# The number of triangles in a tringulation of a set of points in the plane

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Benützung der zu $\tilde{v}_1$ parallelen Elationsachse durch $P$ kann der Parabelscheitel nach Abschnitt 3 konstruiert werden.

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# The number of triangles in a triangulation of a set of points in the plane1)

## 1. Introduction

Our terminology and notation will be standard except as indicated. A good reference for undefined graph theoretic terms is [3].

In [1, 2] the authors discussed the question of the number of 3-cycles which could be present in a planar graph on $p$ points. In this paper, we want to consider essentially the same question when the $p$ points are in *fixed positions* in the plane. We will show that this restriction does not limit the possible range of the number of 3-cycles unless the $p$ points are arranged in a unique, easily characterized configuration.

## 2. Statement of the problem and main results

Begin with a set $P$ of $p \geq 5$ points in the plane, with no three of the points collinear. Suppose we draw straight line segments between pairs of points in $P$ subject only to the restriction that these segments do not intersect except at the points of $P$ themselves, until it is impossible to add more segments in this manner. We call this collection of line segments a *triangulation* of $P$ (since all the finite regions into which these segments divide the plane are triangles). We will generally use $T$ to denote a triangulation of $P$. Note in particular that the line segments comprising the boundary of the convex hull of $P$ will be included in every triangulation $T$ of $P$.

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We want to consider the number of triangles (or 3-cycles) in various triangulations of $P$. To this end, let $h$ denote the number of extreme points of the convex hull of $P$. (For brevity in the sequel, we will term this collection of $h$ points the *extreme points* of $P$.) In any triangulation $T$ of $P$, it follows by Euler's well-known formula that the number of 3-cycles each of which bounds a region (i.e., contains points of $P$ in either its interior or exterior, but not both) will be precisely $2p$-$h$-2 if $h>3$, and $2p$-4 if $h=3$. In addition, however, a triangulation $T$ of $P$ may have 3-cycles containing points of $P$ in both their interior and exterior. We will call such 3-cycles *separating*. In [1, 2], it was shown that the number of separating 3-cycles must be between 0 and either $p-h$ if $h>3$ or $p-4$ if $h=3$. Our goal is to show that except for the two cases described in the statement of the theorem below, it is always possible to triangulate $P$ so as to obtain any number of separating 3-cycles in the indicated range.
We begin with the following result.

**Lemma.** *Let P be a set of $p\geq 5$ points in the plane, with no three of the points collinear. Suppose P has h extreme points. Then there is a triangulation T of P without separating triangles, unless $h=3$ and P has an extreme point x such that $P-x$ has $p-1$ extreme points (see fig. 1). In this exceptional case, any triangulation of P contains exactly $p-4$ separating triangles.*
*Moreover, if $h<p$ and it is possible to triangulate P without separating triangles, then it is possible to obtain such a triangulation with no 'chords' between extreme points of P (i.e., with no line segments between extreme points of P except those comprising the boundary of the convex hull of P).*

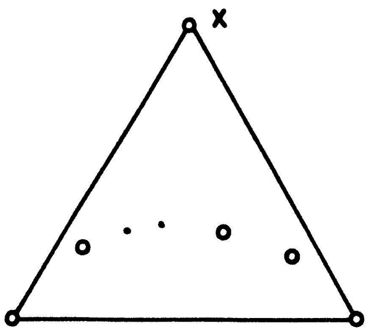

Figure 1

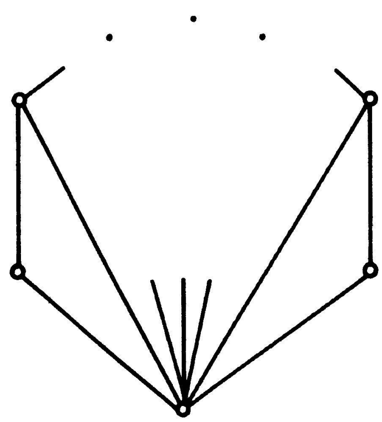
Figure 2

Proof: Observe first that if $h=p$, the desired triangulation is trivial (see fig. 2). Hence we assume $h<p$ in the rest of the proof.
Noting that the lemma is readily verified for $p=5$, we proceed by induction on $p$. Suppose first that $h\geq 4$. It is then easy to see that we can choose an extreme point $x$ of $P$ such that $P-x$ contains say $h'\geq 4$ extreme points. If $h'<p-1$, then by our induction hypothesis, we can triangulate $P-x$ without separating triangles or chords between extreme points of $P-x$. It is then a simple matter to obtain the desired triangulation of $P$ (see fig. 3). If $h'=p-1$, then $P-x$ has extreme points $y, z$ positioned as shown in figure 4. (If $y$ (resp., $z$) did not exist, we would have $h=p$ (resp., $h=3$), contrary to what we have assumed.) It is then a simple matter to complete the desired triangulation of $P$ (see fig. 4).

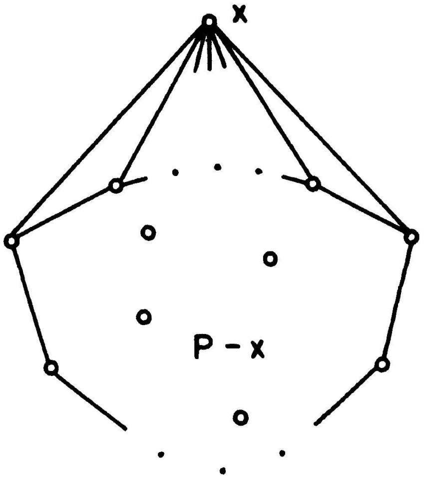


Figure 3

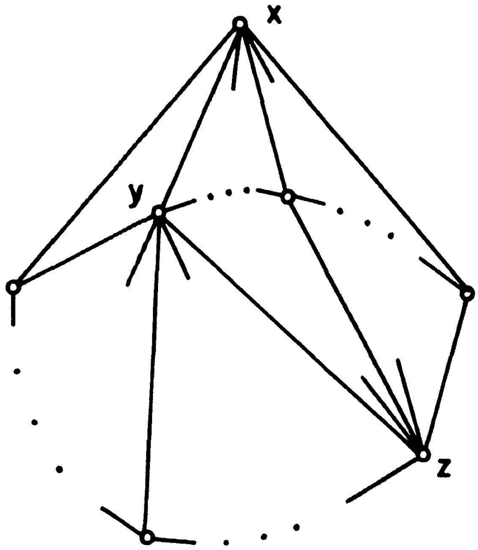


Figure 4

Suppose, therefore, that $h=3$. Let $x_1, x_2, x_3$ be the extreme points of $P$. If $P-x_i$ has $h_i$ extreme points, where $4 \leq h_i < p-1$, for some $i$, then we can triangulate $P$ without separating triangles as in the last paragraph. Otherwise, $P-x_i$ has three extreme points for each $i$. In particular, let $x_2, x_3$ and say $x_4$ be the extreme points of $P-x_1$. Suppose first that $P-x_1-x_i$ does not have $p-2$ extreme points, for $i=2,3,4$. Then by the induction hypothesis, we can triangulate $P-x_1$ without separating 3-cycles. Call this triangulation $T'$. It is easy to see that there will be a point $x$ in the interior of $x_2x_3x_4$ such that $xx_ix_4$ is a nonseparating 3-cycle of $T'$, and $xx_ix_1x_4$ is convex, for either $i=2$ or 3; without loss of generality, suppose this occurs for $i=2$ (see fig. 5a). We obtain a triangulation $T$ of $P$ without separating 3-cycles from $T'$ as follows: Remove the line segment $x_2x_4$ from $T'$, and add the segments $x_1x, x_1x_2$, $x_1x_3$ and $x_1x_4$ (see fig. 5b). It is easily seen that the only way $T$ could contain a separating 3-cycle is if the line segment $xx_3$ belonged to $T$. But in that case, either $xx_2x_3$ or $xx_3x_4$ would be a separating 3-cycle in $T'$, a contradiction.
Suppose, therefore, that $P-x_1-x_i$ has $p-2$ extreme points, for some $i=2,3$ or 4. We need to consider essentially two cases. If $P-x_1-x_2$ has $p-2$ extreme points

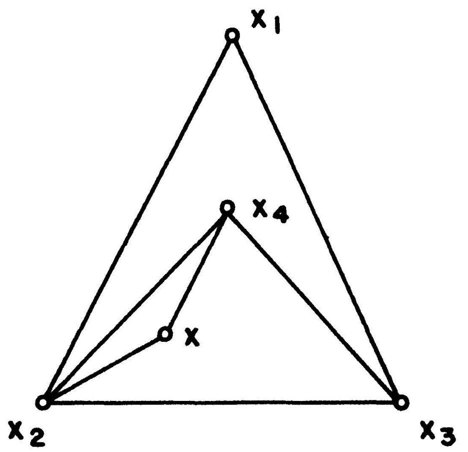


Figure 5a

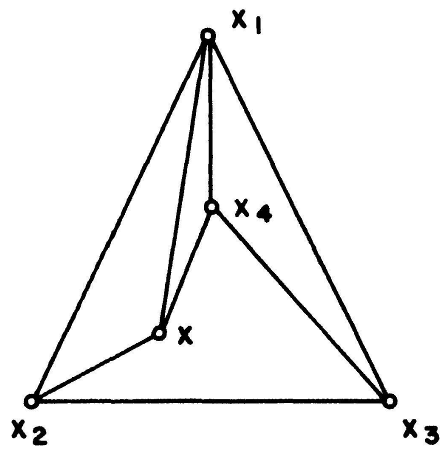


Figure 5b

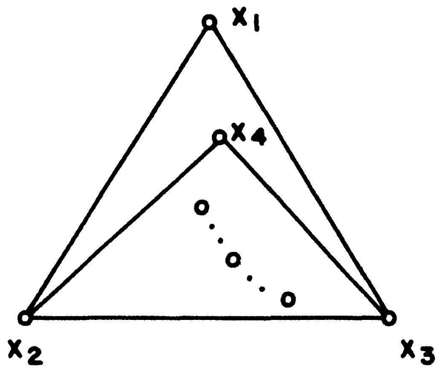


Figure 6a

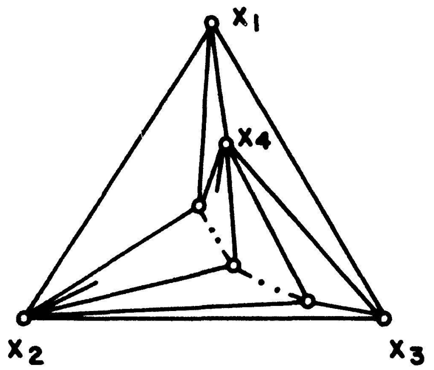


Figure 6b

(see fig. 6a), then we can triangulate $P$ without separating 3-cycles as shown in figure 6b (assuming of course that $P-x_2$ does not have $p-1$ extreme points).

On the other hand, if $P-x_1-x_4$ has $p-2$ extreme points (see fig. 7a), we can triangulate $P$ without separating triangles as shown in figure 7b. This completes the proof for all but the exceptional case.

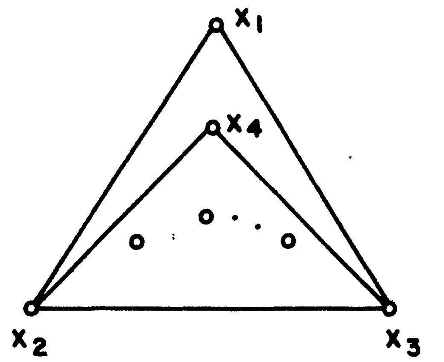


Figure 7a

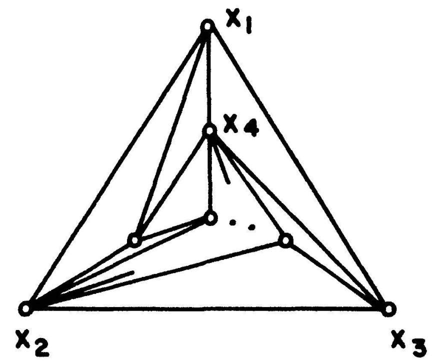


Figure 7b

Consider, therefore, the situation when $h=3$ and $P$ has an extreme point $x$ such that $P-x$ contains $p-1$ extreme points. Let $y,z$ be the other extreme points of $P$. Then in any triangulation $T$ of $P$, there will be a point $w$ inside $xyz$ such that $wyz$ is a nonseparating 3-cycle in $T$ (see fig. 8). It is then easy to see that the line segment $wx$ must also belong to $T$.

Consider the sets of points $P_1$ and $P_2$ inside or on the 3-cycles $wxy$ and $wxz$, respectively. It is easy to see that $P_i-x$ has $|P_i|-1$ extreme points, for $i=1,2$. If

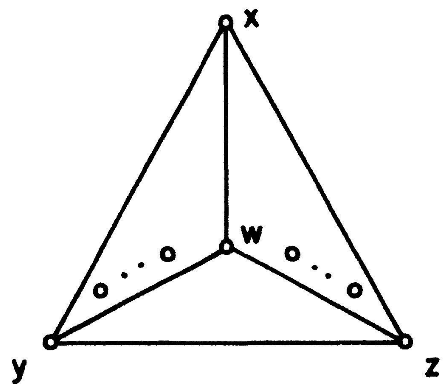


Figure 8

$|P_i| \geq 5$ for $i=1,2$, it follows by the induction hypothesis that any triangulation $T_i$ of $P_i$ must contain $|P_i|-4$ separating 3-cycles. Moreover, the only other separating 3-cycles in $T$ (besides those in $T_1$ and $T_2$) would be $wxy$ and $wxz$. Hence the number of separating 3-cycles in $T$ will be

$$(|P_1|-4)+(|P_2|-4)+2=p-4,$$

as asserted. The cases when $|P_i|=3$ or 4 are similar, and are, therefore, omitted. This completes the proof of the lemma.

We can now state our main result.

**Theorem.** *Let $P$ be a set of $p \geq 5$ points in the plane with no three of the points collinear. Suppose $P$ has $h$ extreme points. Let $\Delta(T)$ denote the number of 3-cycles in a triangulation $T$ of $P$. Then*

1. *If $h \geq 4$,*

$$2p-h-2 \leq \Delta(T) \leq 3p-2h-2.$$

*Moreover, if $a$ is any number in the indicated range, there is a triangulation $T$ of $P$ with $\Delta(T)=a$.*

2. *If $h=3$,*

$$2p-4 \leq \Delta(T) \leq 3p-8.$$

*Moreover, if $a$ is any number in the indicated range, then there is a triangulation $T$ of $P$ with $\Delta(T)=a$ unless either*

a) *$a=3p-9$, or*

b) *$P$ has an extreme point $x$ such that $p-x$ has $p-1$ extreme points. (In this case, $\Delta(T)=3p-8$ for every triangulation $T$ of $P$.)*

Proof: For case 2a, it was shown in [1] that a maximal planar graph on $p$ points cannot contain exactly $3p-9$ 3-cycles. Moreover, case 2b was covered in the preceeding lemma. It only remains, therefore, to treat the nonexceptional cases.

Suppose first, therefore, that $h \geq 4$ as in case 1. Choose any $(3p-2h-2-a)$ nonextreme points of $P$, and let $P'$ denote these points together with the $h$ extreme points of $P$. Triangulate $P'$ without separating 3-cycles (this is possible by the lemma). At this stage, we have exactly $\big(2(3p-h-2-a)-h-2\big)$ 3-cycles. Then recursively join each of the remaining $(a-2p+h+2)$ points of $P-P'$ by line segments to the three points of the triangle in which it occurs. The resulting triangulation of $P$ contains $2(3p-h-2-a)-h-2+3\ (a-2p+h+2)=a$ 3-cycles as desired.

Thus suppose $h=3$ as in case 2, but $a \neq 3p-9$. If $a=3p-8$, recursively draw line segments between each nonextreme point of $P$ (taken in any order) and the three points of the triangle in which it occurs. If $a \leq 3p-10$, let us suppose for the moment that we can choose a set $S$ of three nonextreme points in $P$ such that $S$, together with the extreme points of $P$, comprise a set of six points not of the type excluded by case 2b. Then choose arbitrarily an additional $3p-10-a$ nonextreme points of $P$. These additional points, together with $S$ and the extreme points of $P$, will from a set $P'$ of $3p-4-a$ points which again are not of the type excluded by case 2b. We can, therefore, triangulate $P'$ without separating 3-cycles; at this stage we have exactly $(2(3p-4-a)-4)$ 3-cycles. Now recursively draw line segments between each of the remaining $(a-2p+4)$ points of $P-P'$ and the three points of the triangle in which it occurs to obtain a triangulation of $P$ with exactly $2(3p-4-a)-4+3(a-2p+4)=a$ 3-cycles.

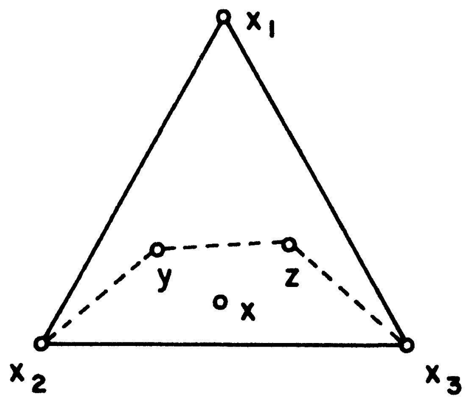


Figure 9

To complete the proof, we need to establish the existence of the set $S$. Let the extreme points of $P$ be $x_1,x_2,x_3$. Suppose that $P-x_i$ has three extreme points, say $y_i,x_j$ and $x_k$, for each $i$. Then we can take $S=\{y_1,y_2,y_3\}$. Otherwise, suppose that say $P-x_1$ has at least four extreme points. Since $P-x_1$ does not have $p-1$ extreme points, let $x$ be a nonextreme point of $P-x_1$. Then it is easy to see there exist extreme points $y,z$ of $P-x_1$ such that $x$ is positioned in the interior of the convex quadrilateral $x_2yzx_3$ (see fig. 9). We can then take $S=\{x,y,z\}$.
This completes the proof of the theorem.

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