Dissections of a Polygon

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Dissections of a Polygon

1. Suppose that a plane oval O (a convex, open, non-empty set) is intersected by c lines forming c dissecting chords, and suppose these chords intersect in p points of O through each of which only two of the chords pass (fig. 1). We do not insist that each two of the chords intersect. Then the number R of disjoint (open) regions formed inside O is given by the pretty formula

$$R=1+c+p. \tag{1}$$

This formula, apparently first given in this setting by Norman Bauman [3], is analogous to a result proved in 1966 by Alfred Brousseau [2] for lines in the plane. A more extensive study of (1) and related formulas appears in [1].
For the sake of completeness, we include a proof of (1), by induction on c. The formula is correct for $c=0$. Suppose a new chord C is added to a set of c chords for which the formula holds, and suppose that C meets those chords in $k \geq 0$ points in O. Since the chord C is a segment, ray, or whole line, it is divided into $k+1$ disjoint (open) pieces by the k points of intersection. Each of these $k+1$ pieces divides a previous region into two parts. Consequently, adding C creates exactly $k+1$ new regions. The right side of (1) plainly increments by exactly $k+1$ when C is added, and it follows that (1) holds for the augmented arrangement also. This proves (1) by induction.
It is worth remarking that simple-connectivity, not convexity, is the crucial assumption; one can prove that (1) holds for any simply-connected domain in the plane dissected by cross-cuts (see [6]).
In this note we use (1) to solve some natural polygon dissection problems.
2. Let $V_1, V_2, \ldots, V_n$ be the vertices of a convex n-gon Π, named in cyclic order, and put $V_{n+1}=V_1$. For each $i=1,2,\ldots,n$, select $x_i \geq 0$ points on the (open) edge $V_i V_{i+1}$ of Π. We divide the interior of Π (which is an oval) into regions by drawing all of the line segments determined by the vertices and the points chosen on the edges (fig. 2). If no three of these chords meet in a point inside the n-gon, how many regions are formed?
The solution is readily found using (1).

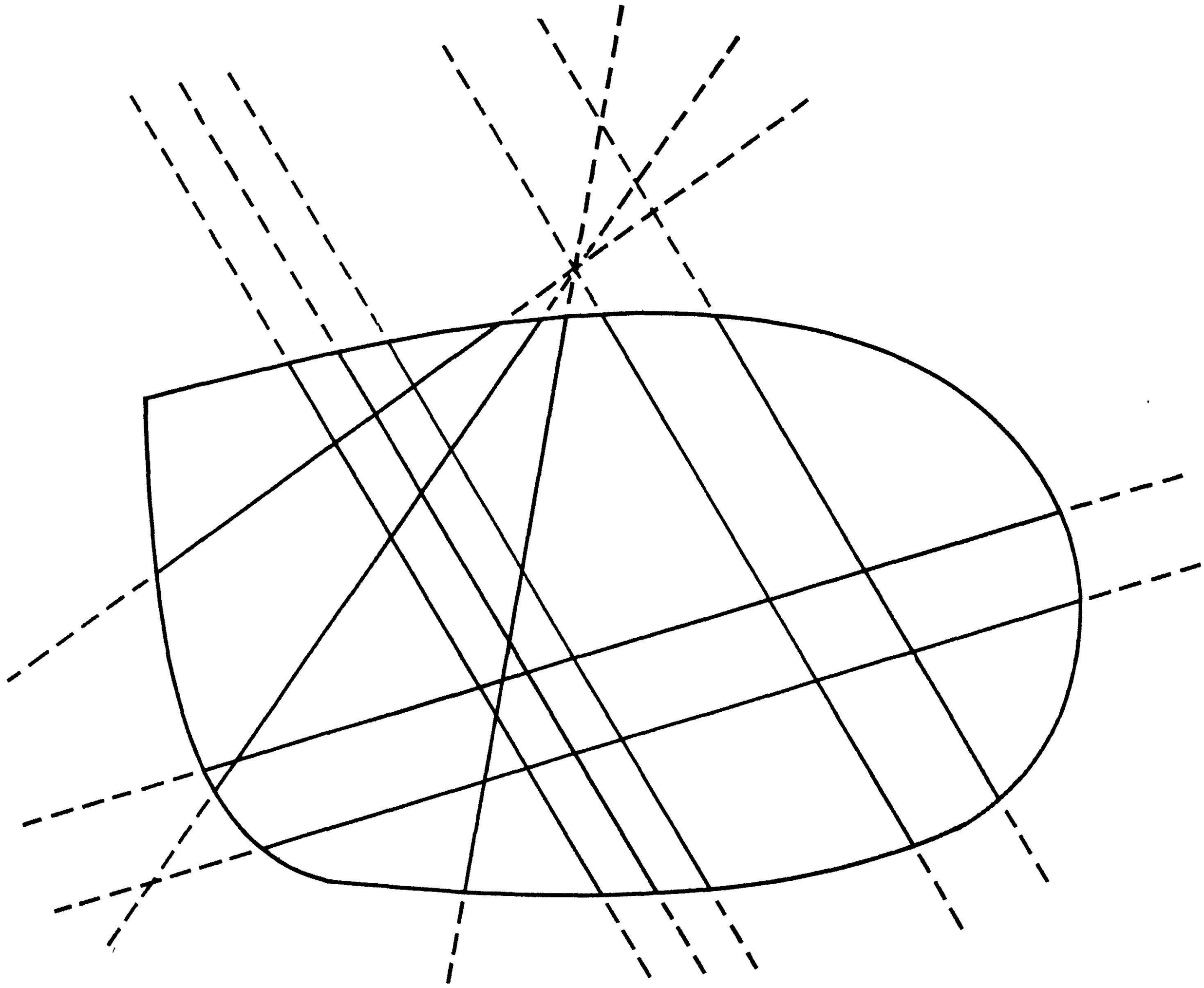

Figure 1. An oval dissected into 33 regions by a family of 10 lines.

Write σ_j for the sum of the $\binom{n}{j}$ products of the x_i's taken j at a time, with the usual combinatorial conventions that $\sigma_j = 0$ and $\binom{n}{j} = 0$ when $j > n$.

It is easy to count the chords of various kinds. To facilitate the descriptions, we write V to refer to a vertex and E to refer to an edge point. There are clearly

$$c_1 = \binom{n}{2} - n$$

chords of the form VV, the diagonals of Π. Each point E on an edge can be joined to any of the far $n-2$ vertices, so there are

$$c_2 = (n-2)\sigma_1$$

chords of the form VE. And there evidently are

$$c_3 = \sigma_2$$

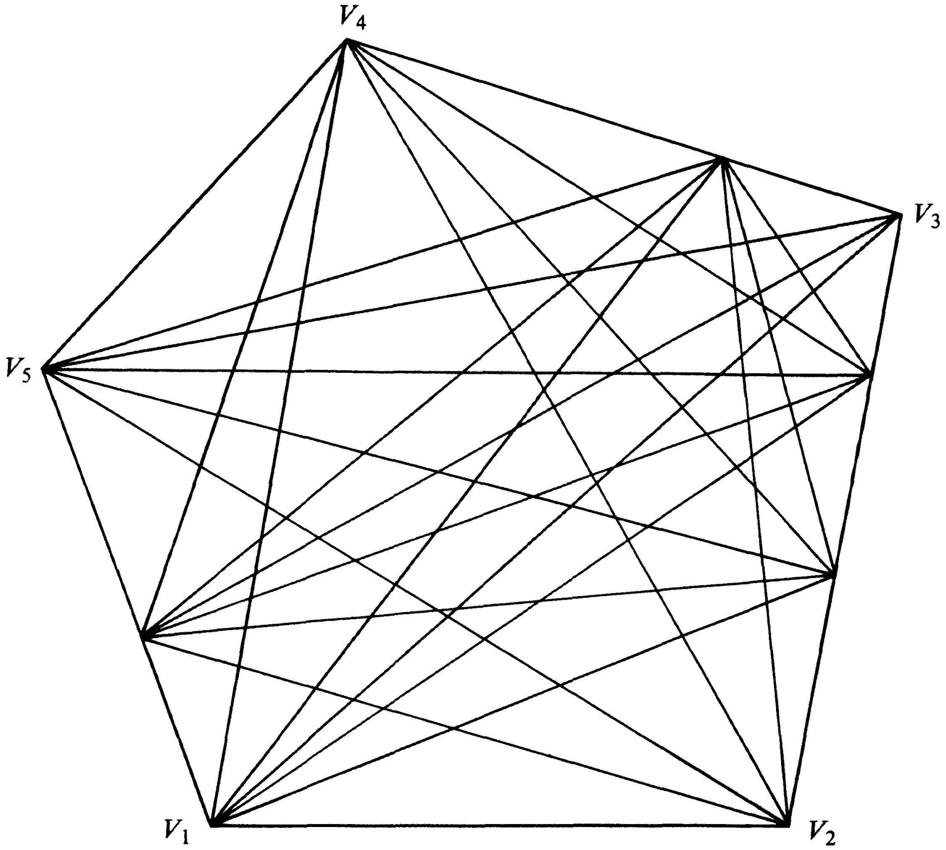

Figure 2. A dissected pentagon with data 0, 2, 1, 0, 1 divided into 116 regions.

chords of the form EE. Consequently, there are

$$c = c_1 + c_2 + c_3 = \binom{n}{2} - n + (n-2)\sigma_1 + \sigma_2$$

chords in this dissected oval.

To count the points of intersection, we rely on the observation that four points on Π, no three of which are collinear, determine six chords exactly two of which meet inside Π to make one point of intersection. There are, in all, $n+\sigma_1$ vertices and edge points on the n-gon, so there would be

$$\binom{n+\sigma_1}{4}$$

points determined inside Π but for the collinearities. We count the points lost due to collinearity and subtract them off.

Four of the $n+\sigma_1$ points may be selected on the same (closed) side in

$$\sum_{i=1}^{n} \binom{x_i+2}{4}$$

ways, and three of the points may be chosen on one (closed) side and the fourth elsewhere in

$$\sum_{i=1}^{n} \binom{x_i+2}{3} (n-2+\sigma_1-x_i)$$

ways. Consequently, the c chords meet to form

$$p = \binom{n+\sigma_1}{4} - \sum_{i=1}^{n} \left[\binom{x_i+2}{4} + \binom{x_i+2}{3} (n-2+\sigma_1-x_i) \right]$$

points inside Π.

It follows from (1) that there are

$$R = 1-n+\binom{n}{2} + (n-2)\sigma_1+\sigma_2+\binom{n+\sigma_1}{4}$$
$$-\sum_{i=1}^{n} \left[\binom{x_i+2}{4} + \binom{x_i+2}{3} (n-2+\sigma_1-x_i) \right]$$

regions formed inside the partitioned polygon.

3. Other natural polygon dissection problems can be posed in this setting and solved using (1). Instead of partitioning the interior of Π by drawing all possible segments determined by the vertices and edge points, we could use only certain of these segments.

There are seven natural sets of dissecting chords to consider, indicated symbolically by (VV), (VE), (EE), (VV, VE), (VV, EE), (VE, EE), and (VV, VE, EE); and there are seven associated partition problems. In the (VE, EE)-problem, for example, all the line segments that join a vertex to an edge point or two edge points are drawn. The (VV)-problem is the classical case of an n-gon dissected by its diagonals. The situation studied in section 2 above is the (VV, VE, EE)-problem.

We use (1) to solve all of these problems. The chords of various kinds have already been counted. It is a little more complicated to count the points.

We say that a point P determined inside Π is of *type* WX-YZ provided the two chords that meet at P are of types WX and YZ. Thus every interior point of intersection is of one of the six types VV-VV, VV-VE, VV-EE, VE-VE, VE-EE, and EE-EE. We count the points of each type separately.

There evidently are

$$p_1 = \binom{n}{4}$$

points of type VV-VV.

A point E can be selected in σ_1 ways, and then three vertices V (not including both endpoints of the edge on which E was chosen) can be picked in

$$\binom{n-2}{3} + 2\binom{n-2}{2} = \binom{n}{3} - \binom{n-2}{1}$$

ways. So there are

$$p_2=\left[\binom{n}{3}-n+2\right]\sigma_1$$

points of type *VV-VE.*
Points of type *VV-EE* are obtained by choosing two edge points on different edges and then choosing two vertices, one on each side of the line joining the two chosen edge points (but choosing both vertices of either of the two edges that contain the chosen edge points is not permitted). So there clearly are

$$p_3=\sum_{i<j}[(j-i)(n-j+i)-2]x_ix_j$$

$$=\sum_{i<j}(j-i)(n-j+i)x_ix_j-2\sigma_2$$

points of type *VV-VE.*
Points of type *VE-VE* arise in two different ways. The two edge points can be chosen on the same edge and any two of the $n-2$ far vertices chosen, which can be done in

$$\binom{n-2}{2}\sum_{i=1}^{n}\binom{x_i}{2}$$

ways; or the two edge points can be chosen on different edges and the two vertices chosen on the same side of the line through them, which can be done in

$$\sum_{i<j}\left[\binom{j-i}{2}+\binom{n-j+i}{2}\right]x_ix_j$$

$$=\binom{n}{2}\sigma_2-\sum_{i<j}(j-i)(n-j+i)x_ix_j$$

ways. So in all there are

$$p_4=\binom{n-2}{2}\sum_{i=1}^{n}\binom{x_i}{2}+\binom{n}{2}\sigma_2-\sum_{i<j}(j-i)(n-j+i)x_ix_j$$

points of type *VE-VE.*
Points of type *VE-EE* also arise in two ways: the three edge points can come from three different edges, or two edge points can come from the same edge and the third from another edge. There are clearly $n\sigma_3$ points of the first sort and

$$(n-2)\sum_{i=1}^{n}\binom{x_i}{2}(\sigma_1-x_i)$$

points of the second. So there are

$$p_5 = n\sigma_3 + (n-2)\sum_{i=1}^{n}\binom{x_i}{2}(\sigma_1 - x_i)$$

points of type *VE-EE.*

There are three ways that points of the type *EE-EE* arise, depending on whether the four edge points are chosen on different sides, which can be done in σ_4 ways; or two are chosen on the same side and the other two on different sides, which can be accomplished in

$$\sum_{i=1}^{n}\binom{x_i}{2}\sum_{\substack{j<k\\ j,k\neq i}} x_j x_k = \sum_{i=1}^{n}\binom{x_i}{2}(\sigma_2 - \sigma_1 x_i + x_i^2)$$

ways; or two points are chosen on each of two sides, which can be accomplished in

$$\sum_{i<j}\binom{x_i}{2}\binom{x_j}{2}$$

ways. So in all there are

$$p_6 = \sigma_4 + \sum_{i=1}^{n}\binom{x_i}{2}(\sigma_2 - \sigma_1 x_i + x_i^2) + \sum_{i<j}\binom{x_i}{2}\binom{x_j}{2}$$

points of type *EE-EE.*

It is an easy matter now to use (1) to write formulas for the number R of regions formed in each of the various cases.

(*VV*). Here $c = c_1$ and $p = p_1$, and so

$$R = 1 + \binom{n}{2} - n + \binom{n}{4} = \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4},$$

a familiar formula (see [4], [5] p. 99–107, and [7] p. 13, 108–112).

(*VE*). Here $c = c_2$ and $p = p_4$, and so $R = 1 + c_2 + p_4$.

(*EE*). Here $c = c_3$ and $p = p_6$, and so $R = 1 + c_3 + p_6$.

(*VV, VE*). In this case we have $c = c_1 + c_2$ chords, which meet to form all points of the types *VV-VV, VV-VE,* and *VE-VE.* So $p = p_1 + p_2 + p_4$, and $R = 1 + c_1 + c_2 + p_1 + p_2 + p_4$.

(*VV, EE*). Here we have $c = c_1 + c_3$ chords, which meet to form all points of the types *VV-VV, VV-EE,* and *EE-EE.* So $p = p_1 + p_3 + p_6$, and $R = 1 + c_1 + c_3 + p_1 + p_3 + p_6$.

(*VE, EE*). There are $c = c_2 + c_3$ chords, which meet to form all points of the types *VE-VE, VE-EE,* and *EE-EE.* So $p = p_4 + p_5 + p_6$, and $R = 1 + c_2 + c_3 + p_4 + p_5 + p_6$.

(*VV, VE, EE*). In this problem there are $c = c_1 + c_2 + c_3$ chords, which meet to form all points of all six types. So $p = p_1 + p_2 + p_3 + p_4 + p_5 + p_6$, and $R = 1 + c_1 + c_2 + c_3 + p_1 + p_2$

$+p_3+p_4+p_5+p_6$. It is not entirely obvious that this formula is the same as the formula obtained in section 2 above for this problem.

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Extremaleigenschaften rotationssymmetrischer Kegelstümpfe im gewöhnlichen Raum (1. Teil)

In dieser Note behandeln wir ein Extremalproblem, Teilproblem eines viel allgemeineren, unter III zu erläuternden Extremalproblems über konvexe Rotationskörper, und letzteres ordnet sich einem Hauptproblem über allgemeine konvexe Körper unter, das unter II kurz dargelegt wird. Die gesonderte Behandlung des Kegelstumpfproblems rechtfertigt sich durch die mit gesicherten Teilresultaten untermauerte Vermutung, dass die Extremalkörper unseres Spezialproblems mit Minimumeigenschaften dieselben im Problem III wenigstens teilweise beibehalten. Weitergehende Untersuchungen bezüglich der notwendigen Bedingungen für Extrema im Problem III lassen sogar den Schluss zu, dass die Aussonderung der Kegelstümpfe nicht zu umgehen ist.

Unsere Ausführungen beziehen sich ausschliesslich auf den gewöhnlichen Raum und die gewöhnliche Ebene.