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Autor(en): **Bottema, O.**

Objekttyp: **Article**

Zeitschrift: **Elemente der Mathematik**

Band (Jahr): **24 (1969)**

Heft 1

PDF erstellt am: **18.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-26641

Nach zweimaliger Anwendung des Induktionsschrittes erhalten wir die Gültigkeit von (5), (6) und (7) für beliebige $A, B \in \mathfrak{P}_1$, also für beliebige Polyeder, womit die ausstehenden Beweise vollständig erbracht sind. H. E. DEBRUNNER, Bern

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A Theorem of BOBILLIER on the Tetrahedron

1. Almost one and a half centuries ago BOBILLIER [1] gave the following theorem: *any plane through the midpoints of two opposite edges of a tetrahedron divides it in two parts of equal volume.* This statement may be found in some texts for secondary schools and in books of higher level such as MOLENBROEK [2], HADAMARD [3], HOLZMÜLLER [4], ALTSHILLER-COURT [5] and the *Exercices* of F.G.M. [6]. The last two authors add a generalization on which we will return at the end of this paper.

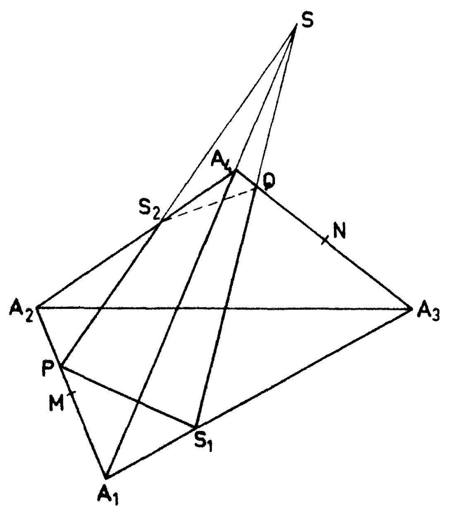

Figure 1

We consider (Fig. 1) an arbitrary transversal PQ of the opposite edges $A_1 A_2$ and $A_3 A_4$ of the tetrahedron $A_1 A_2 A_3 A_4$ and study the ratio of the volumes of the two parts in which it is divided by a variable plane α through $P Q$. P is given by the ratio $A_1 P : P A_2 = p$ and Q by $A_3 Q : Q A_4 = q$, so that $p > 0$, $q > 0$. Two sets of planes α have to be considered, one consisting of planes (such as in Fig. 1) which intersect the edges $A_1 A_3$ and $A_2 A_4$ in S_1 and S_2 respectively; the other set consists of planes having points of intersection S_1' and S_2' with $A_1 A_4$ and $A_2 A_3$. We consider for the time being the first set; a plane of this set is given by the position of S_1, that is by $A_1 S_1 : S_1 A_3 = x$, $x \geq 0$.

$S_1 Q$ and $S_2 P$ have a point of intersection S, lying on the line $A_1 A_4$. MENELAUS' theorem, applied to the triangle $A_1 A_3 A_4$ gives us

$$S A_1 : S A_4 = q\,x : 1\,. \tag{1}$$

If $q\,x > 1$ we have the situation as shown in Fig. 1, S lying on the extension of $A_1 A_4$; for $q\,x = 1$ S is at infinity and for $q\,x < 1$ S is on the extension of $A_4 A_1$. In all three cases the following derivation is essentially the same. If we apply MENELAUS to the triangle $A_1 A_2 A_4$ we obtain

$$A_4 S_2 : S_2 A_2 = p : q\,x\,. \tag{2}$$

One of the parts in which α divides the tetrahedron is the polyhedron $A_1 P S_1 A_4 Q S_2$, which is the difference between the tetrahedra $S\,A_1\,P\,S_1$ and $S\,A_4\,Q\,S_2$ having respectively the trihedra A_1 and A_4 in common with $A_1 A_2 A_3 A_4$. If the volume of the latter is unity, the volume J_1 of $A_1 P S_1 A_4 Q S_2$ is therefore

$$\left.\begin{aligned} J_1 &= \frac{q\,x}{q\,x-1}\cdot\frac{x}{x+1}\cdot\frac{p}{p+1} - \frac{1}{q\,x-1}\cdot\frac{p}{q\,x+p}\cdot\frac{1}{q+1} \\ &= \frac{p}{q\,x-1}\cdot\frac{(q+1)\,q\,x^2\,(q\,x+p)-(p+1)\,(x+1)}{(p+1)\,(q+1)\,(q\,x+p)\,(x+1)}\,. \end{aligned}\right\} \tag{3}$$

As could be expected $q\,x - 1$ is a factor of the numerator and the result is

$$J_1 = p\,\frac{q\,(q+1)\,x^2 + (p+1)\,(q+1)\,x + (p+1)}{(p+1)\,(q+1)\,(q\,x+p)\,(x+1)}\,. \tag{4}$$

The other part has the volume $J_2 = 1 - J_1$ and we obtain for the required ratio

$$\frac{J_1}{J_2} \equiv f(x, p, q) = \frac{p}{q}\cdot\frac{q\,(q+1)\,x^2 + (p+1)\,(q+1)\,x + (p+1)}{(q+1)\,x^2 + (p+1)\,(q+1)\,x + p\,(p+1)}\,. \tag{5}$$

So much for the first set of planes α. We denote the planes of the second set by the ratio $A_1 S_1' : S_1' A_4 = y$, $y \geq 0$. The two sets have two planes in common: $x = 0$ and $y = 0$, giving the plane $A_1 A_2 Q$ and $x = \infty$, $y = \infty$ the plane $A_3 A_4 P$. It is obvious that we obtain the ratio according to which the planes of the second set divide the tetrahedron if we replace in (5) x by y and q by q^{-1}. Moreover we have to keep in mind that 'the' ratio of two volumes is an ambiguous concept, because it can be interchanged with its inverse. As will be clear from the figure, in order that the ratio for $x = 0$ and $y = 0$ be the same, we must take for the ratio of the second set

$$J_2'/J_1' \equiv g(y, p, q) = f^{-1}(y, p, q^{-1})\,. \tag{6}$$

By (5) and (6) for all planes α through PQ the ratio of the volumes is given. We remark that $f(\infty, p, q) = p$, $g(\infty, p, q) = 1/p$, so that there is a discontinuity for the plane through $A_3 A_4$. That is what we expect: if x goes from ∞ to 0 and then y from 0 to ∞ we have at the end the same plane as at the start, but it has rotated through an angle π.

2. We discuss now our results (5) and (6).

For $p = q = 1$ we see that f and g are independent of x and y, and each equal to *one*. That is BOBILLIER's theorem.

If P does not coincide with the midpoint M of $A_1 A_2$, neither Q with the midpoint N of $A_3 A_4$, then we may without any loss of generality suppose $p > 1$, $q < 1$, because this can always be arrived at by interchanging A_1 and A_2, or A_3 and A_4 if necessary. The derivative of f reads, if $N(x)$ stands for the denominator of (5):

$$\frac{df(p,q,x)}{dx} = \frac{p}{q} \cdot N^{-2}(x)\,(p+1)\,(q+1)\{(q+1)\,x+(p+1)\}\{(q-1)\,x+(p-1)\} \tag{7}$$

and its sign is therefore that of the last factor.

Hence, putting

$$x_0 = (p-1)/(1-q)\,, \tag{8}$$

x_0 being a positive number, we have

$$\frac{df}{dx} > 0 \quad \text{for} \quad 0 \le x < x_0\,, \quad \frac{df(x_0)}{dx} = 0\,, \quad \frac{df}{dx} < 0 \quad \text{for } x > x_0.$$

The function f is therefore increasing from the value $f(0) = 1/q$ to its maximum

$$\left.\begin{aligned} m = f(x_0) &= \frac{p}{q} \cdot \frac{p\,q + p - 3\,q + 1}{-p\,q + 3\,p - q - 1} \\ &= \frac{p}{q} \cdot \frac{2\,(p-1) + 2\,(1-q) - (p-1)\,(1-q)}{2\,(p-1) + 2\,(1-q) + (p-1)\,(1-q)} \end{aligned}\right\} \tag{9}$$

and then decreasing to $f(\infty) = p$. The minimum value of $f(x)$ being the smaller of the numbers $1/q$ and p, each more than one, there is among the planes of the first set no one which bisects the tetrahedron.

The derivative of $f(y, p, q^{-1})$ is obtained by replacing in (7) q by q^{-1} and is therefore always positive; that means that $f(y, p, q^{-1})$ increases if y goes from zero to infinity. The conclusion is that $g(y, p, q)$ is a decreasing function of y, starting with the value q^{-1} and ending with p^{-1}. Hence there is one value of y, the positive root of the quadratic equation $g(y, p, q) = 1$ for which the corresponding plane divides the tetrahedron in two equal parts. We remark that

$$\frac{df(0)}{dx} = \frac{(q+1)\,(p-1)}{p\,q}\,, \qquad \frac{dg(0)}{dy} = \frac{(q+1)\,(p-1)}{p\,q^2}\,;$$

hence the derivation of our ratio is discontinous for α coincident with $A_1 A_2 Q$.

Summing up we have the following statements. *If in the tetrahedron* $A_1 A_2 A_3 A_4$ *we consider the pencil of planes* α *through the transversal* $P\,Q$ (P *on* $A_1 A_2$, $A_1 P : P A_2 = p > 1$, Q *on* $A_3 A_4$, $A_3 Q : Q A_4 = q < 1$), *starting with the plane* $A_3 A_4 Q$, *rotating it around* $P\,Q$ *through the angle* π *and ending therefore at the initial position, then the ratio of the volumes of the parts of the tetrahedron* (taken in a certain order) *starts with the value* p, *increases to the value* m *given by* (9) *and then decreases to the value* p^{-1}.

There is always one and only one plane through PQ *which bisects the tetrahedron.*

Independently of the order of the two parts the maximum of the ratio equals m and the minimum is m^{-1}.

The border cases $p = 1$ or $q = 1$ are easily dealt with. For $p = q = 1$ we have $m = 1$.

3. An attractive generalization of BOBILLIER's theorem given by LEVY [7] has been reproduced several times in the course of the years, inter alia by F.G.M. and by ALTSHILLER-COURT. It reads as follows: if a transversal $P\,Q$ divides $A_1 A_2$ and $A_3 A_4$

in the same ratio, then each plane through PQ divides the tetrahedron in the same ratio. That would mean that $f(x)$, given by (5) would be independent of x if $p = q$. This is immediately seen not to be the case, f being a constant only if $p = q = 1$. Therefore the generalization can not be correct. If we check the proof given by the authors just mentioned we see that they consider the two parts of the tetrahedron (Fig. 1) as the sum of A_1. $P S_1 Q S_2$ and $A_1 A_4 Q S_2$ and that of A_2. $P S_1 Q S_2$ and $A_2 A_3 Q S_1$. The first terms have the ratio $A_1 P : P A_2 = p$. The second terms are tetrahedra which have respectively the trihedra A_4 and A_3 in common with $A_1 A_2 A_3 A_4$; hence their ratio is

$$\frac{A_4 S_2}{A_4 A_2} \cdot \frac{A_4 Q}{A_4 A_3} : \frac{A_3 S_1}{A_3 A_1} \cdot \frac{A_3 Q}{A_3 A_4}$$

that is for $p = q$ equal to

$$\frac{1}{1+x} \cdot \frac{1}{p+1} : \frac{1}{1+x} \cdot \frac{p}{p+1} = 1 : p\,.$$

The second terms have therefore the same ratio as the first but unfortunately in the wrong order, so that the conclusion is not valid. The generalization is obviously too good to be true.

4. We consider in five-dimensional affine space a simplex $A_1 A_2 \ldots A_6$ and the midpoints P_1, P_3, P_5 of three mutually skew edges $A_1 A_2$, $A_3 A_4$, $A_5 A_6$. If we introduce barycentric coordinates $x_i (i = 1, \ldots, 6)$, then $P_1 = (110000)$, etc. Hence the equation of any four-dimensional space V through P_1, P_3 and P_5 reads

$$\lambda_1 (x_1 - x_2) + \lambda_2 (x_3 - x_4) + \lambda_3 (x_5 - x_6) = 0\,. \tag{10}$$

The point of intersection of V and the line $A_i A_j$ will be denoted by S_{ij}. Without loss of generality we may suppose $\lambda_i > 0$. Hence A_1, A_3, A_5 are on one side of V and A_2, A_4, A_6 on the opposite side. A point S_{ij} is on an edge if i and j have different parity. One of the two parts in which the simplex is divided by V is the polyhedron $A_1 A_3 A_5 S_{ij}$, $i \not\equiv j \pmod 1$, which is the sum of the three simplices

$$A_1 S_{12} S_{14} S_{16} S_{13} S_{15}\,, \quad A_3 S_{34} S_{36} S_{32} S_{35} S_{31}\,, \quad A_5 S_{56} S_{52} S_{54} S_{51} S_{53}\,. \tag{11}$$

From (10) it follows that $S_{14} = (\lambda_2\, 0\, 0\, \lambda_1\, 0\, 0)$ and thus $A_1 S_{14} : A_1 A_4 = \lambda_1 : (\lambda_1 + \lambda_2)$; furthermore $S_{13} = (-\lambda_2\, 0\, 0\, \lambda_1\, 0\, 0)$ and therefore $A_1 S_{13} : A_1 A_3 = \lambda_1 : (\lambda_1 - \lambda_2)$. If the volume of the simplex A_i is unity then that of the first simplex of (11) is

$$\frac{1}{2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 - \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 - \lambda_3} \cdot = \frac{1}{2} \cdot \frac{\lambda_1^4}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}$$

and as

$$\frac{\lambda_1^4}{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)} + \frac{\lambda_2^4}{(\lambda_2^2 - \lambda_3^2)(\lambda_2^2 - \lambda_1^2)} + \frac{\lambda_3^4}{(\lambda_3^2 - \lambda_1^2)(\lambda_3^2 - \lambda_2^2)} = 1 \tag{12}$$

the sum of the three simplices (11) is seen to be 1/2. The argument can be applied to any space with an odd number of dimensions. Hence we have proved the following generalization of BOBILLIER's theorem: *If in a space of $(2n-1)$-dimensions a simplex $A_1 A_2 \ldots A_{2n}$ is given, P_i $(i = 1, 3, \ldots 2n-1)$ being the midpoints of a set of n mutually skew edges such as $A_{2k-1,\,2k} (k = 1, \ldots n)$, then all (∞^{n-1}) $(2n-2)$-dimensional spaces through P_i divide the simplex in two parts of equal volume.*

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Anschauliche Behandlung eines Verzweigungsprozesses (branching process)

Verzweigungsprozesse (branching processes) geben eine mathematische Darstellung der Entwicklung einer «Bevölkerung» (population), bestehend aus irgendwelchen Elementen, die sich nach Wahrscheinlichkeitsgesetzen fortpflanzen und nach Wahrscheinlichkeitsgesetzen sterben. Sowohl die Elemente dieser Gesamtheit als auch die Art des Fortpflanzungsvorganges können in sehr verschiedener Art gewählt werden; indessen dürfen sich die Glieder gegenseitig weder hemmen noch fördern. T. E. HARRIS hat in den letzten Jahren in [2] eine zusammenfassende Theorie dieser Prozesse gegeben. – In den folgenden Zeilen soll für einen besonders einfachen Verzweigungsprozess, den Galton-Watson-Prozess, zunächst ein Urnenschema entwickelt werden; aus diesem sollen in anschaulicher Weise einige Folgerungen gezogen werden, die nachher vor allem auf das Problem des Aussterbens der Geschlechter angewendet werden.

1. Ein Urnenschema

Wir denken uns eine mit Kugeln gefüllte Urne. Jede Kugel trage eine nichtnegative ganze Zahl z als Nummer; im übrigen seien alle Kugeln gleich und $0 \leqq z \leqq \omega$. Die Wahrscheinlichkeit, aus der gut durchmischten Urne eine Kugel mit der Zahl z als Nummer zu ziehen, sei p_z. Es ist dann $\sum\limits_{z=0}^{\omega} p_z = 1$, und es sei $p_0 \neq 0$ und $p_0 \neq 1$. Nun werde folgendes Spiel gespielt:

1. Akt: Es wird eine Kugel gezogen und ihre Nummer $z = z_1$ notiert. Dann wird sie zurückgelegt, und die Urne wird wieder gut durchmischt.

2. Akt: Nun werden nacheinander z_1 Kugeln gezogen. Dabei legen wir jede Kugel, nachdem wir ihre Nummer notiert haben, wieder zurück und mischen, bevor die nächste Kugel gezogen wird («Ziehen mit Zurücklegen»). Wir bilden die Summe z_2 der in diesem Akt notierten Nummern.

3. Akt: In analoger Weise ziehen wir jetzt z_2 Kugeln, wieder mit Zurücklegen, und bilden die Summe z_3 ihrer Nummern, *usw.*

Sobald eine der Summen $z_i = 0$ ist, brechen wir das Spiel ab; wir setzen in naheliegender Weise in diesem Falle $z_{i+k} = 0$ für $k = 1, 2, 3, \ldots$.

Wir stellen jetzt die Frage: Wie gross ist die Wahrscheinlichkeit q_n, dass $z_n = 0$ ist? *q_n ist also die Wahrscheinlichkeit dafür, dass unser Spiel spätestens mit dem n-ten Akt abbricht.*