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Short note

Inequalities of Levin–Stečkin, Clausing and Chebyshev revisited

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Abstract. We prove the Levin–Stečkin inequality using Chebyshev's inequality and symmetrization. Symmetry and a slightly modified Chebyshev inequality are also the key to an elementary proof of Clausing's inequality.

1 Introduction

It seems that the Levin–Stečkin inequality appeared first in an appendix to the Russian edition of the famous Hardy, Littlewood and Pólya Bible on inequalities [3]. The translator (Levin) enumerates the appendices written by Stečkin, by himself and by both of them. The inequality we consider here comes from Appendix I written by Stečkin. But the English version of the appendix [4] did probably not make this distinction clear enough, so most of the inequalities from that Appendix cited in the literature are called Levin–Stečkin. The one we deal in this paper reads as follows.

Theorem 1.1 (Levin–Stečkin's inequality). *If a function* $p\colon [0,1] \to \mathbb{R}$ *satisfies the conditions*

(1) p *is non-decreasing in* $[0, 1/2]$,

(2) p *is symmetric, i.e.,* $p(x) = p(1-x)$,

then for every convex function φ *the following inequality holds*

$$\int_0^1 p(x)\varphi(x)dx \le \int_0^1 p(x)dx \int_0^1 \varphi(x)dx. \tag{1}$$

The original proof is elementary, but quite complicated. Recently Mercer ([5]) published a proof that uses the notion of extremal points of the set of concave positive functions satisfying $\int_0^1 f(x)dx \le 1$. His method, not very elementary, has an advantage: it provides a simple proof of the Clausing inequality.

Theorem 1.2 (Clausing's inequality [2]). *Let p be a nonnegative function on* $[0, 1]$ *satisfying the following conditions:*

- *p is non-decreasing on* $[0, 1/2]$,
- *p is symmetric.*

Then for every concave, positive function φ the inequality

$$\int_0^1 p(x)\varphi(x)dx \leq \int_0^1 \varphi(x)dx \int_0^1 4\min\{x, 1-x\}p(x)dx \tag{2}$$

holds.

Both inequalities make the reader think of the inequality of Chebyshev, linking the mean of the product of functions with the product of their mean values.

Theorem 1.3 (Chebyshev's inequality). *If the functions $f, g : [a, b] \to \mathbb{R}$ are monotone in the same direction, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

The inequality is reversed if the monotonicities are opposite.

Our aim is to give elementary proofs of Levin–Stečkin's and Clausing's inequalities. The proofs we offer here are sponsored by the word *symmetrization*.

2 The Levin–Stečkin inequality

We prove this inequality in two steps: firstly we show that Theorem 1.1 is valid for symmetric convex functions:

Lemma 2.1. *Under the assumptions of Theorem* 1.1 *if φ is symmetric and convex, then the inequality* (1) *holds.*

Proof. A symmetric convex function is non-increasing in the interval $[0, 1/2]$, thus by Chebyshev's inequality we get

$$\int_0^1 p(x)dx \int_0^1 \varphi(x)dx = \left(\int_0^{1/2} p(x)dx + \int_{1/2}^1 p(x)dx\right)\left(\int_0^{1/2} \varphi(x)dx + \int_{1/2}^1 \varphi(x)dx\right)$$
$$= 4\int_0^{1/2} p(x)dx \int_0^{1/2} \varphi(x)dx \geq 2\int_0^{1/2} p(x)\varphi(x)dx = \int_0^1 p(x)\varphi(x)dx$$

and we are done. □

We shall consider now an arbitrary φ, but the symmetry keeps playing the main role.

Proof of the Levin–Stečkin inequality. Note that for convex φ the function $\frac{\varphi(x)+\varphi(1-x)}{2}$ is convex and symmetric, so we can use Lemma 2.1

$$\begin{aligned}\int_0^1 p(x)\varphi(x)dx &= \frac{\int_0^1 p(x)\varphi(x)dx + \int_0^1 p(1-x)\varphi(1-x)dx}{2}\\ &= \int_0^1 p(x)\frac{\varphi(x)+\varphi(1-x)}{2}dx\\ &\leq \int_0^1 p(x)dx \int_0^1 \frac{\varphi(x)+\varphi(1-x)}{2}dx = \int_0^1 p(x)dx \int_0^1 \varphi(x)dx.\end{aligned}$$

This concludes the proof of the Levin–Stečkin inequality. □

Note. The above theorem is valid for a much broader class of functions φ. In fact φ may be a function that is v-shaped (i.e., decreases on the left part of the interval and increases on the right) and its symmetrization is v-shaped as well. The proof is essentially the same. We leave the details to the reader.

3 Chebyshev's inequality

To prove the Clausing inequality we need a slightly stronger version of Chebyshev's inequality, where the monotonicity of one function is replaced by a weaker condition. Note that this result is somewhat similar to the result of Brunn [1].

Definition 1. We shall say that a function f belongs to the class $M^+(a,b)$ if it is Riemann integrable in $[a,b]$ and there is a $c \in [a,b]$ such that for all $a \leq x \leq b$ the following inequality holds

$$\left(f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right)(x-c) \geq 0. \tag{3}$$

We say that f belongs to $M^-(a,b)$ if the inequality in (3) is opposite.

Obviously every non-decreasing function belongs to the class M^+ (take

$$c = \sup\left\{t : f(t) < \tfrac{1}{b-a}\textstyle\int_a^b f(t)dt\right\})$$

and the non-increasing functions belong to M^-, but these classes are much broader (e.g., $\sin \in M^-(0, 2\pi)$).

Theorem 3.1. *If $g : [a,b] \to \mathbb{R}$ is non-decreasing and $f \in M^+(a,b)$ or g is non-increasing and $f \in M^-(a,b)$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \frac{1}{b-a}\int_a^b g(x)dx.$$

Exchanging M^+ and M^- toggles the inequality.

Proof. Let $f \in M^+$ and g be non-decreasing (the proof in other cases is similar). Denote $f^* = \frac{1}{b-a}\int_a^b f(x)dx$ and let c be the point from Definition 1. Then

$$\begin{aligned}\int_a^b \left(f(x)-f^*\right) g(x)dx &= \int_a^b \left(f(x)-f^*\right)(g(x)-g(c))dx \\ &= \int_a^b \left(f(x)-f^*\right)(x-c) \times \frac{g(x)-g(c)}{x-c}dx \geq 0,\end{aligned}$$

which is equivalent to the required inequality. □

4 Clausing's inequality

Now we have all the tools needed to present an elementary proof of a generalization of the Clausing inequality.

Theorem 4.1. *Let p, q be nonnegative functions on* $[0, 1]$ *satisfying the following conditions:*

- *p and q are symmetric (i.e., $p(x) = p(1-x)$),*
- *p is increasing on* $[0, 1/2]$,
- *q is convex on* $[0, 1/2]$,
- *$q(0) = 0$ and $\int_0^1 q(x)dx = 1$.*

Then for every concave function φ with $\varphi(0) + \varphi(1) \geq 0$ the inequality

$$\int_0^1 p(x)\varphi(x)dx \leq \int_0^1 \varphi(x)dx \int_0^1 p(x)q(x)dx \tag{4}$$

holds.

Proof. Let $\int_0^1 \varphi(x)dx = K$. The Hermite–Hadamard inequality yields $K \geq 0$.
Assume first that φ is symmetric. The inequality (4) can be rewritten as

$$0 \leq \int_0^{1/2} [Kq(x) - \varphi(x)]\, p(x)dx. \tag{5}$$

Let us investigate the function $u(x) = Kq(x) - \varphi(x)$ on the interval $[0, 1/2]$. Symmetry of φ implies $\varphi(0) \geq 0$, thus u is convex, $u(0) \leq 0$ and $\int_0^{1/2} u(x)dx = 0$. Therefore it belongs to the class M^+, and by Theorem 3.1 $\int_0^{1/2}[Kq(x) - \varphi(x)]p(x)dx \geq 0$ which proves (5).
Now let φ be arbitrary. We have

$$\begin{aligned}\int_0^1 p(x)\varphi(x)dx &= \int_0^1 p(x)\frac{\varphi(x)+\varphi(1-x)}{2}dx \\ &\leq \int_0^1 \frac{\varphi(x)+\varphi(1-x)}{2}dx \int_0^1 p(x)q(x)dx \quad \text{(by (4))} \\ &= \int_0^1 \varphi(x)dx \int_0^1 p(x)q(x)dx\end{aligned}$$

which completes the proof. □

The function $q_0(x) = 4\min\{x, 1-x\}$ is a borderline between admissible qs and sample functions φ. Setting $\varphi = q_0$ in (4) we obtain

$$\int_0^1 p(x)q_0(x)dx \le \int_0^1 p(x)q(x)dx$$

which means that q_0 provides the best bound in (4).

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