Kleine Mitteilungen

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Kleine Mitteilungen

Proof of a Conjecture of H. Hadwiger

As part of a research problem [2], Hadwiger conjectured that every simple closed curve in E^3 admits a nontrivial inscribed parallelogram. Schnirelman's method [4] [1] leads immediately to the following result:

Theorem: *Every simple closed* C^2 *curve in* E^3 *admits a nontrivial inscribed rhombus.*

Outline of proof: The statement for plane curves has been proved by Schnirelman [4] [1]. Every simple closed curve in E^3 is homotopic to a plane Jordan curve. If the curve in E^3 is not knotted, the homotopy is in fact an isotopy. If the curve is a knot, it may be deformed into a plane Jordan curve through a C^2-homotopy $F(\alpha, t)$, $0 \leq \alpha \leq 2\pi$, $0 \leq t \leq 1$, for which $F(\alpha, t_0)$ is a simple closed curve except for finitely many values t_0 for which $F(\alpha, t_0)$, $0 \leq \alpha \leq 2\pi$, is a curve with one simple transversal selfintersection. Because of the compactness of the sets involved, a given smooth homotopy can be locally modified to satisfy the given conditions. The parametrization can be chosen so that the Jacobian matrix of F is nowhere singular. The theorem will be proved if we can show that it holds for all curves $F(\alpha, t)$, $t_0 \leq t < t_0 + \varepsilon$ if it holds for $F(\alpha, t_0)$.

By hypothesis, there exist four distinct parameter values $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ so that for $F_i = F(\alpha_i, t_0)$ we have

$$|F_1 - F_2| = |F_2 - F_3| = |F_3 - F_4| = |F_4 - F_1| \; (\neq 0) \qquad (1)$$
$$\det\,(F_1 - F_2,\ F_1 - F_3,\ F_1 - F_4) = 0$$

where det denotes the determinant. The problem is to find four points F_i^* on $F(\alpha, t)$, $t_0 \leq t \leq t_0 + \varepsilon$, that also satisfy conditions (1). We develop in a Taylor polynomial,

$$F_i^* = F_i + \frac{\delta F_i}{\delta x_i}\Delta\alpha_i + \frac{\delta F_i}{\delta t}\Delta t + o(\Delta\alpha_i, \Delta t)$$

introduce the expression in (1) and develop as well. An appropriate form of the inverse function theorem says that under our differentiability assumptions the $\Delta\alpha_i$ can be found if the linearized problem obtained by putting all $o(\Delta\alpha_i, \Delta t) = 0$, can be solved. From (1) one obtains a system of four nonhomogeneous linear equations (that can immediately be written down) for the four unknowns $\Delta\alpha_i$ $(i = 1, 2, 3, 4)$. The matrix of the system has the form

$$\begin{bmatrix} \varphi_{12} & -\varphi_{21}+\varphi_{23} & -\varphi_{32} & 0 & - \\ 0 & \varphi_{23} & -\varphi_{32}+\varphi_{34} & -\varphi_{43} & \\ -\varphi_{14} & 0 & \varphi_{34} & -\varphi_{43}+\varphi_{41} & \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \end{bmatrix}$$

where

$$\varphi_{ij} = |F_i - F_j|^{-1} \left(\frac{\delta F_i}{\delta \alpha_i}, F_j \right)$$

and the μ_i are determinant expressions derived from the last equation (1). Parentheses denote the euclidean scalar product. In the generic case, the rank of the matrix is 4 and, therefore, the problem has a unique solution. A dimension argument [4, 1 p. 107] shows that the curves for which a rhombus can be found for $t_0 \leq t < t_0 + \varepsilon$ are dense in the space of all C^2 curves. A standard convergence argument then shows that the prolongation property is true for all curves in question. We are not concerned with uniqueness since all plane C^2 Jordan curves admit a one-parameter family of rhombuses [4]. It is clear that the deformation of a nondegenerate rhombus will yield a nondegenerate rhombus since otherwise the curvature of the curve cannot be bounded, see [1] p. 109. The argument breaks down if $F(\alpha, t)$ has a double point. However, since the intersection is transversal, a plane through two points of $F(\alpha, t_0)$ that are close to the double point of $F(\alpha, t)$ and whose parameter values are close to one another, will intersect the other arc passing close to the double point at most in one point close to that double point. Therefore, the rhombus inscribed in $F(\alpha, t)$ cannot have edges of length zero.

The method of proof is very powerful for this kind of problem. In a recent dissertation [5], a student of mine, Mrs. Tropper, has used the method to prove, among other things, the following:

For $n \geq 3$, there are infinitely many regular crosspolytopes (for $n = 3$, regular octahedra) inscribed in any surface C^2-diffeomorphic to the sphere S^{n-1} in E^n.

n chords inscribed in a convex hypersurface in E^n are said to form an n-uple of conjugate diameters if support planes at the endpoints of one chord parallel and parallel to the directions of the $n - 1$ other chords of the n-uple. For $n = 2$, the existence of conjugate diameters has been proved by Heil and Krautwald [3]. The result of [5] is:

Every convex hypersurface in E^n, $n \geq 3$, admits infinitely many distinct n-uples of conjugate diameters.

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A Note on a Problem in the Theory of Sequences

1. **Introduction.** It is a well-known fact in analysis that the existence of

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k \tag{1}$$

is sufficient to imply $\lim_{k\to\infty} a_k = 0$. That condition (1) is not a necessary condition is illustrated by the sequence $a_k = 1/k$. It is only reasonable, therefore, to try to find conditions weaker than (1) that would guarantee $\lim_{k\to\infty} a_k = 0$. The aim of this paper is to present a theorem giving one such condition, namely:

Theorem 1. *Let (a_k) be a sequence of complex numbers, such that*

$$\lim_{n\to\infty} \sum_{k=[\lambda n]+1}^{n} a_k \tag{2}$$

exists for every fixed $\lambda \in (0,1)$. Then $\lim_{k\to\infty} a_k = 0$.

It should be noted that the existence of limit (2) for a single $\lambda \in (0,1)$ is generally not sufficient to guarantee that the sequence (a_k) converges to zero. This can be seen, for instance, by choosing $\lambda = {}^1/_2$ and considering the sequence (a_n) defined as follows:

$$a_n = \begin{cases} 1 & \text{if } n = 2^m \\ -{}^1/_{2^{m-1}} & \text{if } n = 2^m + 2j + 1, \quad j = 0, \dots, 2^{m-1} - 1 \quad (m = 1, 2, \dots) \\ 0 & \text{if } n = 2^m + 2j, \qquad j = 1, \dots, 2^{m-1} - 1 \end{cases}$$

The sequence (a_n) clearly does not converge to zero. On the other hand, if $n = 2^m + 4j$ or $n = 2^m + 4j + 3$, where $j = 0, \dots, 2^{m-2} - 1$, it is easy to prove that

$$\sum_{k=[n/2]+1}^{n} a_k = 0 \quad \text{for } n \geqslant 2.$$

Using this result, we can easily deduce that

$$\left| \sum_{k=[n/2]+1}^{n} a_k \right| = {}^1/_{2^{m-1}}$$

if $n = 2^m + 4j + 1$ or $n = 2^m + 4j + 2$, where $j = 0, \dots, 2^{m-2} - 1$. Hence

$$\left| \sum_{k=[n/2]+1}^{n} a_k \right| \leq 4/n \to 0 \ (n \to \infty).$$

In fact, for any integer $k > 2$, we can define a sequence (a_n) such that (2) holds for $\lambda = 1/k$, but $\lim_{n\to\infty} a_n \neq 0$. The sequence (a_n) defined as follows has these properties:

$$a_n = \begin{cases} k - 2 & \text{if } n = k^m \\ -1 & \text{if } n = k^m q, \qquad 2 \leq q \leq k - 1 \qquad\qquad (m = 1, 2, \dots) \\ 0 & \text{if } n = k^m q + r, \ 1 \leq q \leq k - 1 \ \text{ and } \ 1 \leq r \leq k^m - 1 \end{cases}$$

In this case, it can be verified that

$$\sum_{j=[n/k]+1}^{n} a_j = 0 \quad \text{for} \quad n \geqslant k\,.$$

The situation is quite different if we consider irrational numbers in (0,1), as the following theorem shows.

Theorem 2. *Let* (a_n) *be a sequence of complex numbers such that* (2) *is true for* $\lambda = \xi$ *and* $\lambda = 1 - \xi$, *where* ξ *is an irrational number in* (0,1). *Then* $\lim_{n\to\infty} a_n = 0$.

Theorem 1 is clearly a corollary of Theorem 2.

We might mention here that Theorem 1 has an application in the theory of regularly varying sequences, and it was in the context of a problem in this field that Theorem 1 was formulated (see [1]).

Some questions unresolved in this paper are the following: What can be said about the sequence (a_n), if (2) is true for

(1) two or more distinct *rational* numbers in (0, 1) ?

(2) every *rational* number $\lambda \in (0, 1)$?

(3) a single *irrational* number $\xi \in (0, 1)$?

2. *Proof of Theorem* 2. The proof of Theorem 2 is based on the following number-theoretic result:

Lemma. *Let* $\lambda \in (0, 1)$ *and let n be an integer. Then at least one of the following three statements is true:*

(i) $[\lambda n] = \lambda n$

(ii) $[\lambda n] = [\lambda (n - 1)]$

(iii) $[(1 - \lambda) n] = [(1 - \lambda) (n - 1)]$

Proof. Since $[\lambda n] \leq \lambda n < [\lambda n] + 1$, we can write

$$\lambda n = [\lambda n] + \varrho \tag{3}$$

where $0 \leq \varrho < 1$. If $\varrho = 0$, then (i) is true. Therefore, we can assume $0 < \varrho < 1$. If $\lambda \leq \varrho$, then, by (3),

$$\lambda (n - 1) = \lambda n - \lambda = [\lambda n] + \varrho - \lambda$$

where $0 \leq \varrho - \lambda < 1$. Hence $[\lambda(n - 1)] = [\lambda n]$, and consequently (ii) holds. – Finally, if $\varrho < \lambda$, then, by (3),

$$(1 - \lambda) n = n - [\lambda n] - \varrho = n - 1 - [\lambda n] + 1 - \varrho = n_\lambda + 1 - \varrho$$

where $n_\lambda = [(1 - \lambda) n]$. Therefore,

$$(1 - \lambda) (n - 1) = (1 - \lambda) n - (1 - \lambda) = n_\lambda + (1 - \varrho) - (1 - \lambda) = n_\lambda + \lambda - \varrho$$

where $0 < \lambda - \varrho < 1$. Hence $[(1 - \lambda) (n - 1)] = [(1 - \lambda)n]$ and so (iii) holds.

Proof of Theorem 2. Let

$$A_n(\lambda) = \sum_{k=[\lambda n]+1}^{n} a_k$$

and let $\xi \in (0, 1)$ be irrational. Since the sequences $(A_n(\xi))$ and $(A_n(1-\xi))$ converge to finite limits, they are Cauchy sequences. Hence there exists $N_{\varepsilon,\xi}$ such that for $n > N_{\varepsilon,\xi}$ we have $|A_n(\xi) - A_{n-1}(\xi)| < \varepsilon$ and $|A_n(1-\xi) - A_{n-1}(1-\xi)| < \varepsilon$. Since ξ is irrational, $[\xi n] \neq \xi n$ and, consequently, by the previous lemma, we know that either (ii) $[\xi n] = [\xi (n-1)]$ or (iii) $[(1-\xi)n] = [(1-\xi)(n-1)]$ is true. In the first case, we have

$$|a_n| = \left| \sum_{k=[n\xi]+1}^{n} a_k - \sum_{k=[(n-1)\xi]+1}^{n-1} a_k \right| = \left| A_n(\xi) - A_{n-1}(\xi) \right| < \varepsilon .$$

In the second case, we have

$$|a_n| = \left| \sum_{k=[n(1-\xi)]+1}^{n} a_k - \sum_{k=[(n-1)(1-\xi)]+1}^{n-1} a_k \right| = \left| A_n(1-\xi) - A_{n-1}(1-\xi) \right| < \varepsilon$$

Thus, for $n > N_{\varepsilon,\xi}$ we have $|a_n| < \varepsilon$ and the theorem is proved.

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Elementarmathematik und Didaktik

Ein reduziertes Erzeugenden-System der Kongruenzgruppe in der Ebene

1. Die Gruppe der Kongruenz-Abbildungen der Ebene

Unter einer *Isometrie* oder *Kongruenz-Abbildung* γ in der Ebene versteht man das Produkt aus endlich vielen Geraden-Spiegelungen. Bezeichnet σ_g die Spiegelung an der Geraden g, dann ist also

$$\gamma = \sigma_a \circ \sigma_b \circ \sigma_c \circ \ldots \circ \sigma_n . \tag{1}$$

Die Achsenspiegelung ist eine involutorische Abbildung; wird die Bildfigur einer gegebenen Urfigur an der gleichen Achse gespiegelt, so ergibt sich wieder die Urfigur. Das Produkt jeder Spiegelung mit sich selbst ist daher die *identische Abbildung* ι, die jeden Punkt der Ebene auf sich selbst abbildet:

$$\sigma_a \circ \sigma_a = \iota . \tag{2}$$

Hieraus folgt, dass die Kongruenz-Abbildung γ nach (1) eine *inverse Abbildung* γ^{-1} besitzt:

$$\gamma^{-1} = \sigma_n \circ \ldots \circ \sigma_c \circ \sigma_b \circ \sigma_a . \tag{3}$$

In den Produkten $\gamma \circ \gamma^{-1}$ und $\gamma^{-1} \circ \gamma$ sind nämlich Produkte von je zwei gleichen Spiegelungen vorhanden, die man wegen (2) jeweils streichen kann. Man erhält dann schliesslich

$$\gamma \circ \gamma^{-1} = \iota \quad \text{und} \quad \gamma^{-1} \circ \gamma = \iota . \tag{4}$$