Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 75 (2020)

Heft: 4

Artikel: On the inequalities of Grüss-ebyshev and Kantorovich : a probabilistic approach

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DOI: https://doi.org/10.5169/seals-880894

Elem. Math. 75 (2020) 172 – 175
0013-6018/20/040172-4
DOI 10.4171/EM/419

Elemente der Mathematik

Short note

On the Inequalities of Grüss–Čebyshev and Kantorovich: A Probabilistic Approach

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Abstract. First we recall the original form of inequalities found by P.L. Čebyshev in 1882, G. Grüss in 1935 and V.L. Kantorovich in 1948. Then we formulate generalized versions of these inequalities in the language of probability theory which allows to prove them by simple probabilistic arguments. A further moment inequality of this type rounds off this note.

1 Introduction and Results

V. Pták, see [6], provided a very short proof of the following inequality which has been first proved by V.L. Kantorovich, see [3] or [8], p. 85.

Kantorovich Inequality: For any real numbers $x_1, \dots, x_n$ satisfying $0 < x_1 < \cdots < x_n < \infty$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$

$$\left(\sum_{i=1}^{n} \lambda_i\, x_i\right)\left(\sum_{i=1}^{n} \frac{\lambda_i}{x_i}\right) \leq \frac{(x_1 + x_n)^2}{4\, x_1\, x_n}\,. \tag{1}$$

In the particular case $\lambda_1 = \cdots = \lambda_n = \frac{1}{n}$ the inequality (1) has been shown by P. Schweitzer, see [7], by applying the fact that the function $\phi(x_1, \dots, x_n) = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$ is strictly Schur-convex and decreasing in each coordinate, see [5], p. 71. In Theorem 1 we formulate a slightly more general version of (1) emphasizing its probabilistic nature, where its proof is similarly short as that of (1) given in [6].

To state the next two inequalities we define for any two bounded functions $f, g : [a, b] \mapsto \mathbb{R}$ with $-\infty < a < b < \infty$

$$\Delta_{a,b}(f, g) := \frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\mathrm{d}x \int_a^b g(x)\mathrm{d}x\,. \tag{2}$$

Čebyshev Inequality: If f and g are Lipschitz continuous on $[a, b]$ with Lipschitz constants L_f and L_g, respectively, then

$$|\,\Delta_{a,b}(f, g)\,| \leq \frac{1}{12}\,(b-a)^2\, L_f\, L_g\,, \qquad \text{see [1].} \tag{3}$$

Grüss Inequality: If f and g are Riemann integrable on $[a, b]$ with bounded oscillations $\operatorname{osc}_a^b(f) := \sup\{f(x) : x \in [a, b]\} - \inf\{f(x) : x \in [a, b]\}$ and $\operatorname{osc}_a^b(g)$, respectively, then

$$|\,\Delta_{a,b}(f, g)\,| \leq \frac{1}{4} \operatorname{osc}_a^b(f) \operatorname{osc}_a^b(g)\,, \qquad \text{see [2]}. \tag{4}$$

Remarks: The factors 1/12 in (3) and 1/4 in (4) cannot be replaced by smaller constants. For $a = 0, b = 1$ this is easily seen by inserting $f(x) = g(x) = 2x - 1$ in (3) and $f(x) = g(x) = \operatorname{sign}(2x - 1)$ in (4). E. Landau, see [4], pointed out that (4) holds in case of continuous functions f, g with the best possible factor 4/45 instead of 1/4. More about Čebyshev-type and Grüss-type inequalities including historical notes the reader can find in [1], Chapter 3 and references therein. Among others there are provided discrete and operator versions of (3) and (4) in different settings.

In what follows, let X and Y denote real-valued random variables (short: rv's) on some probability space $[\Omega, \mathcal{F}, \mathsf{P}]$, where E, Var and Cov designate expectation, variance and covariance, respectively. We remind the reader that $\mathsf{Cov}(X, Y) := \mathsf{E}(XY) - \mathsf{E}X\,\mathsf{E}Y$ and $\mathsf{Var}(X) := \mathsf{Cov}(X, X)$.

Theorem 1 (Probabilistic Kantorovich-Type Inequality). *Let X be an rv taking values only in $[a, b]$ for $a, b \in \mathbb{R}$ such that $0 < a < b < \infty$. Then*

$$\mathsf{E}(X)\,\mathsf{E}\Big(\frac{1}{X}\Big) \leq \frac{(a + b)^2}{4\,a\,b}\,. \tag{5}$$

Equality holds in (5) iff $\mathsf{P}(X = a) = \mathsf{P}(X = b) = 1/2$. Obviously, (1) follows from (5) if $\lambda_i = \mathsf{P}(X = x_i)$ for $i = 1, \ldots, n$ and $a = x_1 < \cdots < x_n = b$.

The key to formulate probabilistic versions of (3) and (4) consists in the simple fact that $\Delta_{a,b}(f, g)$ coincides with $\mathsf{Cov}(f(X), g(X))$ when X is uniformly distributed on $[a, b]$.

Theorem 2 (Probabilistic Čebyshev- and Grüss-Type Inequality). *Let X be an rv taking values in a finite or infinite interval $[a, b]$ such that $\mathsf{E}X^2 < \infty$. If $f, g : [a, b] \mapsto \mathbb{R}$ are Lipschitz continuous functions on $[a, b]$ with Lipschitz constants L_f, L_g, then*

$$|\,\mathsf{Cov}(f(X), g(X))\,| \leq L_f\, L_g\, \mathsf{Var}(X) \leq L_f\, L_g\, (\mathsf{E}X - a)(b - \mathsf{E}X)\,. \tag{6}$$

If $f, g : [a, b] \mapsto \mathbb{R}$ are Borel-measurable functions on $[a, b]$ with bounded oscillations $\operatorname{osc}_a^b(f)$ and $\operatorname{osc}_a^b(g)$, then

$$|\,\mathsf{Cov}(f(X), g(X))\,| \leq \frac{1}{4} \operatorname{osc}_a^b(f) \operatorname{osc}_a^b(g)\,. \tag{7}$$

Finally, we state an estimate of the absolute central moment $\mathsf{E}|X - \mathsf{E}X|^p$ of order $p \geq 2$.

Theorem 3. *Let the rv X take values in a finite interval $[a, b]$ with $\mu := \mathsf{E}X$. Then*

$$\mathsf{E}|\,X - \mu\,|^p \leq (\max\{\mu - a, b - \mu\})^{p-2}\, (\mu - a)(b - \mu) \tag{8}$$

with equality for $p = 2$ iff $\mathsf{P}(X \in \{a, b\}) = 1$ and for $p > 2$ iff $\mathsf{P}(X = a) = \mathsf{P}(X = b) = 1/2$.

2 Proofs of the Theorems 1, 2, and 3

Proof of Theorem 1. Putting $Y = X/\sqrt{a\,b}$ implies that $\mathsf{P}(\sqrt{a/b} \le Y \le \sqrt{b/a}) = 1$ and $\mathsf{E}(X)\ \mathsf{E}(1/X) = \mathsf{E}(Y)\ \mathsf{E}(1/Y)$. Making use of the well-known inequality $\sqrt{u\,v} \le (u+v)/2$ for any real two numbers $u, v > 0$ (with equality iff $u = v$) we obtain the relation

$$\mathsf{E}(Y)\,\mathsf{E}\Big(\frac{1}{Y}\Big) \le \frac{1}{4}\Big(\mathsf{E}(Y)+\mathsf{E}\Big(\frac{1}{Y}\Big)\Big)^2 = \frac{1}{4}\Big[\sqrt{\frac{b}{a}}+\sqrt{\frac{a}{b}}-\mathsf{E}\Big(\frac{1}{Y}\Big(Y-\sqrt{\frac{a}{b}}\Big)\Big(\sqrt{\frac{b}{a}}-Y\Big)\Big)\Big]^2 .$$

Since the expectation term and also the whole term within the brackets are non-negative it is clear that the right-hand side of the latter relation is less than or equal to $(\sqrt{b/a} + \sqrt{a/b})^2/4 = (a+b)^2/4ab$ which proves (5). Equality in this estimate holds iff the expectation term disappears and this takes place iff the rv Y is concentrated on the numbers $\sqrt{a/b}$ and $\sqrt{b/a}$, say $p := \mathsf{P}(Y = \sqrt{a/b}) = 1 - \mathsf{P}(Y = \sqrt{b/a})$. Then equality in the first estimate, that is $\mathsf{E}(Y) = \mathsf{E}(1/Y)$, is equivalent with

$$p\sqrt{\frac{a}{b}} + (1-p)\sqrt{\frac{b}{a}} = p\sqrt{\frac{b}{a}} + (1-p)\sqrt{\frac{a}{b}} .$$

This equation is satisfied just for $p = 1/2$ completing the proof of Theorem 1. □

Proof of Theorem 2. The inequality of Cauchy–Schwarz–Bunyakovsky, see [8], yields $|\,\mathsf{Cov}(f(X), g(X))\,| \le \sqrt{\mathsf{Var}(f(X))}\,\sqrt{\mathsf{Var}(g(X))}$. We estimate the variance $\mathsf{Var}(f(X))$ in two ways: First by using $|f(y) - f(x)| \le L_f\,|y - x|$ and the probability distribution $P_X(\cdot) = \mathsf{P}(\{\omega \in \Omega : X(\omega) \in (\cdot)\})$ on the Borel sets of $\mathbb{R}$ we get

$$\begin{aligned}\mathsf{Var}(f(X)) = \mathsf{E}\big(f(X) - \mathsf{E}f(X)\big)^2 &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \big(f(y) - f(x)\big)^2 P_X(\mathrm{d}y)\,P_X(\mathrm{d}x)\\ &\le \frac{L_f^2}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (y-x)^2 P_X(\mathrm{d}y)\,P_X(\mathrm{d}x) = L_f^2\ \mathsf{Var}(X)\end{aligned}$$

implying $|\,\mathsf{Cov}(f(X), g(X))\,| \le L_f\,L_g\ \mathsf{Var}(X)$. This combined with (8) for $p = 2$ completes the proof of (6).

Second from the estimate $\mathsf{Var}(f(X)) \le \mathsf{E}\big(f(X) - c\big)^2$ for any $c \in \mathbb{R}$ we obtain for the special value $c_0 = (\inf_{a\le x\le b} f(x) + \sup_{a\le x\le b} f(x))/2$ that $\sup_{a\le x\le b}|\,f(x) - c_0\,| \le \mathrm{osc}_a^b(f)/2$ and thus

$$\sqrt{\mathsf{Var} f(X)} \le \mathrm{osc}_a^b(f)/2$$

proving (7). □

Proof of Theorem 3. We rewrite and estimate $\mathsf{E}|\,X - \mu\,|^p$ as follows

$$\begin{aligned}\mathsf{E}|\,X - \mu\,|^p &= \int_a^{\mu} (\mu - x)^p\,P_X(\mathrm{d}x) + \int_{\mu}^{b} (x - \mu)^p\,P_X(\mathrm{d}x)\\ &\le (\max\{\mu - a, b - \mu\})^{p-2}\ \mathsf{Var}(X) .\end{aligned}$$

To complete the proof of (8) we show that

$$\begin{aligned}\mathsf{Var}(X) &= \mathsf{E}\Big(X - \frac{a+b}{2}\Big)^2 + \mathsf{E}\Big[(X-\mu)^2 - \Big(X - \frac{a+b}{2}\Big)^2\Big] \\ &= \mathsf{E}\Big(X - \frac{a+b}{2}\Big)^2 - \Big(\mu - \frac{a+b}{2}\Big)^2 \\ &\le \Big(\frac{b-a}{2}\Big)^2 - \Big(\mu - \frac{a+b}{2}\Big)^2 = (b-\mu)(\mu-a)\,.\end{aligned}$$

On the other hand, the relation

$$0 = (b-\mu)(\mu-a) - \mathsf{E}(X-\mu)^2 = \mathsf{E}(X-a)(b-X)$$

is valid iff $\mathsf{P}(X \in \{a,b\}) = 1$. This means that $\alpha = \mathsf{P}(X=a) = 1 - \mathsf{P}(X=b)$ and $\mu = \alpha\, a + (1-\alpha)\, b$ for some $0 \le \alpha \le 1$. A short calculation reveals that equality in (8) is only possible for $\alpha = 1/2$. □

References

[1] DRAGOMIR, S.S. *Operator Inequalities of the Jensen, Čebyšev and Grüss Type*, Springer, New York, 2012

[2] GRÜSS, G. Über das Maximum des absoluten Betrages von

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x,$$

Math. Z. **39** (1935), 215–226.

[3] KANTOROVICH, L.V. Functional analysis and applied mathematics (in Russian), *Uspechi matem. nauk* **3** (1948), 89–185.

[4] LANDAU, E. Über einige Ungleichungen von Herrn G. Grüss, *Math. Z.* **39** (1935), 742–744.

[5] MARSHALL, A.W. AND OLKIN, I. *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[6] PTÁK, V. The Kantorovich inequality, *Amer. Math. Monthly* **102** (1995), 820–821.

[7] SCHWEITZER, P. An inequality concerning the arithmetic mean (in Hungarian), *Math.-Phys. Lapok* **23** (1914), 257–261.

[8] STEELE, J.M. *The Cauchy–Schwarz Master Class*, Cambridge University Press, New York, 2004.

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