$ and Γ is a point circle) and $A=\lambda I$.
The lines $\mathfrak{E}_1$ and $\mathfrak{E}_2$ determined by the eigenvectors are commonly called principal axes. In summary, the circle Γ associated with A geometrically determines $\tau,\sigma,\Delta,\lambda_1,\lambda_2,\mathfrak{E}_1,\mathfrak{E}_2$ as shown in figures 1-3. A related method [2] shows how even the image Ax of a given vector x can be geometrically determined; however, an auxiliary set of axes is required in that construction.

2. Construction of the roots of a quadratic polynomial

In notation consistent with the preceding section, let us suppose the polynomial, in the variable λ, is written in the form $\lambda^2-\tau\lambda+\sigma$. If A is defined by

$$[A]\equiv\begin{bmatrix}\tau & -1\\ \sigma & 0\end{bmatrix},$$

we see τ and σ are, respectively, the trace and determinant of A, and so $\lambda^2-\tau\lambda+\sigma=0$ is its characteristic equation. The zeros are thus determined geometrically by constructing the circle Γ whose diameter has the endpoints (τ,σ) and $(0,1)$. Figure 4 illustrates the case x^2+x-2.

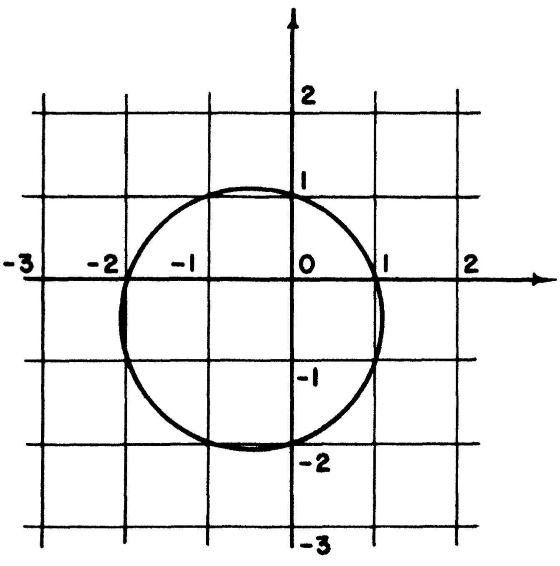

Figure 4

This method is not new but is attributed to the Scottish essayist and historian Thomas Carlyle (1795-1881). In his book, 'Elements of Geometry', Sir John Leslie comments on this geometric solution of quadratics as follows: 'The solution of this important problem now inserted in the text was suggested to me by Mr. Thomas Carlyle, an ingenious young mathematician, and formerly my pupil' (see Eves [3], p. 80). It is not clear whether or not Carlyle realized his method would construct the complex roots as well. It is also of interest to compare our work to the early attempt of John Wallis [4] to construct the imaginary roots of quadratic polynomials; while he failed to identify the imaginary axis explicitly, his reference to a 'plain' in which the roots can be found is the first reference to a geometrical interpretation of the complex numbers.

3. Level curves of quadratic functions

Let $q(\xi,\eta)=\alpha\xi^2+2\beta\xi\eta+\delta\eta^2+2\mu\xi+2\nu\eta$, where $\alpha,\beta,\delta,\mu,\nu$ are constants. It is assumed not each of α,β,δ is zero. Our goal is to geometrically determine the nature of the level curves $q(\xi,\eta)=\kappa$, κ a constant, and devise a procedure to sketch the level curves rapidly yet accurately.
We begin by defining

$$[A]\equiv\begin{bmatrix}\alpha & \beta\\ \beta & \delta\end{bmatrix},\qquad p\equiv\mu i+\nu j,\qquad x\equiv\xi i+\eta j,$$

which allows us to express $q(x)\equiv q(\xi,\eta)$ in the basis free form

$$q(x)=\langle x,Ax\rangle+2\langle x,p\rangle;$$

as usual, A is the linear transformation which $[A]$ defines. Next consider the translation $\hat{x}\equiv x-t$ where $\hat{x}$ represents a new variable vector, viewed as eminating from a new origin at t. Defining $\hat{q}$ by $\hat{q}(\hat{x})\equiv q(\hat{x}+t)=q(x)$, we find

$$\hat{q}(\hat{x})=\langle\hat{x},A\hat{x}\rangle+2\langle\hat{x},At+p\rangle+\langle t,At\rangle+2\langle t,p\rangle.$$

Here the symmetry of A was used to show $\langle t,Ax\rangle=\langle x,At\rangle$. The symmetry also shows the existence of a basis $\mathfrak{B}=(e_1,e_2)$ of orthonormal eigenvectors of A. If $[A,\mathfrak{B}]$ represents A in this basis, we have the expressions

$$[A,\mathfrak{B}]=\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix},\qquad p=\hat{\mu}e_1+\hat{\nu}e_2,\qquad \hat{x}=\hat{\xi}e_1+\hat{\eta}e_2.$$

It has been shown how $\lambda_1,\lambda_2,e_1,e_2$ are geometrically determined by the circle Γ corresponding to $[A]$. The components $\hat{\mu}$ and $\hat{\nu}$ are likewise geometrically determined, since they are the orthogonal projections of p onto the e_1 and e_2 directions.
We now take two cases:

I. $\sigma\equiv\det A\neq 0$.
Equivalently, this is the case where Γ intersects the ξ-axis at non-zero points λ_1 and λ_2. By choosing

$$t = -A^{-1}p = -\lambda_1^{-1}\hat{\mu}e_1 - \lambda_2^{-1}\hat{\nu}e_2$$

we get

$$\hat{q}(\hat{x}) = \langle \hat{x}, A\hat{x} \rangle - \langle p, A^{-1}p \rangle = \lambda_1 \hat{\xi}^2 + \lambda_2 \hat{\eta}^2 - \lambda_1^{-1}\hat{\mu}^2 - \lambda_2^{-1}\hat{\nu}^2.$$

Thus the level curves $\hat{q}(\hat{x})$ = constant is a family of ellipses ($\lambda_1\lambda_2 > 0$) or conjugate hyperbolas ($\lambda_1\lambda_2 < 0$) centered at $x = t$ and whose axes are the (orthogonal) principal axes of A. If Γ is a point circle, then $\lambda_1 = \lambda_2$, and the level curves are concentric circles.

Figure 5 illustrates the case $q(\xi, \eta) = 5\xi^2 + 6\xi\eta + 5\eta^2 + 4\xi + 12\eta$.

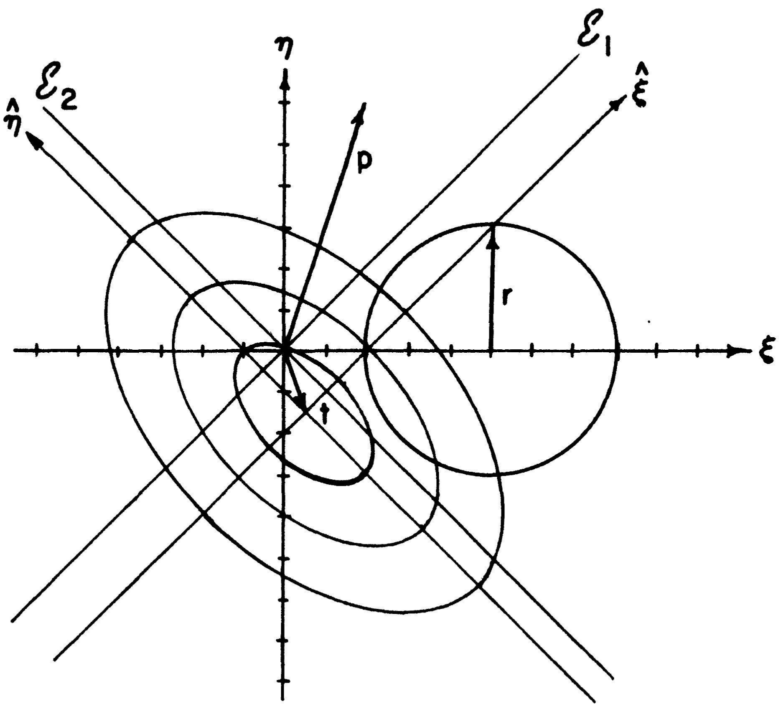

Figure 5

In the hyperbolic case, the circle Γ, being centered on the ξ-axis, will (in view of fig. 1) intersect the vertical η-axis at the two points $\pm\sqrt{-\sigma}\, j, \sigma = \det A$. Recalling the vector $b = \delta i - \beta j$ used in section 2 to define an endpoint of a diameter of Γ, define the vectors

$$v_1 \equiv b + \sqrt{-\sigma}\, j, \qquad v_2 \equiv b - \sqrt{-\sigma}\, j.$$

Then

$$\langle v_1, Av_1 \rangle = [\delta, -\beta + \sqrt{-\sigma}] \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \begin{bmatrix} \delta \\ -\beta + \sqrt{-\sigma} \end{bmatrix} = 0,$$

and similarly $\langle v_2, Av_2 \rangle = 0$. The vectors v_1 and v_2 thus define the two asymptotic directions for the family of hyperbolas. They are determined by Γ, b, and the inter-

section points of Γ with the η-axis in an analogous manner to that by which Γ, $a = \alpha i + \gamma j$, and the intersection points of Γ with the ξ-axis determine the principal axes.
The level curves of $q(\xi, \eta) = 2\xi^2 + 4\xi\eta - \eta^2$ are sketched in figure 6.

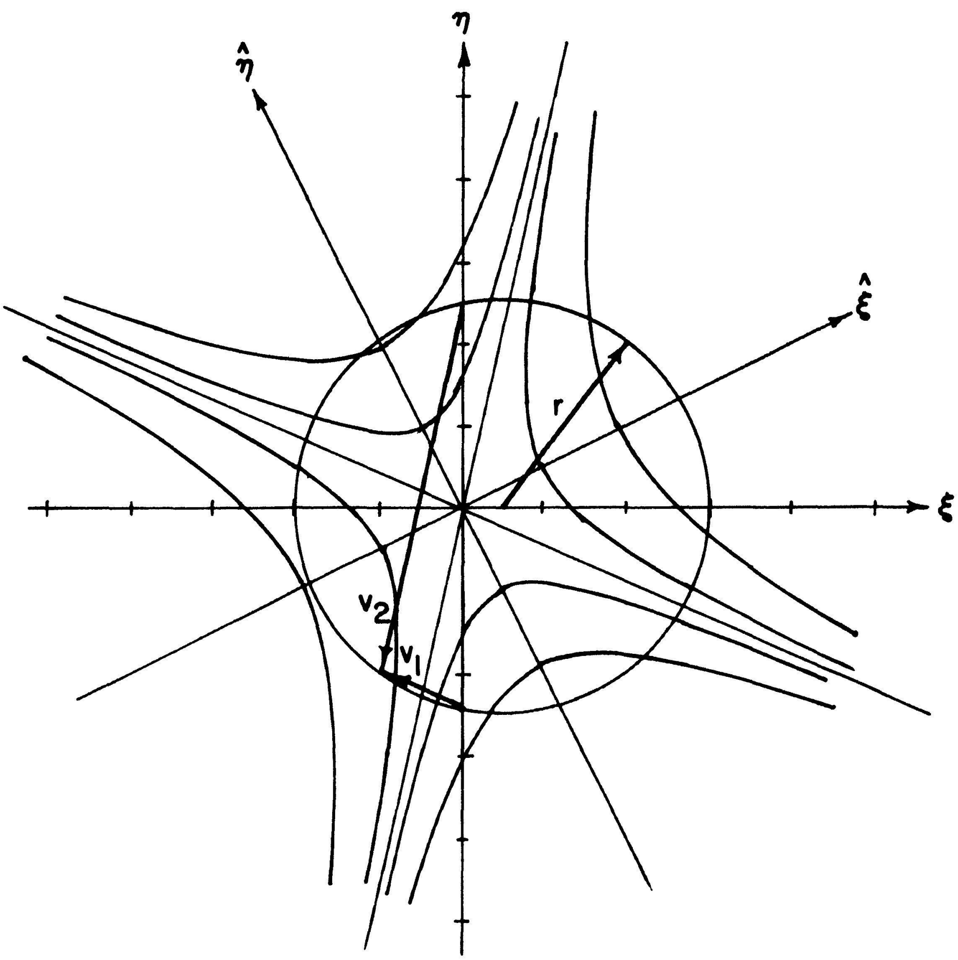

Figure 6

II. $\sigma \equiv \det A = 0$.
Here Γ intersects the ξ-axis in a non-zero eigenvalue λ; let e denote a corresponding unit eigenvector and $\mathfrak{E}$ denote the principal axis. The second eigenvalue is zero; let f denote its corresponding unit eigenvector and $\mathfrak{F}$ the principal axis. Letting $p = \hat{\mu}\hat{e} + \hat{v}f$ and $t = -\lambda^{-1}\hat{\mu}e$, we find

$$\lambda\hat{\xi}^2 + 2\hat{v}\hat{\eta} = \kappa + \lambda^{-1}\hat{\mu}^2 .$$

If p is parallel to e, which is geometrically determined by the circle Γ, then $\hat{v} = 0$ and the locus is a family of straight lines parallel to the principal axis $\mathfrak{F}$. If p is not in the e direction, then $\hat{v} \neq 0$ and the family of level curves consists of congruent parabolas. The common axis of these parabolas is parallel to $\mathfrak{F}$ and has been translated in the e direction from $x = 0$ a distance $-\lambda^{-1}\hat{\mu}$.
Since addition, subtraction, multiplication, division, and square roots of lengths can all be constructed with straightedge and compass, the various axes and translation

vectors defining any level curve can be constructed in the Euclidean sense. In practice, a combination of both geometric and arithmetic methods is most convenient to sketch any level curve quickly yet accurately.

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REFERENCES

1 D.W. DeTemple: A Geometric Method of Phase Plane Analysis. Am. Math. Monthly, *87*, 102–112 (1980).
2 D.W. DeTemple: A Graphical Analysis of 2×2 Matrices. Math. Notes from Washington State University, vol.22, No.1 (1979).
3 Howard Eves: An Introduction to the History of Mathematics, 4th ed. Holt, Rinehart and Winston, New York 1976.
4 John Wallis: Algebra, 1673 (an English translation of the relevant chapters appears in: D.E. Smith: A Source Book in Mathematics. McGraw-Hill, New York 1929).

Kleine Mitteilungen

An identity involving Ramanujan's sum

Let f be an arithmetical function and let $f' = \mu * f$ denote the Dirichlet convolution of f and the Möbius function μ:

$$f'(n) = \sum_{d\mid n} \mu(d) f\left(\frac{n}{d}\right), \qquad n \geqslant 1 \cdot$$

A Ramanujan series is a series of the form

$$\sum_{q=1}^{\infty} a_q c_q(n),$$

where $c_q(n)$ is Ramanujan's sum,

$$c_q(n) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} \exp\left(2\pi i \frac{hn}{q}\right),$$

and where

$$a_q = \sum_{m=1}^{\infty} \frac{f'(mq)}{mq}.$$

H. Delange proved [1] the following.