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Diese letzte Tatsache legt die Frage nahe, ob es sogar eine Funktion $f \in \mathfrak{J}[a, b]$ gibt, für die $\mathcal{H}[f]$ dicht ist in $[a, b]$. Bislang konnte ich jedoch dieses Problem nicht entscheiden.

Es mag noch eine Verallgemeinerung des beschriebenen Rekursionsverfahrens zur Gewinnung der F_k angegeben werden.

Dazu gehe man von einer streng monoton gegen Null konvergierenden Zahlenfolge a_l mit $a_1 = 1$ aus und definiere ausgehend von $F_0 = F$

$$F_k(x) = \begin{cases} 1 \text{ für } x = 0 \\ \frac{1}{l} F_{k-1}\left(\frac{x - a_{l+1}}{a_l - a_{l+1}}\right) \quad \text{für } a_{l+1} \leqslant x \leqslant a_l \quad \text{und} \quad l \in N . \end{cases}$$

Die so gegebenen $\mathcal{H}[F_k]$ besitzen ganz analoge Eigenschaften wie im behandelten Fall $a_l = 1/l$. Ausserdem wächst $\mathcal{H}[F_k]$ für $k \to \infty$ «um so stärker» an, «je langsamer» a_l gegen Null konvergiert.

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LITERATURVERZEICHNIS

[1] GELBAUM-OLMSTED, Counterexamples in Analysis (Holden-Day Inc., San Francisco 1964).

Kleine Mitteilungen

A New Condition for Consecutive Primitive Roots of a Prime

In [1] we have shown that if $p > 3$ is a prime such that $\varphi(p-1)/(p-1) > 1/3$ then there is at least one pair of consecutive primitive roots modulo p. In this note we shall give a new condition that there be consecutive primitive roots for primes of the form $4n+1$.

Theorem 1. If the prime $p = 4n + 1$ has 2 as a primitive root then there is at least one pair of consecutive primitive roots modulo p.

Proof. If g is a primitive root of a prime p, then the congruence $g x \equiv 1 \pmod p$ has $x \equiv g^{p-2}$ for its unique solution; since $(p-2, p-1) = 1$, x is also a primitive root of p. If $p \equiv 1 \pmod 4$, it is well known that if g is a primitive root of p, then so is $-g$.

Let $p = 4n + 1$ be a prime and 2 a primitive root of p. Then -2 is also a primitive root and the congruence $(-2)\, g \equiv 1 \pmod p$, has a primitive root g for its solution. It follows then that $2(g+1) \equiv 1 \pmod p$, so that $g + 1$ is also a primitive root of p.

The following theorem may also be of interest here.

Theorem 2. If the prime $p = 4n + 1$ has exactly one pair of consecutive primitive roots, then 2 is a primitive root of p.

Proof. Let $0 < g < g + 1 < p$ be consecutive primitive roots of p. Then $0 < p - (g+1) < p - g < p$ are also consecutive primitive roots of p. Since there is exactly one pair of consecutive primitive roots of p, we obtain $p - (g+1) = g$, or $(-2)\, g \equiv 1 \pmod p$. Thus -2, and hence 2, is a primitive root of p.

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REFERENCE

[1] E. VEGH, *Pairs of Consecutive Primitive Roots Modulo a Prime*, Proc. Am. Math. Soc. *19*, 1169–1170 (1968).

A property of the Unitary Analogue of Ramanujan's Sum

A divisor $d > 0$ of the positive integer n is called unitary if $d\delta = n$ and $(d, \delta) = 1$. We write $d \| n$. For integers $a, b, b > 0$, let $(a, b)_*$ denote the greatest divisor of a which is a unitary divisor of b. Eckford Cohen ([1], § 2) defined the unitary analogue $c^*(m, n)$ of Ramanujan's Sum as

$$c^*(m, n) = \sum_{\substack{1 \leq x \leq n \\ (x, n)_* = 1}} e(m\,x, n) , \tag{1}$$

where $e(m, n) = \exp(2\,\pi\,i\,m/n)$, and established that

$$c^*(m, n) \text{ is multiplicative as a function of } n , \tag{2}$$

$$\varphi^*(n) \equiv c^*(0, n) = \sum_{d \| n} d\,\mu^*(n/d) , \tag{3}$$

$$\mu^*(n) \equiv c^*(1, n) = (-1)^r , \tag{4}$$

where r is the number of distinct prime factors of n.

Further, he established the following evaluation of $c^*(m, n)$:

$$c^*(m, n) = \sum_{\substack{d \mid m \\ d \| n}} d\,\mu^*(n/d) . \tag{5}$$

In this note, we establish the following:

Theorem. $c^*(m, n) = \dfrac{\varphi^*(n)\,\mu^*(n/a)}{\varphi^*(n/a)}$, where $a = (m, n)_*$.

Proof. Since a is the greatest divisor of m which is a unitary divisor of n, we can write $n = a\,N$, where $(a, N) = 1$. By (5), we have

$$c^*(m, n) = \sum_{d \| a} d\,\mu^*(n/d) = \sum_{\substack{cd = a \\ (c, d) = 1}} d\,\mu^*(a\,N/d) = \sum_{\substack{cd = a \\ (c, d) = 1}} d\,\mu^*(c\,N) .$$

Now, $(a, N) = 1$ and $c\,d = a$ imply that $(c, N) = 1$; so that by (2), (3) and (4),

$$c^*(m, n) = \sum_{\substack{cd = a \\ (c, d) = 1}} d\,\mu^*(c)\,\mu^*(N) = \mu^*(N) \sum_{d \| a} d\,\mu^*(a/d) = \mu^*(N)\,\varphi^*(a) .$$

Since $n = a\,N$ and $(a, N) = 1$, $\varphi^*(n) = \varphi^*(a)\,\varphi^*(N)$, so that

$$c^*(m, n) = \frac{\varphi^*(n)\,\mu^*(N)}{\varphi^*(N)} = \frac{\varphi^*(n)\,\mu^*(n/a)}{\varphi^*(n/a)} .$$

Hence the theorem follows.

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REFERENCE

[1] E. Cohen, *Arithmetical Functions Associated with the Unitary Divisors of an Integer*, Math. Zeit. *74*, 66–80 (1960).

Aufgaben

Aufgabe 607. Sind X und Y Teilmengen einer Menge M, so definieren wir $X + Y$ durch $X + Y = (X \cup Y) \setminus (X \cap Y)$, d.h. $X + Y$ besteht aus all den Elementen von M, die in X oder Y, jedoch nicht im Durchschnitt von X und Y liegen. Es sei nun M eine endliche Menge mit $|M| = n$ Elementen. Ferner sei k eine gerade ganze Zahl mit $2 \leqslant k \leqslant n - 1$. Zeige: Es gibt $n - 1$ Teilmengen $B_1, \ldots, B_{n-1}$ von M mit den Eigen-