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Logical equivalence of the fundamental theorems on operators between Banach spaces

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The following theorems belong to the core topics of a first course in functional analysis (where an operator always means a continuous linear mapping).

- **Banach–Steinhaus Theorem (BST):** *Let X be a Banach space, Y a normed space and $\mathfrak{M}$ a family of operators $T : X \to Y$. If $\mathfrak{M}$ is pointwise bounded (i.e., if* $\sup_{T\in\mathfrak{M}} \|Tx\| <$

Die Hauptsätze über Operatoren zwischen Banachräumen gehören zum Kernbestand einer Einführung in die Funktionalanalysis; sie werden zumeist durch Anwendung des Satzes von Baire bewiesen. Der vorliegende Artikel, der unmittelbar aus den Erfahrungen einer Vorlesung und eines Seminars über Funktionalanalysis resultiert, zeigt eine andere – in der gängigen Lehrbuchliteratur nicht beachtete – Möglichkeit der Herleitung auf. Der Satz von Banach–Steinhaus, der mit gänzlich elementaren Methoden bewiesen werden kann, erweist sich nämlich (ebenfalls in elementarer Weise) als logisch äquivalent zu den anderen Hauptsätzen (Satz von der offenen Abbildung, Satz von der Umkehrabbildung, Satz vom abgeschlossenen Graphen). Im Artikel werden ferner einige didaktische Fragen im Zusammenhang mit der Vermittlung der genannten Sätze angesprochen, was vielleicht nützliche Anregungen für die Gestaltung von Lehrveranstaltungen im Bereich der Funktionalanalysis liefert.

∞ *for each fixed* $x \in X$*), then* $\mathfrak{M}$ *is uniformly bounded (i.e.,* $\sup_{T\in\mathfrak{M}} \|T\|_{\mathrm{op}} < \infty$*, where* $\|\cdot\|_{\mathrm{op}}$ *denotes the operator norm).*

- **Open Mapping Theorem (OMT):** *Let* $T : X \to Y$ *be an operator between Banach spaces. If* T *is surjective, then* T *is open.*
- **Inverse Mapping Theorem (IMT):** *Let* $T : X \to Y$ *be an operator between Banach spaces. If* T *is bijective, then* T^{-1} *is continuous.*
- **Closed Graph Theorem (CGT):** *Let* $T : X \to Y$ *be a linear mapping between Banach spaces. If the graph* $\{(x, Tx) \mid x \in X\}$ *of* T *is closed as a subspace of* $X \times Y$*, then* T *is continuous.*

In the summer semester 2018, the second author taught a functional analysis class which was attended (or rather suffered through?) by the first author. The approach taken in this class was as follows: (BST) was derived in two ways, using both the elementary proof found in [5] and the standard proof based on Baire's theorem which can be found in nearly all functional analysis textbooks (see, for example, [3], [9] or [10]); then (OMT) was derived, again using the standard proof based on Baire's theorem (loc. cit.), and finally the implications (OMT) $\Rightarrow$ (IMT) $\Rightarrow$ (CGT) were proved, thereby establishing all four theorems. Many of the students (including the first author) did not like the application of Baire's theorem, which seemed abstract and unintuitive, and since (BST) could be derived without invoking Baire's theorem, the students asked whether or not this would also be possible for the other theorems. Moreover, the question arose which of the four theorems could be derived from which of the other ones.

One year later, in the summer semester 2019, the second author offered a seminar on selected topics in functional analysis, which was attended by a relatively large portion of the previous audience (including the first author), and this offered a perfect opportunity to explore the above questions (which formed the first author's seminar assignment) in a student-teacher cooperation which included a mixture of literature and internet research. The present paper is the outcome of this cooperation. Its modest purpose is two-fold: to present the final answers (which were scattered in the literature) in an organized and succinct way which may serve as a help to others in preparing functional analysis classes, and to remark on some of the didactical issues which turned up in the process.

It is clear that (BST), dealing with a family of operators, differs in character from the other three theorems, which deal with a single operator. Thus it is natural to first concentrate on these three theorems which, as is well known, are all equivalent (see [2]). It seemed didactically worthwhile to not just prove the three necessary implications to establish the equivalence of these theorems, but to derive an explicit proof for each of the six possible implications, which gave a good opportunity to review the pertinent facts on quotient spaces (a concept not too well liked by most students), i.e., the fact that if U is a closed subspace of a Banach space X then X/U is itself a Banach space with the quotient norm $\|[x]\| := \inf_{u\in U} \|x - u\|$ and that the quotient map $\pi : X \to X/U$, which assigns to each element $x \in X$ its coset $[x] = x + U$, is both open and continuous.

For completeness' sake, let us give quick proofs for all implications.

Theorem. (OMT) $\Leftrightarrow$ (IMT).

Proof. $\Rightarrow$ Assume that $T : X \to Y$ is a bijective operator. Then T is surjective, hence open due to (OMT), and the openness of T is equivalent to the continuity of T^{-1}.

$\Leftarrow$ Let $T : X \to Y$ be a surjective operator. Then the mapping $\widehat{T} : X/\ker(T) \to Y$ given by $\widehat{T}([x]) := Tx$ is a well-defined bijective operator. Then $\widehat{T}^{-1}$ is continuous due to (IMT), so that $\widehat{T}$ is open. But then $T = \widehat{T} \circ \pi$ is open (as a composition of open mappings), where $\pi : X \to X/\ker(T)$ denotes the quotient map. $\square$

Theorem. (IMT) $\Leftrightarrow$ (CGT).

Proof. $\Rightarrow$ Assume that $T : X \to Y$ is a linear mapping whose graph $\Gamma \subseteq X \times Y$ is closed and hence is a Banach space with the norm induced by the product norm on $X \times Y$. Consider the projection operators

$$p_1 : \begin{array}{ccc} \Gamma & \to & X \\ (x, Tx) & \mapsto & x \end{array} \qquad \text{and} \qquad p_2 : \begin{array}{ccc} \Gamma & \to & Y \\ (x, Tx) & \mapsto & Tx \end{array}$$

and note that p_1 is bijective. Then p_1^{-1} is continuous by (IMT). But then $T = p_2 \circ p_1^{-1}$ is also continuous, being the composition of continuous mappings.

$\Leftarrow$ Let $T : X \to Y$ be a bijective operator and let Γ be the graph of T. Then T can be written as the composition

$$\begin{array}{ccccc} X & \xrightarrow{i} & \Gamma & \xrightarrow{p_2} & Y \\ x & \mapsto & (x, Tx) & \mapsto & Tx \end{array}$$

of the embedding i, which is a homeomorphism, and the projection p_2, which is open. Then $T = p_2 \circ i$ is open, being the composition of open mappings, and this means that T^{-1} is continuous. $\square$

Theorem. (OMT) $\Leftrightarrow$ (CGT).

Proof. $\Rightarrow$ Let $\Gamma \subseteq X \times Y$ be the graph of $T : X \to Y$; we assume that Γ is closed. Then the projection $p_1 : \Gamma \to X$ is a bijective operator between Banach spaces, hence is open due to (OMT) and thus is even a homeomorphism; in particular, p_1^{-1} is continuous. The projection $p_2 : \Gamma \to Y$ is also continuous. But then so is $T = p_2 \circ p_1^{-1}$, being the composition of continuous mappings.

$\Leftarrow$ Let $T : X \to Y$ be a surjective operator. Then

$$S : \begin{array}{ccc} Y & \to & X/\ker(T) \\ y & \mapsto & [x] \text{ where } Tx = y \end{array}$$

is a well-defined linear mapping. We want to show that the graph of S is closed. To do so, assume that $y_n \to y$ in Y and $Sy_n \to [x]$ in $X/\ker(T)$; we need to show that $[x] = Sy$. Since T is surjective, we can find, for each $n \in \mathbb{N}$, an element $x_n \in X$ such that $y_n = Tx_n$ and hence $Sy_n = [x_n]$. The condition $Sy_n \to [x]$ means that $[x_n] \to [x]$, i.e., $[x_n - x] \to [0]$; hence by the definition of the quotient norm there are elements $\xi_n \in \ker(T)$ such that $\|x_n - x - \xi_n\| \to 0$, hence $x_n - \xi_n \to x$ and therefore $y_n = Tx_n = T(x_n - \xi_n) \to Tx$.

Since also $y_n \to y$, we conclude that $y = Tx$ and thus $Sy = [x]$. This concludes the proof that the graph of S is closed. Since (CGT) holds by assumption, we conclude that S is continuous, so that there is a constant $C > 0$ such that $\|Sy\| \leq C\|y\|$ for all $y \in Y$. Pick an arbitrary element $y \in Y$ with $\|y\| < 1/C$; then $\|Sy\| < 1$, and by definition of the quotient norm there is an element $\xi \in \ker(T)$ with $y = Tx$ and $\|x - \xi\| < 1$. Then $y = Tx = T(x - \xi) \in T\big(B_1^{(X)}(0)\big)$. Since y was arbitrary, we have established the inclusion $B_{1/C}^{(Y)}(0) \subseteq T\big(B_1^{(X)}(0)\big)$, which implies that T is open. □

This leaves us with the question which role (BST) plays in relation to the three theorems considered so far. It seems to be a rather widely held view that (BST) is somewhat more "elementary" than these other theorems and that it is not possible to derive them from (BST), at least not in general (but possibly in a more restricted context like that of Hilbert spaces); see the discussion and references in [2]. It turns out that this view is not correct. In fact, all four theorems are equivalent, i.e., can be derived from each other, as will be explained below. Due to the different character of (BST), it does seem neither natural nor worthwhile to prove each of the possible implications directly. Therefore, we content ourselves with proving the implication (CGT) ⇒ (BST) and (BST) ⇒ (OMT). Once this is done, we know that all four theorems are logically equivalent, and since (BST) can be established in an elementary way, it is clear that all four theorems can be derived without invoking Baire's theorem. (A deeper study leads to the obscurities of ultrabarreled spaces; see (27.26) in [4].) We start by deriving the Banach–Steinhaus Theorem from the Closed Graph Theorem.

Theorem. (CGT) ⇒ (BST).

Proof. Let $\mathfrak{M}$ be a pointwise bounded family of operators $T : X \to Y$ between Banach spaces. Then, for any fixed $x \in X$, the evaluation map

$$\varphi_x : \begin{array}{ccc} \mathfrak{M} & \to & Y \\ T & \mapsto & Tx \end{array}$$

is bounded and hence is an element of the Banach space B of all bounded functions $\varphi : \mathfrak{M} \to Y$, equipped with the norm $\|\varphi\|_\infty := \sup_{T \in \mathfrak{M}} \|\varphi(T)\|_Y$. Thus the linear mapping

$$\Phi : \begin{array}{ccc} X & \to & B \\ x & \mapsto & \varphi_x \end{array}$$

can be introduced.[1] We want to show that the graph of Φ is closed. To do so, we assume that $x_n \to x$ in X and $\varphi_{x_n} \to g$ in B; we then have to show that $g = \varphi_x$. Since $\varphi_{x_n} \to g$, we have

$$\sup_{T \in \mathfrak{M}} \|g(T) - Tx_n\|_Y = \sup_{T \in \mathfrak{M}} \|g(T) - \varphi_{x_n}(T)\|_Y = \|g - \varphi_{x_n}\|_\infty \to 0. \qquad (\star)$$

Now fix $T \in \mathfrak{M}$. Then $(\star)$ implies that $Tx_n \to g(T)$. On the other hand, since $x_n \to x$ and since T is continuous, we also have $Tx_n \to Tx = \varphi_x(T)$. Therefore, $g(T) = \varphi_x(T)$.

[1] If $\mathfrak{M} = \{T_1, T_2, T_3, \ldots\}$ is countable, we can identify B with $\ell^\infty(Y)$ and Φ with the mapping $x \mapsto (T_1x,\ T_2x,\ T_3x,\ \ldots)$, which seemed reassuring to some of the students.

Since T was arbitrary, this implies that $g = \varphi_x$, which is what we had to show. Thus the graph of Φ is closed. Since the validity of (CGT) was assumed, this implies that Φ is continuous. Thus there is a constant $C > 0$ such that

$$C\|x\| \geq \|\Phi(x)\|_\infty = \|\varphi_x\|_\infty = \sup_{T\in\mathfrak{M}} \|\varphi_x(T)\|_Y = \sup_{T\in\mathfrak{M}} \|Tx\|_Y$$

for all $x \in X$ and hence $\|T\|_{\text{op}} \leq C$ for all $T \in \mathfrak{M}$. □

The next theorem establishes the most interesting implication and shows that the supposedly weaker theorem (BST) actually implies the other three theorems. The proof is a slightly expanded version of the one found in [1].

Theorem. (BST) $\Rightarrow$ (OMT).

Proof. Let $T : X \to Y$ be a surjective operator. Denote by $\|\cdot\|_X$ the norm on X, by $\|\cdot\|_Y$ the norm on Y and by B_r and B'_r the open balls of radius r centered at the origin in X and Y, respectively. We show first that there is a number $\delta > 0$ such that

$$B'_\delta \subseteq \overline{T(B_1)}\,. \tag{1}$$

To do so, we introduce for each $n \in \mathbb{N}$ a new norm $\|\cdot\|_n$ on Y via

$$\|y\|_n := \inf\{\|u\|_X + n\|v\|_Y \mid u \in X,\ v \in Y,\ y = v + Tu\}.$$

Since each element $y \in Y$ can be written as $y = v + Tu$ where $v := y$ and $u := 0$, we have $\|y\|_n \leq n\|y\|_Y$ for all $n \in \mathbb{N}$ and all $y \in Y$. We now consider the vector space Z of all sequences $(y_1, y_2, y_3, \ldots)$ with only a finite number of nonzero entries, equipped with the norm

$$\|(y_1, y_2, y_3, \ldots)\|_Z := \sup_{n\in\mathbb{N}} \|y_n\|_n,$$

and consider for each $n \in \mathbb{N}$ the linear mapping $S_n : Y \to Z$ given by

$$S_n(y) := (0, \ldots, 0, y, 0, 0, \ldots)$$

with the element y in the nth position. From $\|S_n(y)\|_Z = \|y\|_n \leq n\|y\|_Y$ we conclude that S_n is continuous with $\|S_n\|_{\text{op}} \leq n$. Moreover, given an arbitrary element $y \in Y$, we can find an element $x \in X$ such that $y = Tx$, because T is surjective by hypothesis. Then, due to the decomposition $y = v + Tu$ where $v = 0$ and $u = x$, we find that

$$\|S_n(y)\|_Z = \|y\|_n \leq \|x\|_X$$

independently of n. Thus $\{S_n \mid n \in \mathbb{N}\}$ is a pointwise bounded family of operators. Invoking (BST), we see that there is a constant $C > 0$ such that $\|S_n\|_{\text{op}} \leq C$ for all $n \in \mathbb{N}$. We now claim that condition (1) is satisfied whenever $\delta < 1/C$. In fact, given $y \in B'_\delta$ so that $\|y\|_Y < \delta$, we have $\|y\|_n = \|S_n(y)\|_Z \leq \|S_n\|_{\text{op}}\|y\|_Y \leq C\|y\|_Y < C\delta < 1$. The definition of the norm $\|\cdot\|_n$ then implies that there are elements $u_n \in X$ and $v_n \in Y$ such that $y = v_n + Tu_n$ and $\|u_n\|_X + n\|v_n\| < 1$, thus $\|u_n\|_X < 1$ (i.e., $u_n \in B_1$) and $n\|v_n\|_Y < 1$, therefore $\|v_n\|_Y < 1/n$ and consequently $v_n \to 0$, which implies $Tu_n \to y$ and hence $y \in \overline{T(B_1)}$. This proves (1). Scaling immediately shows that (1) implies the condition

$$B'_{\delta/2^n} \subseteq \overline{T(B_{1/2^n})} \tag{2}$$

for all $n \in \mathbb{N}$. We claim that

$$B'_{\delta/2} \subseteq T(B_1). \tag{3}$$

To prove this claim, let $y \in B'_{\delta/2}$ so that $\|y\|_Y < \delta/2$. Since $y \in \overline{T(B_{1/2})}$, there is an element $x_1 \in B_{1/2}$ such that $\|y - Tx_1\|_Y < \delta/4$. Since $y - Tx_1 \in B'_{\delta/4} \subseteq \overline{T(B_{1/4})}$, there is an element $x_2 \in B_{1/4}$ such that $\|y - Tx_1 - Tx_2\|_Y < \delta/8$. Continuing in this fashion, we find elements $x_k \in B_{1/2^k}$ such that $\|y - \sum_{k=1}^n Tx_k\|_Y < \delta/2^{n+1}$ for all n. Since $\sum_{k=1}^\infty \|x_k\|_X < \sum_{k=1}^\infty (1/2)^k = 1$ and since X is complete, the series $\sum_{k=1}^\infty x_k$ converges absolutely towards an element $x \in X$ for which $\|x\|_X = \|\sum_{k=1}^\infty x_k\|_X \le \sum_{k=1}^\infty \|x_k\|_X < 1$ so that $x \in B_1$. Moreover, we have $Tx = T(\sum_{k=1}^\infty x_k) = \sum_{k=1}^\infty Tx_k = y$. Thus $y = Tx$ lies in $T(B_1)$. This proves (3), and (3) clearly implies that T is an open mapping. □

All the questions which motivated this paper are now satisfactorily answered. We conclude with three remarks concerning aspects which played a role in both the functional analysis class and the subsequent seminar.

Remark 1: Converses of the theorems. It is worthwhile noting that the converses of both (CGT) and (OMT) hold trivially. For (CGT) this is immediately clear, because if $T : X \to Y$ is continuous then $x_n \to x$ implies $Tx_n \to Tx$ and hence $(x_n, Tx_n) \to (x, Tx)$, which shows that the graph of T is closed. This argument requires neither the completeness of X nor that of Y, hence works for operators between arbitrary normed spaces (not necessarily Banach spaces). Similarly, the converse of (OMT) holds for arbitrary linear mappings (continuous or not) between arbitrary normed spaces (complete or not).

Theorem. *Let $T : X \to Y$ be a linear mapping between normed spaces. If T is open then T is surjective.*

Proof. Since T is open, there is a number $\delta > 0$ such that $B'_\delta \subseteq T(B_1)$. Let $y \in Y$ be an arbitrary element of Y; we must find an element $x \in X$ such that $Tx = y$. If $y = 0$ we can take $x := 0$. Let $y \neq 0$. Choose a number r such that $0 < r < \delta$. Then $ry/\|y\|$ lies in $B'_\delta \subseteq T(B_1)$, which implies that there is an element $\xi \in B_1$ such that $T\xi = ry/\|y\|$. Letting $x := (\|y\|/r)\xi$, we have $Tx = y$. □

Establishing the converses of (CGT) and (OMT) provided a good opportunity to point out the role of completeness and to teach the students to appreciate which statements hold only for Banach spaces and which ones hold for arbitrary normed spaces. For example, (BST) requires completeness in X, but not in Y (and this is crucial in the proof that (BST) implies (OMT), because the space Z used in this proof is *not* complete). It was also important to point out which statements hold not just for linear mappings between normed spaces, but for arbitrary mappings between topological spaces (such as the fact that the openness of a bijection is equivalent to the continuity of its inverse), thereby giving the students a feeling for the right level of generality of mathematical statements.

Remark 2: Intuitive understanding. It is very important for students (and teachers!) to not just understand mathematical theorems in the sense that the proofs can be followed, but to also develop an intuitive understanding of the meaning and significance of these theorems. In this respect, (IMT) can be understood relatively easily: If an operator $T : X \to Y$ between Banach spaces is bijective, then the equation $Tx = y$ has a unique solution x for each given right-hand side y. The fact that (IMT) holds shows that x depends continuously on y, i.e., if y changes only a little then x also changes only a little, a fact which can be easily illustrated for initial value problems and is basic for the development of numerical schemes from functional analytic results. The intuitive meaning of (OMT) is less obvious, but can be stated as follows: The surjectivity of a mapping $T : X \to Y$ can be considered as a purely qualitative existence statement: for each right-hand side y, the equation $Tx = y$ has at least one solution x. Now if T is open, this can be made into a quantitative existence statement by guaranteeing the existence of a solution x together with an estimate for the size of x in terms of the size of the right-hand side.

Theorem. *Let $T : X \to Y$ be a surjective operator between Banach spaces. Then there is a constant C such that for each element $y \in Y$ the equation $Tx = y$ possesses a solution x such that $\|x\| \leq C\|y\|$.*

Proof. Due to (OMT), the mapping T is open; hence there is a number $\delta > 0$ such that $B'_\delta \subseteq T(B_1)$. Letting $K'_\delta := \overline{B'_\delta}$ and $K_1 := \overline{B_1}$, we also have $K'_\delta = \overline{B'_\delta} \subseteq \overline{T(B_1)} \subseteq T(\overline{B_1}) = T(K_1)$. Now let $C := 1/\delta$. If $y \neq 0$ is an arbitrary nonzero element of Y, then $\delta y/\|y\|$ lies in K'_δ; hence there is an element $\xi \in K_1$ such that $\delta y/\|y\| = T\xi$. Then $x := (\|y\|/\delta)\xi$ satisfies $Tx = y$ and $\|x\| \leq \|y\|/\delta = C\|y\|$. If $y = 0$ we can simply choose $x := 0$. □

These ideas are nicely expanded in [6] (in particular Theorem 1.7.12), giving a deeper understanding for the meaning of the Open Mapping Theorem which is not immediately clear from the mere statement of this theorem. See also [7] and [8] for insightful remarks on the Closed Graph Theorem.

Remark 3: Applications. It almost goes without saying that a true understanding of abstract theorems such as the ones discussed in this paper only develops when one sees these theorems "in action" by applying them to concrete problems. Thus the functional analysis class which gave rise to this paper also covered applications such as initial and boundary value problems, numerical integration schemes, Fourier analysis and others. For the students, the blend of abstract theory and concrete applications was one of the main reasons to appreciate the attractivity of functional analysis as a mathematical discipline.

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