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A unifying framework for generalizations of the Eneström–Kakeya theorem

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1 Introduction

The Eneström–Kakeya theorem ([6], [7], and [10]) establishes upper and lower bounds on the moduli of the zeros of a polynomial with positive coefficients that are simple explicit functions of the coefficients. It has been extended and generalized in different ways, a good overview of which can be found in [8], while sharpness of the bounds was considered in [1], [2], and [9]. For additional historical remarks about this pretty theorem we also refer to [16, pp. 271–272].

Here we construct a framework that generates theorems similar to that of Eneström and Kakeya, namely, theorems that derive regions in the complex plane that contain the zeros of polynomials with positive coefficients. These regions, which consist of a single disk or the union of several disks, are explicitly determined, i.e., they do not require numerical methods, and we use the same two basic tools to derive all of them: a family of polynomial

Das klassische Eneström–Kakeya Theorem liefert explizite obere und untere Schranken für die Nullstellen von Polynomen mit positiven Koeffizienten. Der Autor der vorliegenden Arbeit entwickelt einen gemeinsamen Rahmen für diesen Satz und Varianten davon und gelangt auf diese Weise zu einer Theorie, welche deutlich feinere Aussagen erlaubt. So lassen sich explizit Regionen in der komplexen Ebene angeben, welche die Nullstellen enthalten. Diese Regionen sind Kreisscheiben oder die Vereinigung von Kreisscheiben. Die Verallgemeinerung einer Beobachtung von Cauchy und eine Familie von geeigneten Polynom-Multiplikatoren führen dabei zu einem einfachen und transparenten Zugang.

multipliers and a generalization of an observation by Cauchy. This transparent approach is simpler than traditional ones and unifies the derivation of existing and new results.

Although we obtain useful bounds, our main purpose is to show how a classical and elegant theorem is based on another classical result and how generalizations of the latter can be used to improve it in ways that are quite different from existing ones.

The paper is organized as follows. In Section 2 a few results are collected that are needed throughout. In Section 3, we derive zero inclusion regions composed of a single disk centered at the origin, while in Section 4 we obtain disks that are not centered at the origin. In Sections 5 and 6, we derive zero inclusion regions consisting of two and three disks, respectively. Some technical remarks are relegated to an appendix.

2 Preliminaries

We first collect a few theorems and definitions that will be needed further on. Throughout this section, we consider a polynomial $p(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ with complex coefficients, unless specified otherwise.

Definition 2.1. The *Cauchy radius of the kth kind* of p, with $k = 1, \ldots, n$, is defined as the unique positive solution s_k of

$$|a_n|z^n + |a_{n-1}|z^{n-1} + \cdots + |a_{n-k+1}|z^{n-k+1} - |a_{n-k}|z^{n-k} - \cdots - |a_1|z - |a_0| = 0 \; . \quad (1)$$

When $k = 1$, s_1 is simply called the *Cauchy radius* of p.

Definition 2.2. The sets $\Gamma_1(k)$ and $\Gamma_2(k)$ for the polynomial p, with $k = 1, \ldots, n$, are defined as

$$\Gamma_1(k) = \Big\{ z \in \mathbb{C} : |a_n z^k + a_{n-1} z^{k-1} + \cdots + a_{n-k+1} z| \\ \leq |a_n| s_k^k + |a_{n-1}| s_k^{k-1} + \cdots + |a_{n-k+1}| s_k \Big\}$$

and

$$\Gamma_2(k) = \Big\{ z \in \mathbb{C} : |a_n z^k + a_{n-1} z^{k-1} + \cdots + a_{n-k+1} z + a_{n-k}| \\ \leq |a_n| s_{k+1}^k + |a_{n-1}| s_{k+1}^{k-1} + \cdots + |a_{n-k+1}| s_{k+1} + |a_{n-k}| \Big\} \; .$$

The boundaries of $\Gamma_1(k)$ and $\Gamma_2(k)$ are lemniscates. With these definitions, we state the following theorem, which is Theorem 3.2 in [13], where k here corresponds to $n - k$ in that theorem.

Theorem 2.1. *All the zeros of the complex polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ lie in the sets $\Gamma_1(k)$ and $\Gamma_2(k)$. If $\Gamma_1(k)$ or $\Gamma_2(k)$ consists of disjoint regions whose boundaries are simple closed (Jordan) curves and ℓ is the number of foci of its bounding lemniscate that are contained in any such region, then that region contains ℓ zeros of p when it does not contain the origin, and $\ell + n - k$ zeros of p when it does contain the origin.*

The special case $\Gamma_1(1)$ in Theorem 2.1 is the disk defined by $|z| \leq s_1$, where s_1 is the unique positive solution of

$$|a_n|z^n - |a_{n-1}|z^{n-1} - \cdots - |a_1|z - |a_0| = 0\ . \tag{2}$$

This is a classical observation by Cauchy from 1829 ([5], see also, e.g., [11, Th. (27, 1), p. 122 and Exercise 1, p. 126], [15, Theorem 3.1.1], [16, Theorem 8.1.3]).

The special case $\Gamma_2(1)$ in Theorem 2.1 is the disk defined by $|z + a_{n-1}/a_n| \leq s_2 + |a_{n-1}/a_n|$, where s_2 is the unique positive solution of

$$|a_n|z^n + |a_{n-1}|z^{n-1} - |a_{n-2}|z^{n-2} - \cdots - |a_1|z - |a_0| = 0\ .$$

This is Theorem 1 in [12].

As k increases, $\Gamma_1(k)$ and $\Gamma_2(k)$ become too complicated. Instead, we will approximate a region Γ, bounded by a lemniscate of the form $|q(z)| = R$, where $q(z) = z^m + b_{m-1}z^{m-1} + \cdots + b_0$, as follows. Denoting the zeros of q by c_j, we have that

$$|q(z)| = |z^m + b_{m-1}z^{m-1} + \cdots + b_0| = |z - c_1||z - c_2| \ldots |z - c_m|\ ,$$

so that

$$\Gamma = \{z \in \mathbb{C} : |q(z)| \leq R\} = \{z \in \mathbb{C} : |z - c_1||z - c_2| \ldots |z - c_m| \leq R\}\ ,$$

which implies the inclusion

$$\Gamma \subseteq \bigcup_{j=1}^{m} \left\{z \in \mathbb{C} : |z - c_j| \leq R^{1/m}\right\}\ .$$

Although larger than Γ, this union of disks is easier to work with, and is, as we shall see, still useful. It may sometimes be better to allow different disks to have different radii, but in the interest of simplicity we will use the same radius for all disks.

Additional zero inclusion regions can be obtained by applying Theorem 2.1 to the reverse polynomial $p^{\#}(\zeta) = a_0\zeta^n + a_1\zeta^{n-1} + \cdots + a_{n-1}\zeta + a_n$ (with $a_0 \neq 0$), whose zeros are the reciprocals of the zeros of p. Upper bounds on the moduli of the zeros of $p^{\#}$ then lead to lower bounds on the moduli of the zeros of p. We refrain from obtaining such additional regions, as their derivations are straightforward applications of the results for p.

We conclude this section by stating the Eneström–Kakeya theorem ([6], [7], and [10]) with a proof that already contains the ingredients for its generalization.

Theorem 2.2 (Eneström–Kakeya). *All the zeros of the real polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with positive coefficients lie in the disk defined by*

$$\left\{z \in \mathbb{C} : |z| \leq \max_{0 \leq j \leq n-1} \frac{a_j}{a_{j+1}}\right\}.$$

Proof. Consider $(z - \gamma)p(z)$, where $\gamma \in \mathbb{R}$:

$$(z - \gamma)p(z) = a_n z^{n+1} + \sum_{j=1}^{n} (a_{j-1} - \gamma a_j)z^j - \gamma a_0\ . \tag{3}$$

Clearly, any upper bound on the zeros of $(z-\gamma)p(z)$ will also be an upper bound on the zeros of p. From (3) we observe that all the coefficients of $(z-\gamma)p(z)$, except that of z^{n+1}, will be nonpositive if

$$\gamma = \max_{0\le j\le n-1} \frac{a_j}{a_{j+1}} ,$$

while the constant term is negative. For that value of γ, the Cauchy radius of $(z-\gamma)p(z)$ is the unique positive solution of $(z-\gamma)p(z) = 0$ and, since $\gamma > 0$ is a positive zero of $(z-\gamma)p(z)$ and p has no positive zeros, it must be equal to the Cauchy radius. This concludes the proof. □

It is worth pointing out that the Eneström–Kakeya bound is not necessarily better than the Cauchy radius of p, although its obvious advantage is that it is explicit and therefore does not require the solution of a polynomial equation. The following example illustrates the theorem.

Example. Define the polynomials $p_1(z) = 2z^5 + z^4 + 4z^3 + z^2 + 2z + 3$ and $p_2(z) = z^5 + z^4 + 2z^3 + 3z^2 + 2z + 1$. For p_1, whose largest zero has magnitude 1.3921, the Cauchy radius is 1.9242, while the Eneström–Kakeya theorem gives 4.00 as an upper bound, which is worse.

On the other hand, for p_2, where the largest zero has magnitude 1.4013, we obtain 2.4654 and 2.0 for the Cauchy radius and Eneström–Kakeya, respectively. Here, the Eneström–Kakeya bound is better.

There is one clear case where the Eneström–Kakeya theorem is guaranteed to produce a better upper bound than the Cauchy radius, namely, when $a_{n-1}/a_n = \max_{1\le j\le n} a_{j-1}/a_j$. This can be seen by substituting a_{n-1}/a_n in the left-hand side of (2), which yields a negative value, indicating that a_{n-1}/a_n is less than the Cauchy radius. It is also an immediate consequence of Theorem 8.3.1 in [16].

The key ingredients in the proof of Theorem 2.2 are the multiplier $z-\gamma$ and the set $\Gamma_1(1)$, defined by the Cauchy radius, which provides the motivation for generalizing this theorem by changing the multiplier, while using the sets $\Gamma_1(k)$ and $\Gamma_2(k)$ from Theorem 2.1 for $k \ge 1$, leading to inclusion regions consisting of a single disk as well as several disks. To avoid tedious repetition, we will provide one fully worked out theorem for each case, while stating additional results with an explanation of how they are obtained, but without a formal proof.

From here on, we denote by $D(a, r)$ the closed disk with radius r and centered at a.

3 Single disk centered at the origin

In this section we derive two inclusion regions consisting each of a single disk centered at the origin. These two results, one of which is similar to an existing one, are only marginally more complicated than Theorem 2.2, and clearly illustrate how more such inclusion disks can be obtained.

Theorem 3.1. *Let the real polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 3$ have positive coefficients, and let*

$$\gamma_2 = \frac{a_{n-1}}{a_n}\,,\ \gamma_1 = \frac{a_{n-2} - \gamma_2 a_{n-1}}{a_n}\,,$$
$$\gamma_0 = \max\left\{0, \frac{\gamma_1 a_0}{-a_1}, \frac{\gamma_2 a_0 + \gamma_1 a_1}{-a_2}, \max_{0\leq j\leq n-3} \frac{a_j - \gamma_2 a_{j+1} - \gamma_1 a_{j+2}}{a_{j+3}}\right\}\,.$$

Then all the zeros of p lie in the disk $D(0,\mu)$, where μ is the unique positive zero of $z^3 - \gamma_2 z^2 - \gamma_1 z - \gamma_0$.

Proof. Consider $q(z) = (z^3 - \gamma_2 z^2 - \gamma_1 z - \gamma_0)p(z)$, where $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}$:

$$\begin{aligned} q(z) = a_n z^{n+3} &+ (a_{n-1} - \gamma_2 a_n) z^{n+2} + (a_{n-2} - \gamma_2 a_{n-1} - \gamma_1 a_n) z^{n+1} \\ &+ \sum_{j=3}^{n} \left(a_{j-3} - \gamma_2 a_{j-2} - \gamma_1 a_{j-1} - \gamma_0 a_j\right) z^j - (\gamma_2 a_0 + \gamma_1 a_1 + \gamma_0 a_2)\, z^2 \\ &- (\gamma_1 a_0 + \gamma_0 a_1)\, z - \gamma_0 a_0\,. \end{aligned} \tag{4}$$

If we define γ_0, γ_1, and γ_2 as in the statement of the theorem, then the coefficients of the nonleading powers of z in the right-hand side of (4) are all nonpositive, which means, reasoning as in the proof of Theorem 2.2, that the Cauchy radius of q, which is also an upper bound on the moduli of the zeros of p, is the unique positive zero of $z^3 - \gamma_2 z^2 - \gamma_1 z - \gamma_0$. □

The following theorem is obtained similarly, by using a quadratic multiplier of the form $z^2 - \gamma_1 z - \gamma_0$.

Theorem 3.2. *Let the real polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 2$ have positive coefficients and let*

$$\gamma_1 = \frac{a_{n-1}}{a_n} \quad \textit{and} \quad \gamma_0 = \max\left\{0, \max_{0\leq j\leq n-2} \frac{a_j - \gamma_1 a_{j+1}}{a_{j+2}}\right\}\,.$$

Then all the zeros of p lie in the disk $D\left(0, \frac{1}{2}\left(\gamma_1 + \left(\gamma_1^2 + 4\gamma_0\right)^{1/2}\right)\right)$.

Although somewhat dissimilar at first sight, this theorem is essentially Theorem 1 in [3].

Example. We illustrate the theorems in this section with the polynomial $p_3(z) = z^6 + 4z^5 + 2z^4 + 2z^3 + 3z^2 + 6z + 7$. They produce the following upper bounds on the moduli of the zeros:

Cauchy radius:	4.580	Theorem 3.1:	3.788
Eneström–Kakeya theorem:	4.000	Theorem 3.2:	4.000 .

It is, in general, difficult to predict which theorem is preferable. Theorems obtained with higher-order multipliers do not necessarily outperform those obtained with lower-order multipliers, although they frequently do, as is the case here.

4 Single disk not centered at the origin

We now derive additional inclusion regions, consisting, this time, of disks not centered at the origin. We present three theorems, of which one is an existing result, although formulated slightly differently.

Theorem 4.1. *Let the real polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 4$ have positive coefficients, and define*

$$\alpha = \max\left\{\frac{a_{n-2}}{a_n}, \frac{a_{n-3}}{a_{n-1}}\right\}, \quad \beta = \max\left\{0, \max_{0 \leq j \leq n-4} \frac{a_j - \alpha a_{j+2}}{a_{j+4}}\right\},$$
$$\mu = \left(\frac{1}{2}\left(\alpha + \left(\alpha^2 + 4\beta\right)^{1/2}\right)\right)^{1/2}.$$

Then all the zeros of p lie in the closed disk $D(-a_{n-1}/a_n, \mu + |a_{n-1}/a_n|)$.

Proof. Consider $q(z) = (z^4 - \alpha z^2 - \beta)p(z)$:

$$\begin{aligned} q(z) = a_n z^{n+4} &+ a_{n-1} z^{n+3} + (a_{n-2} - \alpha a_n)\, z^{n+2} + (a_{n-3} - \alpha a_{n-1})\, z^{n+1} \\ &+ \sum_{j=4}^{n} \left(a_{j-4} - \alpha a_{j-2} - \beta a_j\right) z^j \\ &- (\alpha a_1 + \beta a_3)\, z^3 - (\alpha a_0 + \beta a_2)\, z^2 - \beta a_1 z - \beta a_0 \,. \end{aligned} \tag{5}$$

We have from equation (5) that all the coefficients of q, except those of z^{n+4} and z^{n+3}, will be nonpositive if α and β are defined as in the statement of the theorem. Then the Cauchy radius of the second kind of q is the unique positive solution of $(z^4 - \alpha z^2 - \beta)p(z) = 0$, which is the unique positive solution μ of $z^4 - \alpha z^2 - \beta$. This quartic is a quadratic in z^2, and its positive zero is given by $\mu = \left(\frac{1}{2}\left(\alpha + \left(\alpha^2 + 4\beta\right)^{1/2}\right)\right)^{1/2}$. Theorem 2.1 with the set $\Gamma_2(1)$ then implies that all the zeros of q, and therefore also all those of p, must lie in the closed disk $D(-a_{n-1}/a_n, \mu + |a_{n-1}/a_n|)$. □

The following theorem is a slightly different version of Corollary 3 of Theorem 3 in [4]. Here, it is obtained with a multiplier of the form $z^2 - \mu^2$ and the set $\Gamma_2(1)$.

Theorem 4.2. *Let the real polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ with $n \geq 2$ have positive coefficients, and let*

$$\mu = \left(\max_{0 \leq j \leq n-2} \frac{a_j}{a_{j+2}}\right)^{1/2}.$$

Then all the zeros of p are included in the closed disk $D(-a_{n-1}/a_n, \mu + |a_{n-1}/a_n|)$.

The following theorem is obtained with a multiplier of the form $z^3 - \gamma_1 z - \gamma_0$ and the set $\Gamma_2(1)$.

Theorem 4.3. *Let the real polynomial* $p(z) = \sum_{j=0}^{n} a_j z^j$ *with* $n \geq 3$ *have positive coefficients, define*

$$\gamma_1 = \frac{a_{n-2}}{a_n}\,, \quad \gamma_0 = \max\left\{0, \max_{0 \leq j \leq n-3} \frac{a_j - \gamma_1 a_{j+2}}{a_{j+3}}\right\}\,,$$

and denote by μ *the unique positive zero of* $z^3 - \gamma_1 z - \gamma_0$. *Then all the zeros of p lie in the closed disk* $D(-a_{n-1}/a_n, \mu + |a_{n-1}/a_n|)$.

Example. To graphically illustrate some of these results, we have chosen the upper bound from Theorem 3.1 and combined it with the inclusion disk from Theorem 4.1 for the polynomial $p_3(z) = z^6 + 4z^5 + 2z^4 + 2z^3 + 3z^2 + 6z + 7$ we encountered at the end of Section 3. The result can be found in Figure 1, where the solid circle centered at the origin is obtained from Theorem 3.1, and the dashed circle indicates the upper bound from the Cauchy radius. The disk obtained from Theorem 4.1, centered at -4, is bounded by the other solid circle. The zeros of p are indicated by black dots. Here the disk from Theorem 4.1 cuts off a significant part of the disk from Theorem 3.1, since the zeros must lie in the intersection of both.

The radii of the inclusion disks for the theorems in this section are as follows:

Cauchy radius of the second kind:	5.475	Theorem 4.2:	5.732
Theorem 4.1:	5.492	Theorem 4.3:	5.618 .

As in Section 3, one discerns a positive trend with increasing degree of the multipliers upon which the theorems are based.

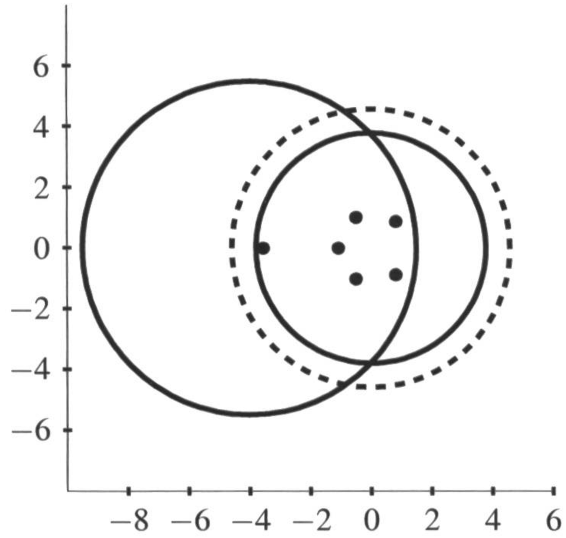

Figure 1 Inclusion regions from Theorems 3.1 and 4.1 for p_3.

So far, we have only used the sets $\Gamma_1(k)$ and $\Gamma_2(k)$ from Theorem 2.1 for $k = 1$. In the following two sections we derive inclusion regions with $k \geq 2$ consisting of two and three disks, respectively.

5 Two disks

In this section we derive zero inclusion regions consisting of two disks. We present two theorems.

Theorem 5.1. *Let the real polynomial* $p(z) = \sum_{j=0}^{n} a_j z^j$ *with* $n \geq 2$ *have positive coefficients, and define*

$$\mu = \left(\max_{0 \leq j \leq n-2} \frac{a_j}{a_{j+2}} \right)^{1/2} \text{ and } R = \mu^2 + \frac{a_{n-1}}{a_n} \mu \ .$$

Then all the zeros of p *are included in the union of disks* $D(0,\, R^{1/2}) \cup D(-a_{n-1}/a_n,\, R^{1/2})$*. If the disks are disjoint, then the disk centered at* $-a_{n-1}/a_n$ *contains one zero and the one centered at the origin contains the remaining* $n-1$ *zeros of* p*.*

Proof. Consider the polynomial $q(z) = (z^2 - \gamma)p(z)$. Then

$$q(z) = a_n z^{n+2} + a_{n-1} z^{n+1} + \sum_{j=2}^{n} (a_{j-2} - \gamma a_j) z^j - \gamma a_1 z - \gamma a_0 \ .$$

If we now choose $\gamma = \mu^2$ with μ as in the statement of the theorem, then all the coefficients of q, other than the two leading ones, are nonpositive, so that s_2, its Cauchy radius of the second kind, is its unique positive solution, namely, μ. The set $\Gamma_1(2)$ in Theorem 2.1 is then given by

$$\Gamma_1(2) = \left\{ z \in \mathbb{C} : \left| z \left(z + \frac{a_{n-1}}{a_n} \right) \right| \leq R \right\} ,$$

with $R = \mu^2 + (a_{n-1}/a_n)\mu$. This set is contained in the following union of two disks:

$$\left\{ z \in \mathbb{C} : |z| \leq R^{1/2} \right\} \bigcup \left\{ z \in \mathbb{C} : \left| z + \frac{a_{n-1}}{a_n} \right| \leq R^{1/2} \right\} .$$

If the disks are disjoint, then, from Theorem 2.1 with $k = 2$, we obtain that the disk centered at the origin contains $1 + (n+2) - 2 = n+1$ zeros, while the disk centered at $-a_{n-1}/a_n$ contains one zero of q. The zeros of q are those of p with the addition of $\pm\mu$. Because the disks are disjoint, all points in the disk centered at $-a_{n-1}/a_n$ have a negative real part, so that μ must lie in the disk centered at the origin, which means that $-\mu$ also lies in that disk. The only zero of q in the disk centered at $-a_{n-1}/a_n$ must therefore be a zero of p, while the other $n-1$ zeros of p lie in the disk centered at the origin. This concludes the proof. □

This theorem is somewhat similar to Theorem 4.2. However, instead of one disk centered at $-a_{n-1}/a_n$, we now have two disks (one centered at the origin and the other at $-a_{n-1}/a_n$) with a smaller radius. That this radius is smaller follows from the fact that

$$\mu^2 + \frac{a_{n-1}}{a_n}\mu \leq \left(\mu + \frac{a_{n-1}}{a_n} \right)^2 \Longrightarrow \left(\mu^2 + \frac{a_{n-1}}{a_n}\mu \right)^{1/2} \leq \mu + \frac{a_{n-1}}{a_n} ,$$

where μ has the same meaning here as in Theorem 4.2.

The following theorem is obtained with the multiplier $z^3 - \mu^3$, combined with the set $\Gamma_2(2)$.

Theorem 5.2. *Let the real polynomial* $p(z) = \sum_{j=0}^{n} a_j z^j$ *with* $n \geq 3$ *have positive coefficients, define*

$$\mu = \left(\max_{0 \leq j \leq n-3} \frac{a_j}{a_{j+3}} \right)^{1/3} \quad \textit{and} \quad R = \mu^2 + \frac{a_{n-1}}{a_n}\mu + \frac{a_{n-2}}{a_n},$$

and let c_1 *and* c_2 *be the zeros of the quadratic* $z^2 + (a_{n-1}/a_n)z + a_{n-2}/a_n$.
Then all the zeros of p are included in the union of disks $D(c_1, R^{1/2}) \cup D(c_2, R^{1/2})$.
If the disks are disjoint, then $c_2 < c_1 < 0$, *the disk centered at* c_2 *contains one zero, and the one centered at* c_1 *contains the remaining* $n-1$ *zeros of p.*

Example. Figure 2 compares the inclusion regions obtained from Theorem 4.2 and Theorem 5.1 for the same polynomial $p_3(z) = z^6 + 4z^5 + 2z^4 + 2z^3 + 3z^2 + 6z + 7$ that we used at the end of Section 3 and in Figure 1. The dotted circle centered at the origin (with radius 3.788) is the boundary of the disk obtained from Theorem 3.1, while the large dashed circle (with radius 5.732) is the boundary of the disk from Theorem 4.2. The two smaller solid circles (with radii 3.151) are the boundaries of the disks from Theorem 5.1. The zeros of p, which necessarily lie in the intersection of the inclusion regions, are indicated by black dots. The Cauchy radius of p_3 is 4.580.

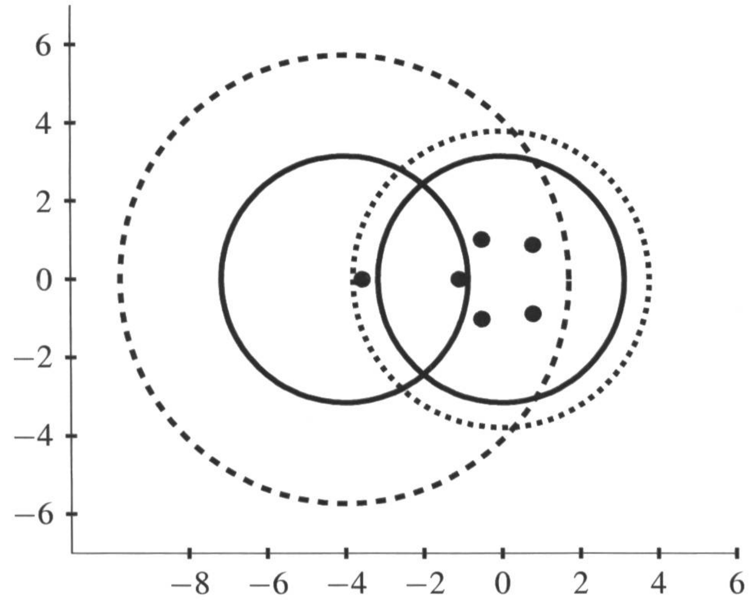

Figure 2 Comparison of Theorems 4.2 and 5.1 for p_3.

Regions composed of two disks are not necessarily smaller than those composed of a single disk, although they frequently are. Moreover, when the disks are disjoint, they provide additional information about the location of the zeros that cannot be obtained from standard generalizations of the Eneström–Kakeya theorem.

6 Three disks

In this section we carry out one more application of Theorem 2.1 to obtain a zero inclusion region consisting of three disks.

Theorem 6.1. *Let the real polynomial* $p(z) = \sum_{j=0}^{n} a_j z^j$ *with* $n \geq 3$ *have positive coefficients, and define*

$$\mu = \left(\max_{0 \leq j \leq n-3} \frac{a_j}{a_{j+3}} \right)^{1/3} \quad \text{and} \quad R = \mu^3 + \frac{a_{n-1}}{a_n}\mu^2 + \frac{a_{n-2}}{a_n}\mu \ .$$

Then all the zeros of p *are included in the union of disks* $D(0, R^{1/3}) \cup D(c_1, R^{1/3}) \cup D(c_2, R^{1/3})$, *where* c_1 *and* c_2 *are the zeros of the quadratic* $z^2 + (a_{n-1}/a_n)z + a_{n-2}/a_n$. *There exist only the following two scenarios for disks to be disjoint.*

(1) *The disk centered at the origin is disjoint from the other two, in which case that disk contains* $n-2$ *zeros of* p, *while the union of the other two contains the two remaining zeros of* p. *If these are also disjoint, then each contains one zero of* p.

(2) *The two disks not centered at the orgin are disjoint, but only one of them is disjoint from the disk at the origin, in which case that disk contains one zero of* p, *while the union of the other two contains* $n-1$ *zeros of* p. *This scenario is only possible when* c_1 *and* c_2 *are real and negative.*

Proof. Consider the polynomial $q(z) = (z^3 - \gamma)p(z)$, which is given by

$$q(z) = a_n z^{n+3} + a_{n-1} z^{n+2} + a_{n-2} z^{n+1} + \sum_{j=3}^{n} \left(a_{j-3} - \gamma a_j\right) z^j - \gamma a_2 z^2 - \gamma a_1 z - \gamma a_0 \ ,$$

with $\gamma = \mu^3$, where μ is as in the statement of the theorem. Then the three leading coefficients of q are positive, while all other coefficients are nonpositive. The Cauchy radius of the third kind of q is then its unique positive zero, which is the unique positive zero $\gamma^{1/3} = \mu$ of $z^3 - \gamma$. This means that the set $\Gamma_1(3)$ in Theorem 2.1 is given by

$$\Gamma_1(3) = \left\{ z \in \mathbb{C} : \left| z \left(z^2 + \frac{a_{n-1}}{a_n} z + \frac{a_{n-2}}{a_n} \right) \right| \leq R \right\} \ ,$$

where $R = \mu^3 + (a_{n-1}/a_n)\mu^2 + (a_{n-2}/a_n)\mu$. This set is contained in the union of three disks:

$$\left\{ z \in \mathbb{C} : |z| \leq R^{1/3} \right\} \bigcup \left\{ z \in \mathbb{C} : |z - c_1| \leq R^{1/3} \right\} \bigcup \left\{ z \in \mathbb{C} : |z - c_2| \leq R^{1/3} \right\} \ .$$

The following scenarios arise when the disks are disjoint. If $a_{n-1}^2/a_n^2 < 4a_{n-2}/a_n$, then c_1 and c_2 are complex conjugate with a negative real part, and the disks centered at these points are either not disjoint or both disjoint from the disk centered at the origin. When they are disjoint, then, by Theorem 2.1, q has $1 + (n+3) - 3 = n + 1$ zeros in the disk centered at the orgin. One of those must be μ, the real positive zero of $h(z) := z^3 - \gamma$. The other two zeros of h have the same modulus as μ and must therefore also lie in that disk, which means that the remaining $n-2$ zeros of q in that disk are zeros of p, and so are the two zeros of q in the union of the two disks centered at c_1 and c_2. If these are also disjoint from each other, then, by Theorem 2.1, each contains one zero of p.

If c_1 and c_2 are not complex, then they are both real and negative. If the disks centered at these points are both disjoint from the disk centered at the origin, then reasoning similarly as before, their union contains two zeros of p; if they are disjoint from each other, then each contains one zero of p. If only one is disjoint from the disk centered at the origin, then it contains one zero of p. □

Examples. The following examples illustrate Theorem 6.1.

- In Figure 3, we compare the inclusion regions obtained from Theorem 5.1 and Theorem 6.1 for the same polynomial $p_3(z) = z^6+4z^5+2z^4+2z^3+3z^2+6z+7$ we used before. The dotted circle centered at the origin (with radius 3.788) is the boundary of the disk obtained from Theorem 3.1. The solid circles in the top figure (with radii 3.151) are those obtained from Theorem 5.1, while those in the bottom figure (with radii 2.507) are obtained from Theorem 6.1. The zeros of p are indicated by black dots. The Cauchy radius in this case is 4.580.

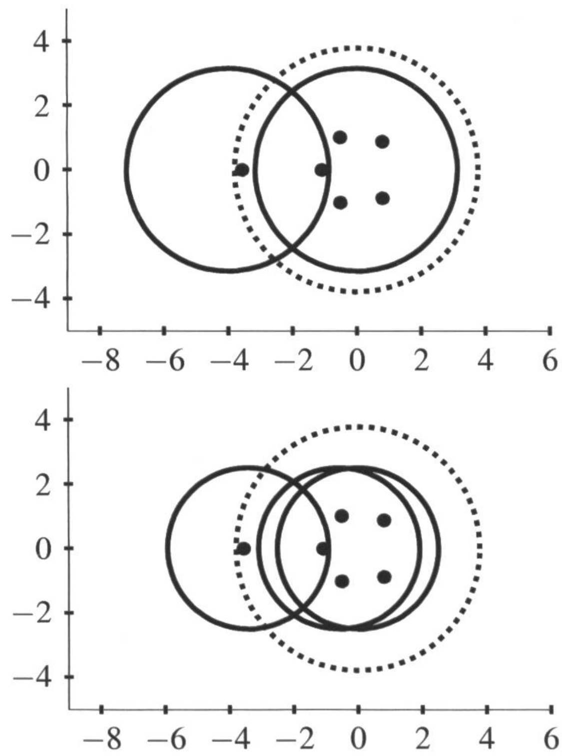

Figure 3 Comparison of Theorems 5.1 and 6.1 for p_3.

- In Figure 4, we carried out the same comparison as in Figure 3 for the polynomial $p_4(z) = z^6+9z^5+5z^4+6z^3+4z^2+8z+1$. The dotted circle centered at the origin (with radius 8.744) is the boundary of the disk obtained from Theorem 3.1. The solid circles in the top figure (with radii 5.012) are those obtained from Theorem 5.1, while those in the bottom figure (with radii 3.552) are obtained from Theorem 6.1. As before, the zeros of p are indicated by black dots. The Cauchy radius of p_4 is 9.592. Here Theorem 6.1 isolates one zero of p_4 from the others.

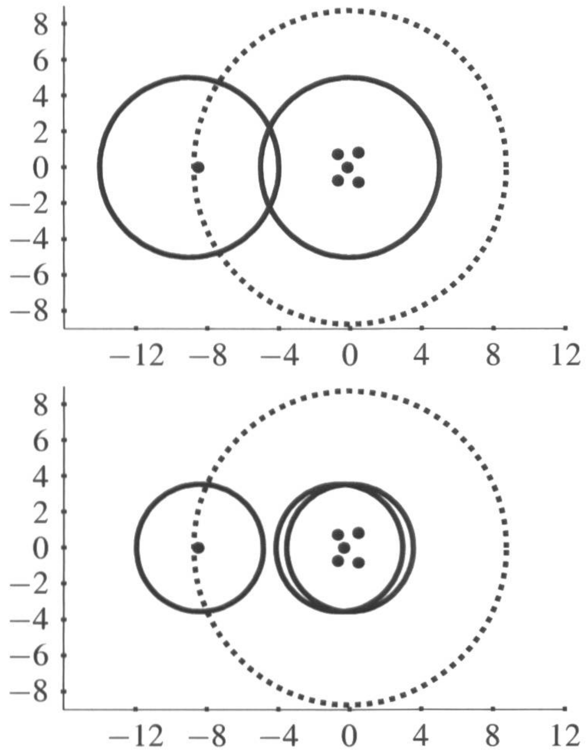

Figure 4 Comparison of Theorems 5.1 and 6.1 for p_4.

Conclusion. We have constructed a framework to derive generalizations of the classical Eneström–Kakeya theorem using two simple tools: polynomial multipliers and a theorem establishing inclusion regions for the zeros of a polynomial. This framework unifies and simplifies the derivation of these generalizations, obtaining new as well as old theorems in the process, while transparently showing how more of them can be generated. One feature of our results, namely, zero inclusion regions consisting of more than one disk, is apparently not found in any of the existing generalizations of this theorem.

Appendix

The following remarks are mainly concerned with relations between the different theorems.

(1) If in Theorem 3.1

$$\max_{0\leq j\leq n-3} \frac{a_j}{a_{j+1}} \leq \frac{a_{n-1}}{a_n} \leq \frac{a_{n-2}}{a_{n-1}} ,$$

then $\gamma_1 \geq 0$, which implies that $\gamma_0 = 0$, and we obtain precisely the result in Theorem 3.2, which in this case yields a smaller upper bound than the Cauchy radius, as explained in (3) below.

(2) In Theorem 3.1, the alternative choice

$$\gamma_2 = \max\left\{\frac{a_{n-1}}{a_n}, \frac{a_{n-2}}{a_{n-1}}\right\}, \ \gamma_1 = 0\,, \ \text{and} \ \gamma_0 = \max\left\{0, \max_{0\le j\le n-3} \frac{a_j - \gamma_2 a_{j+1}}{a_{j+3}}\right\},$$

leads to a similar result, which for brevity's sake we do not pursue.

(3) Theorem 3.2 delivers a better bound than the Cauchy radius when $\gamma_0 = (a_{n-2} - \gamma_1 a_{n-1})/a_n$, which is a consequence of Theorem 3.1 in [14]. We also note that, if $\frac{a_{n-1}}{a_n} = \max_{0\le j\le n-1} \frac{a_j}{a_{j+1}}$, then $\gamma_0 = 0$ and the upper bound from this theorem is identical to the Eneström–Kakeya upper bound, which in this case is better (smaller) than the Cauchy radius (as in the example of Section 3).

(4) In Theorem 4.1, if

$$\frac{a_{n-2}}{a_n} = \max_{0\le j\le n-2} \frac{a_j}{a_{j+2}} \quad \text{or} \quad \frac{a_{n-3}}{a_{n-1}} = \max_{0\le j\le n-2} \frac{a_j}{a_{j+2}}, \tag{6}$$

then $\beta = 0$ and the upper bound becomes $\sqrt{\alpha}$. If, given (6), $a_{n-2}/a_n \ge a_{n-3}/a_{n-1}$, then we obtain the same bound as in Theorem 4.2 and Theorem 4.3 while, if, given (6), $a_{n-3}/a_{n-1} \ge a_{n-2}/a_n$, then we obtain the same bound as in Theorem 4.2, but not necessarily as in Theorem 4.3.

(5) In Theorem 4.3, when $a_{n-2}/a_n = \max_{0\le j\le n-2} a_j/a_{j+2}$, then $\gamma_0 = 0$, in which case this theorem produces the same inclusion disk as in Theorem 4.2.

(6) Variations of Theorem 4.3 can be generated by using the multiplier $z^3 - \gamma_2 z^2 - \gamma_1 z - \gamma_0$ with $\gamma_2 = (1-\varepsilon)a_{n-1}/a_n$ for $0 < \varepsilon \le 1$, which shifts the center of the inclusion disk to $-\varepsilon a_{n-1}/a_n$, making Theorem 4.3 a special case for $\varepsilon = 1$. Similar variations can be considered for other theorems here as well, although we will not pursue them.

(7) The two disks in Theorem 6.1 that are not centered at the origin have the same centers as the disks in Theorem 5.2, but their radii are smaller because

$$\mu^2\left(\mu^2 + \frac{a_{n-1}}{a_n}\mu + \frac{a_{n-2}}{a_n}\right)^2 \le \left(\mu^2 + \frac{a_{n-1}}{a_n}\mu + \frac{a_{n-2}}{a_n}\right)^3$$

implies that

$$\mu^{1/3}\left(\mu^2 + \frac{a_{n-1}}{a_n}\mu + \frac{a_{n-2}}{a_n}\right)^{1/3} \le \left(\mu^2 + \frac{a_{n-1}}{a_n}\mu + \frac{a_{n-2}}{a_n}\right)^{1/2}.$$

(8) If the radius of the disk centered at the origin in the theorems of Sections 5 and 6 is larger than the Cauchy radius, then it will obviously contain all the zeros of the polynomial and the other disk(s) can be ignored. This can be detected by a simple substitution of the radius of that disk in equation (1).

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