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On the color partitions of a graph

Dedicated to Dorwin Cartwright on his 65th birthday

An *m-coloring* of a graph G is a decomposition of its vertex (or point) set $V(G)$ into a union of m disjoint subsets $V_1, V_2, \ldots, V_m$ such that no two points in any one V_i are adjacent. Each of the subsets V_i is a *color set*, which means that all points of V_i may be assigned the same color, whereas points in different color sets are assigned different colors. In the resulting m-coloring of G, no two adjacent points have the same color. The *chromatic number* $\chi(G)$ is the smallest m for which an m-coloring of G exists. We always write $\chi(G)=n$; hereafter we consider only n-colorings of G, i.e., colorings of G with the smallest possible number of colors. Here we are following the notation and terminology of [2].

Let G be a graph with p points and $\chi(G)=n$, and let $V_1 \cup V_2 \cup \cdots \cup V_n$ be an n-coloring of G. Let $|V_i|=p_i$, so that $\sum_{i=1}^{n} p_i = p$. Without loss of generality, we also assume that $p_1 \geq p_2 \geq \cdots \geq p_n$. Then the sequence $(p_1, p_2, \ldots, p_n)$ is called a *color partition* of G. We also define the following invariants: $M = \max\{p_1\}$, $M_0 = \min\{p_1\}$, $m = \max\{p_n\}$, $m_0 = \min\{p_n\}$ where the maxima and minima are taken over all n-color partitions of G.

Our first result will serve to clarify the meaning of the invariant p_1. Recall that an *induced subgraph* of G consists of a subset of the vertex set $V(G)$ and all lines of G joining points of this subset. If an induced subgraph of G contains no lines, no two points of the subgraph are adjacent in G, so all points of the subgraph may be placed in one color set. It might be suspected that M would be the order of the largest totally disconnected induced subgraph of G. In fact, this is not the case.

Theorem 1. *There are graphs G for which M is not the order of the largest induced subgraph of G with no lines.*

Proof: The tree shown in figure 1 gives an example. The largest induced subgraph containing no lines has vertex set $\{v_1, v_2, v_3, v_4\}$, so it is of order 4. A coloring

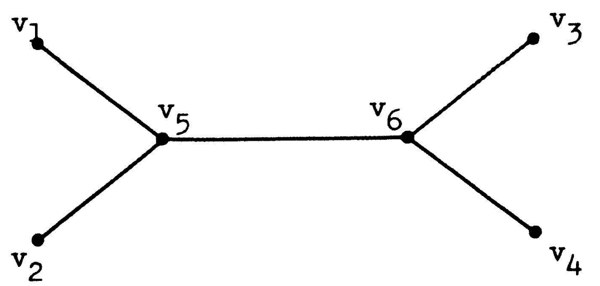

Figure 1

using $\{v_1, v_2, v_3, v_4\}$ as a color set requires three colors. However, there is a unique 2-coloring with color sets $\{v_1, v_2, v_6\}$, and $\{v_3, v_4, v_5\}$, so that $\chi(G)=2$ and $M=3$. □
Since $\sum_{i=1}^{n} p_i$ is a fixed number p for all color partitions of a graph G, one might suspect that every graph G has a color partition which includes both $M=\max\{p_1\}$ and $m_0=\min\{p_n\}$. Again we show that this is not true.

Theorem 2. *There are graphs having no color partition in which both M and m_0 occur.*

Proof: An example is shown in figure 2, where the symbols ●, ○ and ■ represent three colors. Since the graph contains a triangle, $\chi(G)>2$. The 3-coloring in figure 2a verifies that $\chi(G)=3$. Since v_i and v_{i+1}, $i=1$ to 8, cannot have the same color, $M \leq 5$. The (5,2,2) partition in figure 2a shows that $M=5$. The (4,4,1)

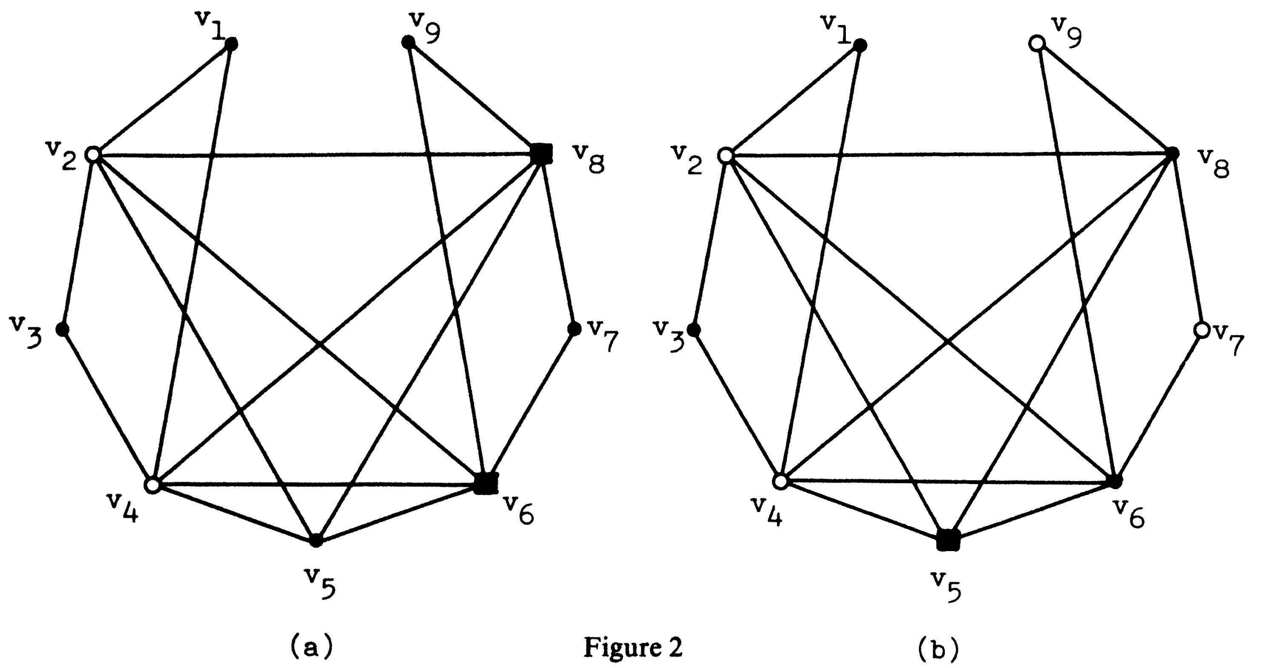

Figure 2

partition in figure 2b shows that $m_0=1$. The only possible color set with five elements is $\{v_1, v_3, v_5, v_7, v_9\}$. It is clear that the coloring in figure 2a is the only 3-coloring having a color set with five elements. For this graph, a partition containing M cannot contain m_0. □
No graph with fewer than nine points could serve to illustrate theorem 2. To see this, note that if a graph illustrates theorem 2, it must have two different partitions $(p_1, p_2, \ldots, p_n)$ and $(r_1, r_2, \ldots, r_n)$ such that all the following conditions hold:

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} r_i = p,$$

$$r_1 < p_1, r_n < p_n, p_1 \geq p_2 \geq \cdots \geq p_n, r_1 \geq r_2 \geq \cdots \geq r_n.$$

It is an elementary number theoretic observation that the smallest p for which all these conditions hold is $p=9$; and in this case, $n=3$, $p_1=5$, $p_2=p_3=2$, $r_1=r_2=4$ and $r_3=1$.

These values gave rise to the (5,2,2) and (4,4,1) partitions used in figure 2.
Recall that the complement $\bar{G}$ of a graph G has the same vertex set as G, but a line is in $\bar{G}$ if and only if it is not in G. The following lemma interprets a color partition of G in terms of $\bar{G}$.

Lemma 3a. *A graph G has a $(p_1,p_2,\dots,p_n)$ color partition if and only if $K_{p_1} \cup K_{p_2} \cup \cdots \cup K_{p_n}$ is a subgraph of $\bar{G}$.*

Proof: No two points in the same color set are adjacent in G. Thus each color set with p_i points induces a complete subgraph K_{p_i} in $\bar{G}$. □
The lemma is illustrated in figure 3. The points v_2 and v_4, which have the same color in G, induce K_2 in $\bar{G}$. The next theorem is an instant corollary of lemma 3a.

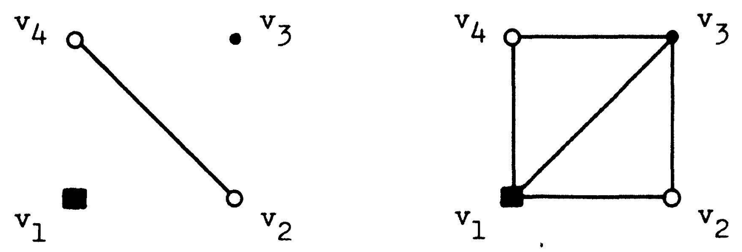

Figure 3

Theorem 3. *The maximum number of lines in a graph with a $(p_1,p_2,\dots,p_n)$ color partition is*

$$\binom{p}{2} - \sum_{i=1}^{n} \binom{p_i}{2}.$$

Proof: By the lemma, $\bar{G}$ must have at least

$$\sum_{i=1}^{n} \binom{p_i}{2}$$

lines.
A graph G is called *uniquely colorable* if there is only one decomposition of $V(G)$ into $n=\chi(G)$ color sets. Cartwright and Harary [1] showed that among all graphs G with p points and a unique coloring into n color sets, the minimum number of lines is $(2p-n)(n-1)/2$. To show that this minimum is attained, they used the graph $K(p-n+1,1,1,\dots,1)$, whose color partition is, of course, $(p-n+1,1,1,\dots,1)$. We show that for any partition $(p_1,p_2,\dots,p_n)$ with $\sum_{i=1}^{n} p_i=p$, there is a uniquely colorable graph having the partition $(p_1,p_2,\dots,p_n)$ and containing exactly $(2p-n)(n-1)/2$ lines.

Theorem 4. *Let $\sum_{i=1}^{n} p_i=p$ with $p_1 \geq p_2 \geq \cdots \geq p_n$. Then the minimum number of lines in a uniquely colorable graph with partition $(p_1,p_2,\dots,p_n)$ is $(2p-n)(n-1)/2$.*

We indicate the proof with an example. By the theorem of [1], it suffices to construct a uniquely colorable graph G with partition $(p_1, p_2, \ldots, p_n)$ having $(2p-n)\,(n-1)/2$ lines. To illustrate the construction, consider the partition (5,4,3). Label the points of the graph $v_1, v_2, \ldots, v_{12}$, and take the color sets to be $C_1 = \{v_1, v_2, v_3, v_4, v_5\}$, $C_2 = \{v_6, v_7, v_8, v_9\}$ *and* $C_3 = \{v_{10}, v_{11}, v_{12}\}$. The lines of G are constructed as follows. Choose a point (say, the first one listed) in each color set, and join that point to every point not in its color set. Thus the point v_1 will be adjacent to all points in $C_2 \cup C_3$. As shown in figure 4, the point v_6 is adjacent to every point not in C_2, and v_{10} is adjacent to every point not in C_3.

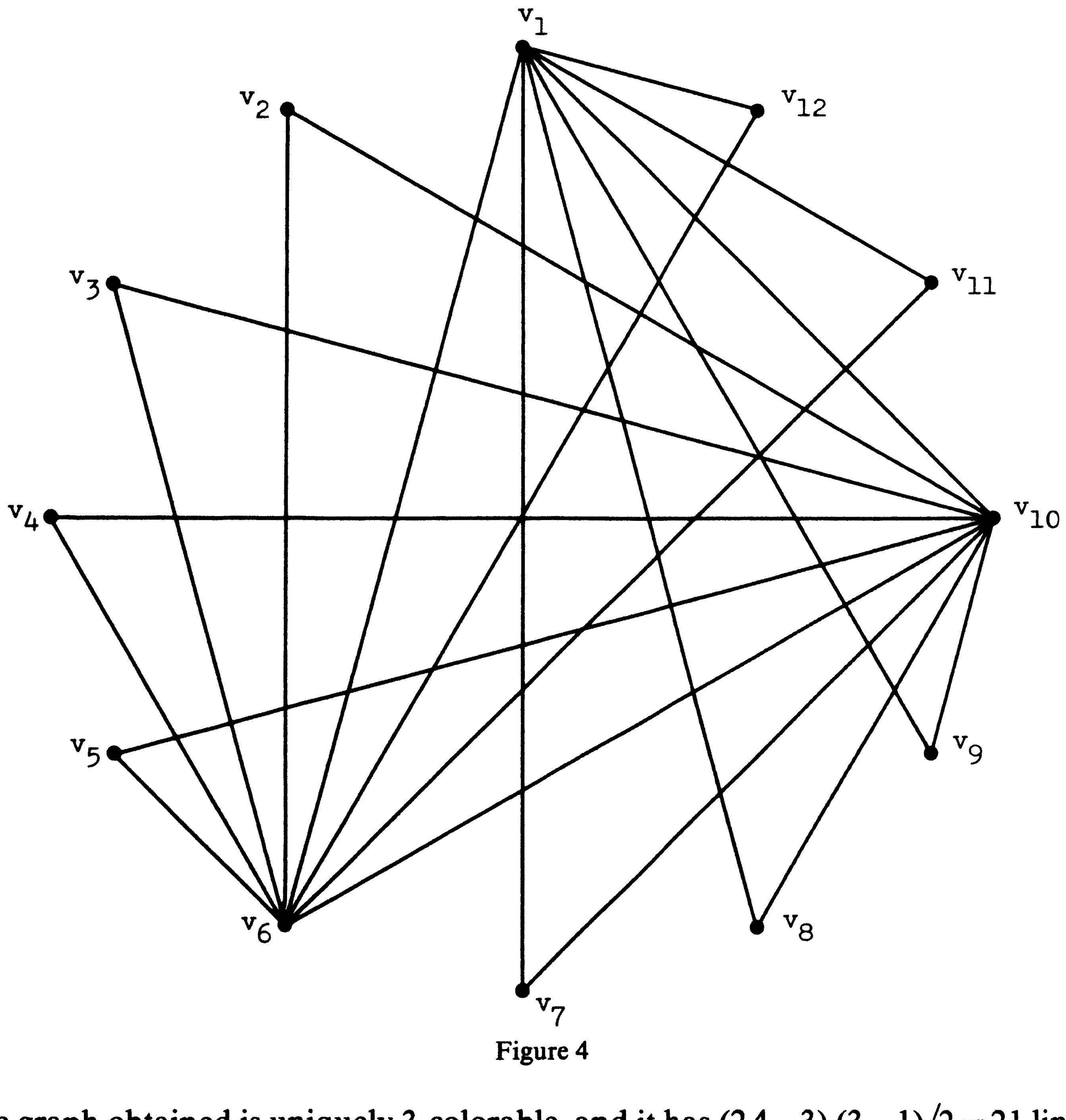

Figure 4

The graph obtained is uniquely 3-colorable, and it has $(24-3)\,(3-1)/2 = 21$ lines. More generally, the selected point in C_i is adjacent to $p - p_i$ points. However, those lines that join the selected points of different color sets have been counted twice. Thus the number of lines drawn (couting duplicates twice) is $\sum_{i=1}^{n}(p-p_i) = np - p$. Since there are $\binom{n}{2}$ such lines that have been counted twice, the number of lines in the graph is $np - p - \binom{n}{2} = (2p-n)\,(n-1)/2$. □

Lemma 3a proves very useful when trying to construct graphs with specified color partitions. In fact, the graph used in the proof of theorem 2 was discovered by using this lemma to construct one having both a (5,2,2) and a (4,4,1) color partition. We shall show the details of the use of the lemma in the construction of the graph used in the proof of the next result.

It is reasonable to suspect that if $M > M_0$ for a given graph G, then for every r with $M \geq r \geq M_0$, there would be some color partition of G for which $p_1 = r$. The next construction will show that this interpolation conjecture is not true.

We shall construct a graph with both an (8,2,2) partition and a (4,4,4) partition, but having no partition with $4 < p_1 < 8$. To do so, note that the complementary graph $\bar{G}$ must contain a subgraph of the form $K_8 \cup K_2 \cup K_2$. Label the points so that the points of the induced K_8 are $v_1, v_2, \ldots, v_8$, the points of the first induced K_2 are v_9 and v_{10}, and the points of the second induced K_2 are v_{11} and v_{12}.

We want to construct the subgraph of the form $K_4 \cup K_4 \cup K_4$ in such a way as to prevent formation of 3-colorings in which a color set has five, six or seven points. To do so, we distribute the points of each K_4 among K_8 and the two K_2's as widely as possible. For example, take the points of the first K_4 to be v_1, v_2, v_9, v_{11}; let the points of the second K_4 be v_3, v_4, v_5, v_{10}; and for the third K_4 use v_6, v_7, v_8 and v_{12}. The union of the lines in all six of these complete subgraphs is the graph $\bar{G}$ which with the corresponding graph G are shown in figure 5.

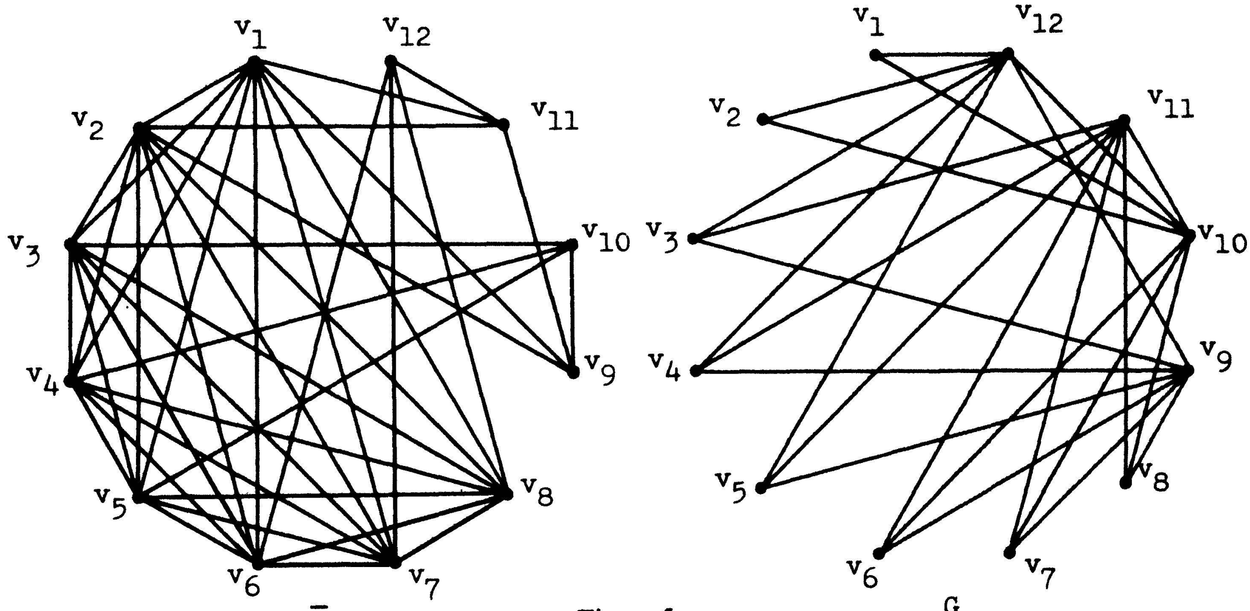

Figure 5

Theorem 5. *There is a graph G with $\chi(G) = 3$ for which the only 3-color partitions are (8,2,2) and (4,4,4).*

Proof: Consider the graph G of figure 5. Since G has a triangle, $\chi(G) > 2$. Figures 6a and 6b show two 3-colorings of G, yielding an (8,2,2) color partition and a (4,4,4) color partition, respectively. It remains to show that there are no other 3-color partitions. It is clear that no color set of G can contain more than eight points and that $\{v_1, v_2, \ldots, v_8\}$ is the only possible 8-point color set. Since v_9 und v_{12} must be in different color sets, and v_{10} and v_{11} must be in different color sets, an (8,3,1) color partition is impossible.

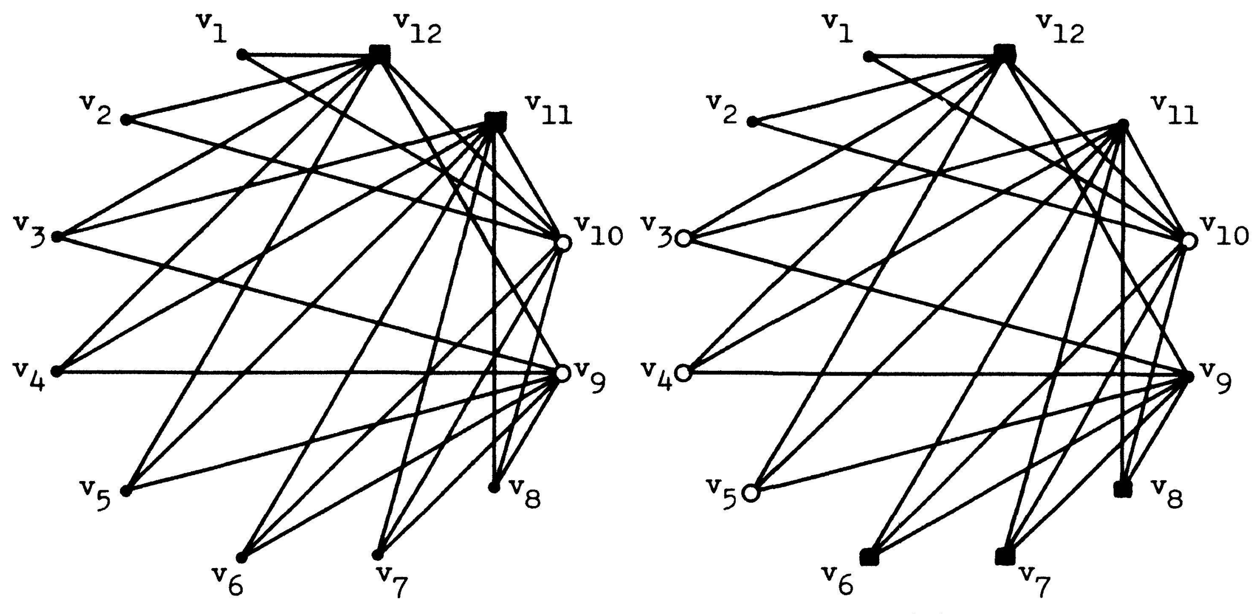

Figure 6

The only points that could possibly be in the same color set with v_9 are v_1, v_2, v_{10} and v_{11}, since these are the only points not adjacent to v_9 in G. However, v_{10} and v_{11} are adjacent in G, so they cannot be in the same color set. It follows that a color set containing v_9 has at most four points. In the same way, one can show that a color set containing v_{10} or v_{11} or v_{12} has at most four points. Therefore in any color partition in which $p_1 > 4$, the color set with p_1 points must be a subset of $\{v_1, v_2, \ldots, v_8\} = V_1$. If $4 < p_1 < 8$, the subgraph induced by $v_9, v_{10}, v_{11}, v_{12}$ and those points of V_1 that are not in the color set with p_1 points must be 2-colorable. However, it is easily verified that the subgraph induced by $v_9, v_{10}, v_{11}, v_{12}$ and any one or more points of V_1 contains a triangle and is therefore not 2-colorable. Then if $p_1 < 8$, it follows that $p_1 = 4$ and (4,4,4) is the only 3-color partition of G with $p_1 = 4$. □

Remark: In view of this theorem there is no r with $M_0 = 4 < r < M = 8$ for which this graph has a 3-color partition with $p_1 = r$. It also follows that for no s with $m_0 = 2 < s < 4 = m$ is there a 3-color partition of G with $p_n = s$.
Cartwright and Harary [1] have shown that in the n-coloring of a uniquely n-colorable graph, the subgraph induced by the union of any two color classes is connected. It is easy to obtain an analogous result for graphs with unique n-color partitions.

Theorem 6. *In any n-coloring of a graph with a unique n-color partition, the union of any two color sets has at most one component in which there are more points of one color than of the other.*

Proof: Suppose C_i and C_j are two color sets of the graph G, and suppose $C_i \cup C_j$ has at least two components in which more points have one color than the other. Then a reversal of the colors of all points within exactly one of these components produces a different n-color partition of G. □

Corollary. *In any n-coloring of a graph with a unique color partition, if two color sets C_i and C_j have the same number of points, then every component of $C_i \cup C_j$ has the same number of points of each color.*

The converse of the theorem is not true. Figure 7 illustrates a 3-chromatic graph having 3-color partitions (3,3,3) and (4,3,2). The color sets for the (3,3,3) partition

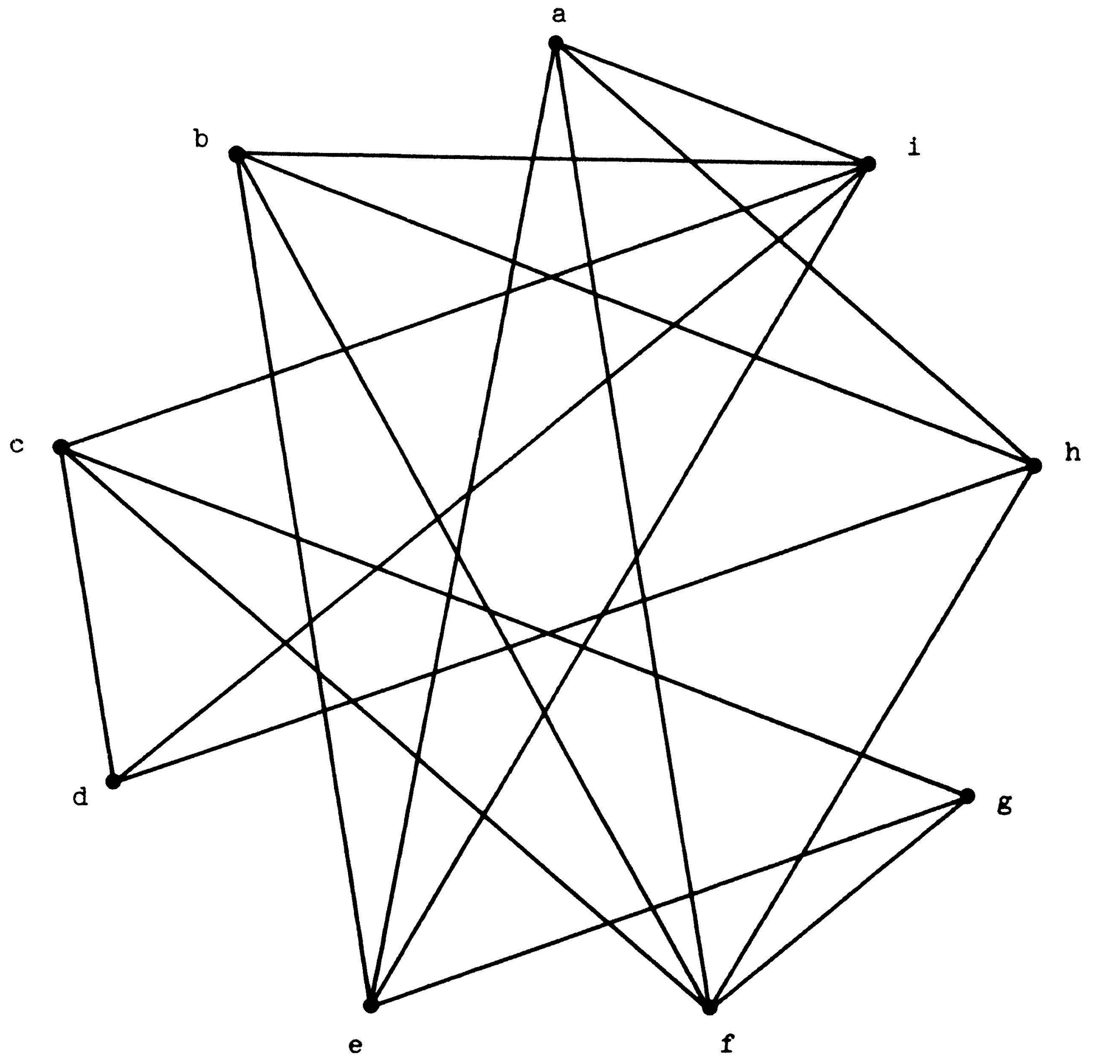

Figure 7

are $\{a,b,c\}$, $\{d,e,f\}$, and $\{g,h,i\}$; for the (4,3,2) partition, they are $\{a,b,d,g\}$, $\{c,e,h\}$ and $\{f,i\}$. These are the only 3-colorings of G. It is routine to verify that in either coloring, the union of any two color sets is connected (thus having only one component), yet the color partition of G is not unique.

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REFERENCES

1 D. Cartwright and F. Harary: On colorings of signed graphs. El. Math. *23*, 85–89 (1969).
2 F. Harary: Graph Theory. Addison-Wesley, Reading, Mass., 1969.

1) Visiting Scholar 1977–9 at the University of Michigan from Southern Illinois University, Carbondale.