Inequalities concerning the inradius and circumradius of a triangle

den, aber ε nächstliegenden Gitterpunkt des Zylinders Z. Unter den Hyperebenen (16) einer Verteilung (2) gibt es also im vorliegenden Fall keine ε beliebig benachbarte Hyperebene. Entsprechendes folgt für jede Hyperebene der Verteilung (2). Somit gilt:

Sind die Verhältnisse der Logarithmen von $|x_1|, \ldots, |x_n|$ *rational, so besitzt die Verteilung* (2) (*und damit auch jede ihrer unendlichen Teilverteilungen*) *keinen eigentlichen Häufungspunkt.*

b) Die Gleichung (17) von ε hat *mindestens zwei Koeffizienten mit irrationalem Verhältnis.* Es sei etwa $\lg|x_1|/\lg|x_2|$ irrational. Die Schnittgerade der Hyperebene ε mit der Koordinatenebene $i_1 i_2$ hat somit die Gleichung

$$i_2 = -\frac{\lg|x_1|}{\lg|x_2|}\, i_1 . \qquad (18)$$

Da die Irrationalzahl $\lg|x_1|/\lg|x_2|$ beliebig genau rational approximiert werden kann, liegt in der $i_1 i_2$-Ebene in jedem um die Gerade (18) abgegrenzten Parallelstreifen mindestens ein Gitterpunkt. In diesem Fall gibt es also in beliebiger Nähe der Hyperebene ε noch weitere Scharebenen, und Entsprechendes folgt für jede Hyperebene der Verteilung (2). Somit gilt:

Ist mindestens ein Verhältnis der Logarithmen von $|x_1|, \ldots, |x_n|$ *irrational, so ist jeder eigentliche Punkt der Verteilung* (2) *Häufungspunkt.* O. GIERING, Stuttgart

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Inequalities Concerning the Inradius and Circumradius of a Triangle

The purpose of this paper is to establish in an elementary manner a number of new inequalities between the inradius and circumradius of a triangle, its area and certain functions of its sides.

These inequalities will be strong enough to permit us to deduce without difficulty several well-known results, and to sharpen some inequalities due to other authors.

1. *Some Identities*

If T be any triangle and A_1, A_2, A_3 its vertices, we shall denote by a_i the side opposite vertex A_i, and by α_i the angle at A_i. Further, let r be the inradius and R the circumradius of T, while Δ denotes its area and $2\,s$ its perimeter. For simplicity, we shall write

$$\sum a_i \text{ for } \sum_{i=1}^{3} a_i, \quad \sum_{i<j} a_i a_j \text{ for } \sum_{1<i<j<3} a_i a_j, \quad \text{and } \prod a_i \text{ for } \prod_{i=1}^{3} a_i .$$

We begin by establishing several useful relations between r, R and the sides of T. It can be shown without difficulty that in any triangle,

$$2\sum_{i<j} a_i\, a_j - \sum a_i^2 = 4\, r\,(4\, R + r)\,. \tag{1}$$

For $r = \Delta/s$ and $4\, r\, R = a_1\, a_2\, a_3/s$, so that

$$4\, r\,(4\, R + r) = \frac{4\, a_1\, a_2\, a_3}{s} + \frac{4\, \Delta^2}{s^2}\,.$$

Since $\Delta^2 = s\,(s - a_1)\,(s - a_2)\,(s - a_3)$, we have

$$\begin{aligned} 4\, r\,(4\, R + r) &= \frac{1}{2\, s}\,[8\, a_1\, a_2\, a_3 + (a_1 + a_2 - a_3)\,(a_2 + a_3 - a_1)\,(a_3 + a_1 - a_2)] \\ &= \frac{1}{2\, s}\,[2\,(a_1 + a_2 + a_3)\,(a_1\, a_2 + a_2\, a_3 + a_3\, a_1) \\ &\quad - (a_1 + a_2 + a_3)\,(a_1^2 + a_2^2 + a_3^2)]\,, \end{aligned}$$

whence identity (1).

Combining this relation with the algebraic identity

$$2\sum_{i<j} a_i\, a_j + \sum a_i^2 = \left(\sum a_i\right)^2, \tag{2}$$

we obtain the relations

$$\left(\sum a_i\right)^2 = 2\sum a_i^2 + 4\, r\,(4\, R + r)\,, \tag{3}$$

and

$$4\sum_{i<j} a_i\, a_j = \left(\sum a_i\right)^2 + 4\, r\,(4\, R + r)\,. \tag{4}$$

2. *Three Inequalities*

Consider triangle HIO, where H, I, O denote the orthocenter, incenter, and circumcenter of T. HOBSON [1][1]) proved the following formula for the distance between H and I (in our notation):

$$IH^2 = 2\, r^2 - 4\, R^2\, \Pi \cos\alpha_i\,. \tag{5}$$

But

$$4\, R^2\, \Pi \cos\alpha_i = \frac{1}{4}\left(\sum a_i\right)^2 - (2\, R + r)^2$$

(see [2] for a proof), and thus

$$IH^2 = 3\, r^2 + 4\, r\, R + 4\, R^2 - \frac{1}{4}\left(\sum a_i\right)^2. \tag{6}$$

This establishes the inequality

$$\left(\sum a_i\right)^2 \leq 4\,(3\, r^2 + 4\, r\, R + 4\, R^2)\,, \tag{7}$$

with equality only in an equilateral triangle, for only then do points I and H coincide.

[1]) Numbers in brackets refer to references, page 131.

Now let G be the centroid of T. According to a well-known theorem of EULER, G lies on OH, between O and H, and $GO: GH = 1:2$. Consequently, we have

$$GI^2 = \frac{2}{3} IO^2 + \frac{1}{3} IH^2 + \frac{2}{9} OH^2 . \tag{8}$$

It is readily shown that

$$\sum a_i^2 = 3 \sum GA_i^2 , \tag{9}$$

and that for any point P in the plane of T,

$$\sum PA_i^2 = \sum GA_i^2 + 3\, PG^2 . \tag{10}$$

If we choose for P the circumcenter O of T, we have, by (9):

$$3\, R^2 = \frac{1}{3} \sum a_i^2 + 3\, OG^2 ,$$

whence, since $OH = 3\, OG$:

$$OH^2 = 9\, R^2 - \sum a_i^2 . \tag{11}$$

Finally, by EULER's formula for the distance separating a triangle's incenter and circumcenter, we have

$$IO^2 = R^2 - 2\, r\, R . \tag{12}$$

By substituting (6), (11) and (12) in formula (8), we obtain

$$GI^2 = r^2 - \frac{1}{12} (\sum a_i)^2 + \frac{2}{9} \sum a_i^2 ,$$

which, by relation (3), is equivalent to

$$36\, GI^2 = (\sum a_i)^2 + 20\, r^2 - 64\, r\, R , \tag{13}$$

whence

$$(\sum a_i)^2 \geq 4\, r\, (16\, R - 5\, r) , \tag{14}$$

with equality only when the triangle is equilateral.

We have thus proved the following inequalities for a triangle with sides a_1, a_2, a_3, inradius r, circumradius R and area Δ:

First Inequality

$$4\, r\, (16\, R - 5\, r) \leq (\sum a_i)^2 \leq 4\, (3\, r^2 + 4\, r\, R + 4\, R^2) . \tag{15}$$

Second Inequality

$$12\, r\, (2\, R - r) \leq \sum a_i^2 \leq 4\, r^2 + 8\, R^2 . \tag{16}$$

Third Inequality

$$r^3\, (16\, R - 5\, r) \leq \Delta^2 \leq r^2\, (3\, r^2 + 4\, r\, R + 4\, R^2) . \tag{17}$$

Indeed, the first inequality is given by (7) and (14), the second follows from the first and from relation (3), while the third is deduced from the first by the identity $2\, \Delta = r\, \Sigma\, a_i$. In all three, equality holds only when the triangle is equilateral.

3. *Two Applications*

3.1. F. LEUENBERGER has proved [3] the following inequality:

$$18\, r\, R \leq \sum_{i<j} a_i\, a_j \leq 9\, R^2 .$$

Using identity (1) and inequality (16), we obtain the stronger result

$$4\, r\, (5\, R - r) \leq \sum_{i<j} a_i\, a_j \leq 4\, (r + R)^2 . \tag{18}$$

This can be combined with the identity

$$\sum_{i<j} a_i\, a_j = 2\, R \sum h_i ,$$

where h_i ($i = 1, 2, 3$) are the altitudes of T, to obtain the inequality

$$\frac{2\, r\, (5\, R - r)}{R} \leq \sum h_i \leq \frac{2\, (r + R)^2}{R} . \tag{19}$$

3.2. Another result due to the same author [4] is

$$\frac{\sqrt{3}}{R} \leq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{\sqrt{3}}{2\, r} .$$

Here it is possible, by applying (18), to sharpen the left side. For we have

$$\sum \frac{1}{a_i} = \frac{\sum_{i<j} a_i\, a_j}{\Pi\, a_i} ,$$

and by the inequality between arithmetic and geometric means,

$$\frac{1}{3} \sum_{i<j} a_i\, a_j \geq (\Pi\, a_i)^{2/3} ,$$

whence

$$(\Pi\, a_i)^{-1} \geq \sqrt{27} \left(\sum_{i<j} a_i\, a_j \right)^{-3/2} .$$

Multiplying both sides by $\sum_{i<j} a_i\, a_j$, we get

$$\sum \frac{1}{a_i} \geq \left(\frac{27}{\sum_{i<j} a_i\, a_j} \right)^{1/2} ,$$

and it follows from (18) that

$$\frac{3\sqrt{3}}{2\, (r + R)} \leq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq \frac{\sqrt{3}}{2\, r} . \tag{20}$$

4. *Corollaries*

Our three inequalities can be used to obtain some well-known results. We shall mention three such examples.

4.1. Inequality (16), together with the classical inequality $2\, r \leq R$, gives us KUBOTA's inequality [5]:

$$36\, r^2 \leq \sum a_i^2 \leq 9\, R^2 .$$

4.2. We can write the right side of (17) as

$$\Delta^2 \leq r^2 \left[\left(4 R^2 + \frac{8}{3} r R + \frac{1}{3} r^2\right) + \frac{1}{3}\left(4 r R + 8 r^2\right)\right].$$

Since $4 r^2 \leq 2 r R \leq R^2$, the value of the second parenthesis never exceeds $4 R^2$, and therefore

$$\sqrt{3}\,\Delta \leq r (4 R + r). \tag{21}$$

But because of identity (3), this is equivalent to

$$4\sqrt{3}\,\Delta \leq (a_1 + a_2 + a_3)^2 - 2 (a_1^2 + a_2^2 + a_3^2),$$

or

$$4\sqrt{3}\,\Delta \leq a_1^2 + a_2^2 + a_3^2 - [(a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2],$$

an inequality which was first proved by P. FINSLER and H. HADWIGER [6].

4.3. Finally, since

$$\prod a_i = \frac{\sum\limits_{i<j} a_i a_j}{\sum 1/a_i},$$

we have by (18) and (20)

$$\left(\frac{\sqrt{3}}{2}\right)^3 \cdot \prod a_i \leq (r + R)^3 \leq \left(\frac{3 R}{2}\right)^3,$$

and therefore

$$\Delta = \frac{a_1 a_2 a_3}{4 R} \leq \frac{\sqrt{3}}{4} (a_1 a_2 a_3)^{2/3},$$

whence the chain of inequalities

$$(\sqrt{3}\,\Delta)^{3/2} \leq \left(\frac{\sqrt{3}}{2}\right)^3 a_1 a_2 a_3 \leq (r + R)^3, \tag{22}$$

parts of which have already been pointed out by L. CARLITZ [7].

In concluding, we note that although (20) and (22) might lead one to suspect that there exists an inequality between Σa_i and $2\sqrt{3}(r + R)$, this is not the case, as the following example shows:

For $a_1 = 3$, $a_2 = 4$, $a_3 = 5$, we have $r = 1$ and $R = 5/2$, and therefore $\Sigma a_i < 2\sqrt{3}(r + R)$, while in the isoceles triangle $a_1 = 2$, $a_2 = a_3 = 6$, we have $r = \sqrt{35}/7$ and $R = 18/\sqrt{35}$, so that $\Sigma a_i > 2\sqrt{3}(r + R)$.

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