Conchoid and Negative Circle

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ARCHIMEDES hat nun das klassische Beweisverfahren für den vorliegenden Zweck etwas modifiziert, indem er nicht von den Differenzen $B - A$, sondern von den Verhältnissen $B:A$ spricht. Statt zu beweisen, dass die Differenz $B - A$ kleiner gemacht werden kann als jede gegebene gleichartige Grösse F, beweist er, dass das Verhältnis $B:A$ kleiner gemacht werden kann als ein gegebenes Streckenverhältnis $b:a$ mit $b > a$. Mit dem Verhältnis kann man etwas leichter rechnen als mit der Differenz, denn die Oberflächen der beiden ähnlichen Rotationskörper verhalten sich wie die Quadrate entsprechender Strecken und die Inhalte wie die Kuben.

Mit dieser modifizierten Exhaustion beweist ARCHIMEDES also, dass die Oberfläche der Kugel gleich vier Grosskreisen ist und der Inhalt viermal so gross wie der eines Kegels, dessen Basis ein Grosskreis und dessen Höhe der Kugelradius ist. Nun ist nach EUDOXOS dieser Kegel gleich einem Drittel des Zylinders mit gleicher Basis und gleicher Höhe; also ist die Kugel gleich 4/3 dieses Zylinders, das heisst gleich 2/3 des ihr umbeschriebenen Zylinders.

Überblicken wir nun den Beweis, so sehen wir, dass er sich in allen grossen Linien als Produkt einer bewundernswert scharfsinnigen, bewussten Überlegung verstehen lässt. Einige Einfälle (wie die Verwandlung des Kegelmantels in eine gleich grosse Kreisfläche und die Benutzung von Verhältnissen statt Differenzen am Schluss) dienen nur zur eleganten Darstellung des Beweises. Wesentlich waren nur zwei Einfälle: erstens das Ziehen der zu einer Seite parallelen Diagonalen in Figur 3, zweitens die Zerlegung des Rotationskörpers in Stücke, die im Mittelpunkt X zusammenkommen und durch Rotation von Dreiecken entstehen. Wir haben aber gesehen, wie man durch bewusste Überlegung die Einfälle geradezu provozieren kann, indem man sich richtig klarmacht, welche Schwierigkeit an der betreffenden Stelle zu überwinden ist und welche Bedingungen die gesuchte Umformung oder Zerlegung zu erfüllen hat.

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Conchoid and Negative Circle

(Continued)

7.

The relation of the negative circle to the pair of conchoids is not one to one, for it depends on the choice of the unit circle v^2. The *form* (as distinct from the size and location) of the Conchoid of NICOMEDES as a Euclidean curve depends, like that of the conic, on a single parameter of zero dimension. It is the ratio, say μ, of the radial parameter to the distance of the node from the base-line. To form a given 'negative circle', any chosen point N along c_0 will serve as node (N' or N'' in figure 3) of the 'auxiliary conchoid' and will in turn determine the circle v^2, i.e. the chosen unit by which the two spaces are related. Moving the diametral point D into the central position M (figure 5) whence proceed the tangents at the extremities L, L of the latus rectum, the end-points (D_1'', etc.) of the rotating satellite line become the upper and lower vertices A, A of the conchoid. It is then easy to see that the condition, relating the auxiliary conchoid to the negative circle which it will help to form, is that

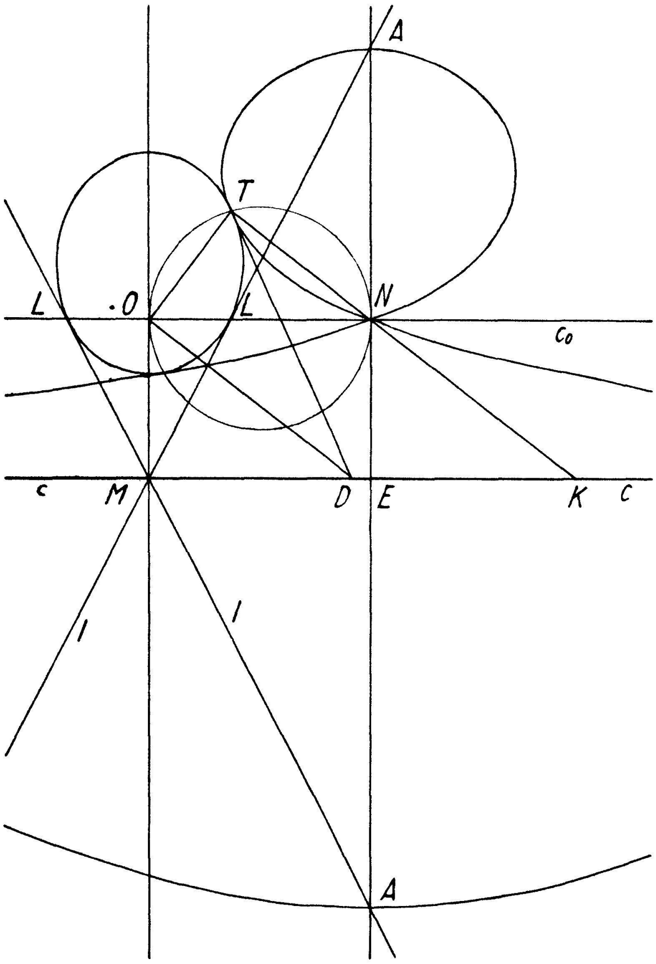

Figure 5

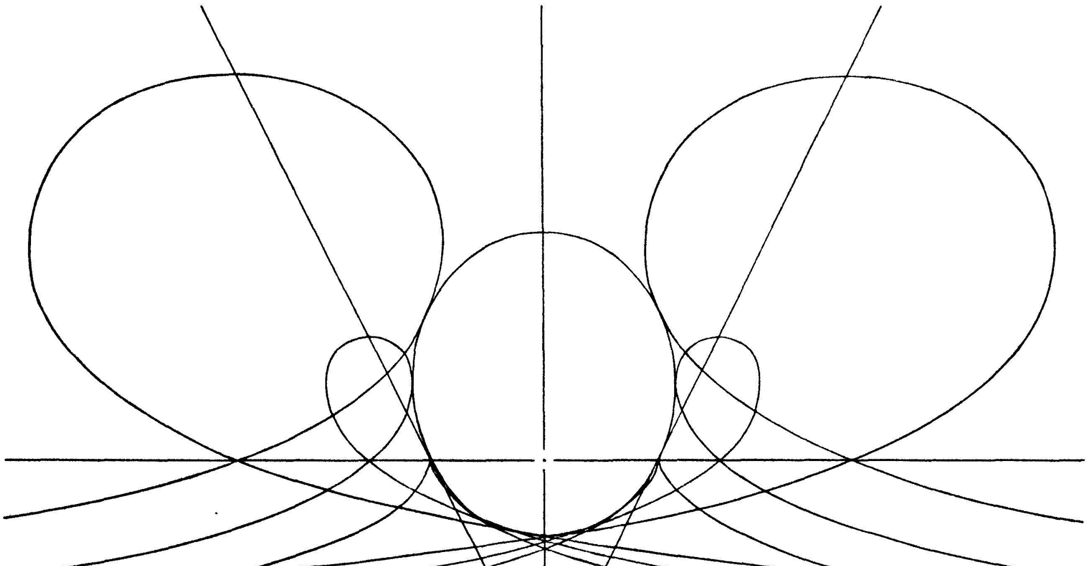

Figure 6

the ratio, $ON:EA$ in figure 5, of the distance of the node from the point-at-infinity to the radial parameter of the conchoid be equal to the eccentricity of the conic, $\varepsilon = m/r$. If the node is on the conic at L, figure 5, a conchoid with a cusp, $\mu = 1$, will be the outcome. If the node is within the conic (in the Euclidean sense), the conchoid will have an isolated point at the node, $\mu < 1$; if outside the conic, it has a real loop, $\mu > 1$. The resulting family of conchoids, with common base-line c and nodes along c_0, auxiliary to a given 'negative circle', will therefore have their vertices along the lines, ML in figure 5, touching the latter at the extremities of the latus rectum. These relationships are illustrated in figure 6 and will be gone into more fully in § 11.

8.

Analytical Confirmation, proving that the diagonals of the moving parallelogram in figures 3 and 4 envelop the required conic.

In homogeneous Cartesians, the corners D_1', etc. have the coordinates:

$$\left.\begin{array}{ll} D_1'', D_1': & \left\{\begin{array}{l} Y_1:Z_1 = -\dfrac{1}{m} + r\sin\theta \\ X_1:Z_1 = -\dfrac{\cot\theta}{m} + r\cos\theta \pm m \end{array}\right. \\ D_2'', D_2': & \left\{\begin{array}{l} Y_2:Z_2 = -\dfrac{1}{m} - r\sin\theta \\ X_2:Z_2 = -\dfrac{\cot\theta}{m} - r\cos\theta \pm m \end{array}\right. \end{array}\right\} \tag{5}$$

whence the diagonals are the lines:

$$D_1''D_2',\ D_1'D_2'': \quad (r\sin\theta)\,X - (r\cos\theta \pm m)\,Y = \pm Z, \tag{6}$$

giving in either case, by partial differentiation with respect to the parameter θ:

$$Y:X = -\cot\theta. \tag{7}$$

By the construction, the point D on c, common to both diagonals, has coordinates X, Y such that $Y:X = +\tan\theta$. Equation (7) therefore confirms that the meeting-point of successive diagonals—parameters $\theta, \theta + d\theta$—is the point T, such that DT subtends a right angle at O, as of course it must do if the lines are to envelop a conic with O as focus and c as directrix.

Eliminating θ between (6) and (7) and omitting the alternative solution $X^2 + Y^2 = 0$, we obtain the envelope:

$$X^2 + Y^2 = \left(\frac{m}{r}\right)^2 \left(Y + \frac{Z}{m}\right)^2, \tag{8}$$

clearly the point-equation of a conic with focus at the origin, eccentricity $\varepsilon = m/r$, and the line c ($Z:Y = -m$) for directrix. Deriving the line-equation by the familiar method, we obtain:

$$x^2 + (y - m)^2 = r^2 z^2, \tag{9}$$

equation to the negative circle with radius r and median line c ($x:y:z = 0:m:1$).

It may be mentioned in passing that if the symbols (X,Y,Z), (x,y,z) in equations (6) and (9) are interchanged, (6) becomes the obvious line-equation of the point-pair, in which the Euclidean circle (9), polar-reciprocal to the given negative circle, meets the moving diameter determined by θ, which is now the angle between this diameter and the axis of Y. Calling the centre of the circle C, the 'eccentricity' $m/r = \varepsilon$ now has the natural and obvious meaning; it is the eccentricity of O as origin, relative to the given circle—the ratio of the distance OC to the radius. At the same time, in the negative space determined by O, this circle counts as a conic, the same eccentricity ε being now the constant ratio of the negative distances of all the tangent lines from o as focal line and C as director-point. ('Focal line' and 'director-point' are the negative-Euclidean equivalents of focus and directrix; the constant ratio is not difficult to prove and throws an interesting light on the relation of conics and Apollonian circles.) The concept of 'eccentricity' thus has a double significance in either case. For the one space—whether positive or negative—for which the curve is a circle, it is the distance, relative to the radius, of the central point or median line from the natural origin, namely the infinitude of the other space, while for the latter it is the eccentricity of the conic according to the classical definition.

Needless to say, in the above construction the roles of the two spaces can also be reversed, though the resulting process will be more difficult and of less practical value. That is to stay, starting from negative space and assuming that a fixed negative or line-to-line distance can be transferred at will, the polar-reciprocal of the conchoid, formed as a linewise curve, will lead to the construction of a Euclidean circle pointwise.

9.

Equation (6) gives for the line-coordinates of the pair of tangents to the negative circle from the diametral point θ:

$$x : y : z = (\pm r \sin\theta) : (m \mp r \cos\theta) : 1\,. \tag{10}$$

This therefore is the parametral equation of the negative circle (9). $\theta = \pi$ gives for the parallel tangents from the (Euclidean) point-at-infinity of c, i.e. the tangents at the ends of the major axis, $x:y:z = 0:(m \pm r):1$. The significance of $m \pm r$ in the measures of negative space is obvious. For $\theta = \pi/2$ we have the tangents (ML in figure 5) et the ends of the latus rectum, $x:y:z = \pm r:m:1$. For the Euclidean length of the semi-latus-rectum this gives $OL = 1/r$, a result which is easily confirmed synthetically. (The 'negative translation', cf. § 2, with fixed point and line O, c_0, transforming c into o, transforms the negative vector cl in figure 5 into the equivalent vector ol', where l' is the Euclidean parallel to OM through L. Hence $r = c\,l = o\,l' = 1/OL$.)

The horizontal and vertical Euclidean lengths in figure 5 and the essential ratios μ, ε—compare the similar triangles AEM, MOL—are as follows:

$$\left.\begin{aligned} &ME = ON = m, \quad MO = EN = \frac{1}{m}\,; \quad EA = r, \quad OL = \frac{1}{r}\,, \\ &\mu = EA:EN = EA:MO; \quad \varepsilon = LO:OM = ME:EA; \quad \varepsilon\,\mu = ME:MO = m^2. \end{aligned}\right\} \tag{11}$$

The radius of the common unit circle v^2 is given by the geometry of the figure; it is $\sqrt{-1}$ times $MO \cdot ON = EA \cdot OL = 1$.

Besides the values $\pi/2$ and 0, for which the parallelogram of figure 3 becomes rectangular and degenerates, there are two other special values of the parameter θ, namely:

$$\text{(i)} \quad \theta = \cos^{-1}(\pm \varepsilon) \quad \text{and (ii)} \quad \theta = \sec^{-1}(\pm \varepsilon). \tag{12}$$

(i) are the values, real and distinct only if $r > m$, for which a diagonal of the parallelogram becomes perpendicular to c and to the sides $2\,m$; this gives the tangents to the *ellipse* at the ends of the minor axis. (ii) are the values, real and distinct only if $r < m$, for which a diagonal is perpendicular to the other pair of sides, $2\,r$. This gives the asymptotes of the *hyperbola*. For the construction of these special tangents the circle on ON or that on EA as diameter (figure 5) may be used, and it is easy to prove that the resulting satellite lines have the required length; the details are here omitted. For $m = r$ (parabola), equations (12) give $\theta = 0$ or π, the special pairs of tangents merging of course into o.

In the chosen coordinate system (origin at O) the auxiliary conchoid has the equation:

$$Z^2 \left\{ r^2 Y^2 - m^2 \left(Y + \frac{Z}{m} \right)^2 \right\} = \left(Y + \frac{Z}{m} \right)^2 (X^2 + Y^2 \pm 2\,m\,X\,Z), \tag{13}$$

the + and − in the final term referring to the two equal conchoids with nodes to the left and right of O respectively, cf. figures 3 and 4.

Conchoid and negative circle touch at the points, real or imaginary, where they meet. The conchoid remains—for Euclidean space—entirely *outside* the conic. This follows readily from the original construction, from the geometry of the respective curves, and from equations (8) and (13). The points of contact, T in figure 5, are the points where the conic meets the circle on ON as diameter. For every value of ε there will be certain osculating conchoids, where the two points of contact merge into one.

Figure 6 indicates how as an outcome of this relation the conchoid appears as envelope of its auxiliary conchoids. Analysis confirms that the envelope of the conchoid-family (§ 7) breaks up into the lines c and c_0, each counting doubly, and the required conic.

10.

We conclude with a brief reference to three-dimensional space and the negative-Euclidean sphere (cf. §§ 1–3). The latter will of course be a quadric surface touching the absolute cone borne by the point-at-infinity O. (This cone, being spherical[1]),

[1]) The term 'spherical cone' to denote any cone in perspective with the *Kugelkreis* is used for example by Prof. L. N. G. FILON in his *Introduction to Projective Geometry* (1935 edition, p. 303). (*Sphärischer Kegel* would presumably be the German equivalent.) L. HEFFTER and L. KOEHLER, *Analytische Geometrie*, Vol. II (G. Braun, Karlsruhe 1923), p. 2, use the term *absoluter Kegel*. Where the geometry of negative-Euclidean space is concerned, it is, however, obviously better to reserve the designation 'absolute' for the unique spherical cone carried by the point-at-infinity, determining the metric of the space. *Kugelkreis* (for which there is no easy English equivalent) is an apt expression for the precise reason that every Euclidean sphere contains this circle and every quadric containing it is a Euclidean sphere. In the same way the negative-Euclidean sphere is determined by the absolute cone in the point-at-infinity, functioning as '*Kugelkegel*'.

might well be named the *Kugelkegel*, by analogy to the *Kugelkreis* in the Euclidean plane-at-infinity ω.)

In its Euclidean aspect the negative sphere will be a spheroid or elliptic paraboloid or hyperboloid of rotation, with O as focus and with the median plane, say γ (polar to O) as directrix-plane. We shall assume γ to be a horizontal plane, beneath O. The spatial forms are easily imagined by rotating figures 4, 5, etc.—taking the plane of the paper henceforward to be vertical—about the vertical axis through O (OM in figure 5). The median line c becomes the horizontal median plane γ. The 'commedial' family of negative spheres appears as a family of spheroids, etc. with common focus O and directrix-plane γ.

The ∞^2 lines of the median plane act as '*diametral lines*', analogous to the ∞^2 diameters—lines through the central point—of a positive-Euclidean sphere. Every diametral line, say d, carries an opposite pair of tangent planes to the negative sphere, say λ_1, λ_2, such that the negative distance $\gamma\lambda_1 = \gamma\lambda_2 = r$ remains constant while d moves freely throughout γ. Corresponding to the constant *outward radial distance* of all the points of a positive sphere from its centre, we have the picture of a constant *inward tangential sweep* of all the planes of the negative sphere as from its median plane.

To form the sphere from this idea we have recourse once more to the construction of *satellite lines*, though these will need renewed definition. To this end, figure 2 (imagined vertical) may be taken as the cross-section of a three-dimensional space, at once positive and negative, with the point-at-infinity of the negative space at O. The broken circle is to represent the common unit sphere of the two spaces, say v^2 (cf. § 1). Let P be the cross-section of a line d, perpendicular to the plane of the paper, bearing two planes λ_1, λ_2, shown in cross-section as the lines l_1, l_2. p will then be the polar line of d; L_1, L_2 the poles of λ_1, λ_2 with respect to v^2. The positive or point-to-point and negative or plane-to-plane distances are thus equated; $L_1L_2 = \lambda_1\lambda_2$. We now rotate p through 90° about an axis through O, parallel to d, i.e. perpendicular to p and to the plane of the paper. Rotating in either of two opposite directions, we obtain the lines p', p'', which we define to be the *principal satellites* of the line d. They obviously meet the planes λ_1, λ_2 in points K_1, K_2 on either side, such that, as in § 3, $K_1K_2 = L_1L_2 = \lambda_1\lambda_2$. The negative distance of any two planes of d can be measured as the positive distance of the points in perspective with them along either of its two 'principal satellites'.

11.

The satellites can now be used for the construction of the negative sphere directly from the idea of constant inward distance of all its tangent planes from the median plane. Before proceeding, we should, however, note two essential differences in the 'satellite' conception, as against the simpler two-dimensional case. The differences are these: (1) The principal satellites of all the lines of space can no longer be obtained by a single pair of correlations. For the rotation leading from p to p' or p'' requires a specific axis through O, the direction of which is determined by that of the original line d. (To speak of correlations at all, the ∞^4 lines of space would require ∞^2 distinct pairs of correlations, corresponding to the ∞^2 sheaves of parallel lines d.) (2) The two *principal* satellites are no longer the only ones, if by 'satellite of d' we mean *any* line such that the negative distance from plane to plane of d equals the positive distance of the points in which these planes meet the line.

In effect, it is obvious that either principal satellite (p' or p'' in figure 2, seen in its three-dimensional aspect) can be moved continuously into ∞^2 other positions, such that the distance K_1K_2 of the points in which it meets the planes λ_1, λ_2 remains unaltered. To begin with, p' can be moved into ∞^1 positions parallel to itself in a plane through p', parallel to d. Secondly, if moved into another parallel plane, nearer to d, it need only be turned through the appropriate angle to keep the distance K_1K_2 still unaltered. Within this nearer plane, it can then again be moved into ∞^1 parallel positions. We thus obtain a special type of line-congruence, the ∞^2 real lines of which are all contained in a set of planes parallel to d, between (and including) the two outermost planes in which the principal satellites p', p'' (figure 2) are situated. It is the *satellite congruence* of the line d with respect to the chosen unit sphere. Conversely and reciprocally, we can determine the ∞^2 real lines for which a given line, say s, will serve as satellite. These and kindred problems lead to a number of interesting examples in the theory of line-congruences, which may be worth investigating.

12.

Returning now to the construction of the negative sphere, only the two *principal* satellites of each diametral line (§ 10) will be required. Figure 3 may now be taken as the cross-section of the spatial figure, with the ellipse (negative circle) representing the negative sphere and the line c, once again, the median plane γ. Let the point D represent a diametral line d, perpendicular to the plane of the paper. d', d'' are then the principal satellites of d, along each of which the constant radius, $K'D_1' =$ etc. $= r$, is measured off in the two opposite directions. $dD_1' = dD_2''$ and $dD_1'' = dD_2'$ are then the tangent planes borne by d. Clearly, if d moves parallel to itself in the plane γ, d', d'' will rotate as before about the fixed points N', N'' (cf. figure 4).

Assigning different directions to d within γ, it is easy to recognize that the principal satellites of all the lines of γ form a line-congruence consisting of all the lines joining the points of the vertical axis through O to the points of an Euclidean circle in the horizontal plane γ_0 (shown in cross-section as c_0, figure 3), namely the circle with O as centre and N', N'' as a pair of diametrically opposite points.

To express this analytically, let X, Y, Z, W be homogeneous Cartesian point-coordinates with O as origin and $W = 0$ as the plane-at-infinity ω. Making the Z-axis vertical, $Z = 0$ is the plane γ_0, and $mZ + W = 0$, or $W:Z = -m$, the plane γ, m being now the negative-Euclidean distance of the median plane inward from ω. The congruence of principal satellites of the diametral lines is then composed of all the lines meeting the vertical axis $X = Y = 0$ and the horizontal circle $Z = 0$, $X^2 + Y^2 = m^2 W^2$. Marking off the constant Euclidean distance r along all these satellites, in either direction as from their common points with γ, we obtain an octic surface, such as results for example by the rotation of the symmetrical pair of conchoids in figure 4 about the vertical axis through O. Every diametral line d in γ is associated with four points of this surface, forming a parallelogram in a vertical plane through O as in figure 3, and the two planes, joining d to the diagonal pairs of these points, are the tangent planes from d to the required sphere. The three-dimensional construction is thus completed.

As will be readily confirmed, the equation of the conchoidal octic surface is as follows:

$$\left[(X^2+Y^2+Z^2+m^2W^2)\left(Z+\frac{W}{m}\right)^2-r^2Z^2W^2\right]^2 = 4\,m^2W^2(X^2+Y^2)\left(Z+\frac{W}{m}\right)^4. \qquad (14)$$

For the plane $Y=0$, this leads back (substituting Z, W for Y, Z respectively) to equation (13) for the symmetrical pair of conchoids as in figure 4. As in figure 6, § 9, a family of surfaces of this type will envelop each negative sphere. Inserting $Z=0$ in (14), we obtain $W=0$ counting quadruply and the nodal circle $X^2+Y^2-m^2W^2=0$ counting doubly; this circle is of course an isolated feature if the plane conchoid sections have isolated points for nodes.

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Kleine Mitteilungen

Drei neue Näherungskonstruktionen für die Quadratur des Kreises

In dieser Zeitschrift[1]) wurden kürzlich mehrere Verfahren mitgeteilt, um zu einem gegebenen Kreis mit Zirkel und Lineal ein nahezu flächengleiches Quadrat zu konstruieren oder, was dasselbe ist, Näherungswerte für

$$\sqrt{\pi}=1{,}772\,453\,85\ldots$$

elementargeometrisch zu realisieren. Wir vermehren diese Möglichkeiten hier um drei einfache und recht genaue Konstruktionen, deren Berechnung mit Hilfe der Satzgruppe des Pythagoras keinerlei Schwierigkeiten bietet.

Wir bezeichnen durchwegs mit O und r Mittelpunkt und Radius des gegebenen Kreises, mit A und B Eckpunkte des Näherungsquadrates.

a) Der Wert

$$\sqrt{17-8\sqrt{3}}=\sqrt{3{,}143\,59\ldots}=1{,}773\,01\ldots$$

ist zwar nur um wenig genauer als die oben erwähnten Näherungen. Hingegen erlaubt er eine bemerkenswert einfache Konstruktion (Figur 1). Mit dem Radius r schlagen wir aus einem beliebigen Peripheriepunkt C des gegebenen Kreises den Bogen ODE, aus dem Schnittpunkt D den Bogen OCE. Wird die Strecke EO über O hinaus um $OF=2r$ verlängert, so schneidet der Kreis mit Zentrum F und Radius $3r$ aus der Geraden CD eine Sehne der Länge $AB=r\sqrt{17-8\sqrt{3}}$, also die gesuchte Quadratseite. Man beachte nämlich, dass die Hälfte dieser Sehne Kathete in einem rechtwinkligen Dreieck mit der Hypotenuse $FA=3r$ und der zweiten Kathete $2r+(r\sqrt{3})/2$ ist.

[1]) E. VOELLMY, *Die Quadratur des Kreises in Näherungskonstruktionen*, El. Math. *5*, 12–15 (1950). A. ZINNIKER, *Zwei neue Näherungskonstruktionen der Kreisquadratur mit Zirkel und Lineal*, El. Math. *6*, 112–113 (1951). Neben den in diesen Mitteilungen zu findenden weitern Literaturvermerken sei u. a. auf TH. VAHLEN, *Konstruktionen und Approximationen* (Teubner, Leipzig 1911), sowie auf ausserordentlich scharfe Näherungen von S. RAMANUJAN, *Modular Equations and Approximations to π*, Quart. J. *45*, 350–372 (1914) hingewiesen.