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An inequality for cevians and applications

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1 The inequality for cevians

In a triangle ABC with sides a, b, c, semiperimeter s, circumradius R and inradius r, let AD be a cevian. We give a lower bound for its length, in terms of its adjacent sides and corresponding angle.

Theorem 1. *For the cevian AD with $BD/DC = \lambda$ and angle $\angle BAC = \alpha$, we have the inequality*

$$AD \geqslant \left(\frac{\lambda}{\lambda+1}b + \frac{1}{\lambda+1}c\right)\cos\frac{\alpha}{2}. \tag{1}$$

Equivalently, if the ratio $BD/BC = k$ is given, with $k' = 1-k$, the inequality is

$$AD \geqslant (kb + k'c)\cos\frac{\alpha}{2}. \tag{2}$$

Cevane oder Ecktransversalen sind Geraden, welche durch die Ecke eines Dreiecks verlaufen. Sie spielen eine zentrale Rolle in der Dreiecksgeometrie. Ein Beispiel ist der bekannte Satz von Ceva, bei dem es darum geht, in welchen Verhältnissen drei kopunktale Ecktransversalen die gegenüberliegenden Dreieckseiten teilen. Es ist aber auch nützlich, Aussagen über die Längen von Ecktransversalen zu finden. Die Autoren der vorliegenden Arbeit geben hier eine allgemeine untere Schranke für diese Längen an. Daraus ergeben sich interessante Folgerungen für spezielle Ecktransversalen sowie für das Produkt der Längen von drei kopunktalen Ecktransversalen.

Proof. The proof is given for the inequality in form (2). Let M be a point on the side AB, such that DM is parallel to CA. We construct two similar right triangles, AMF and DME, with right angles at vertices F and E, and angles equal to $\alpha/2$ at vertices A and D, see Figure 1.

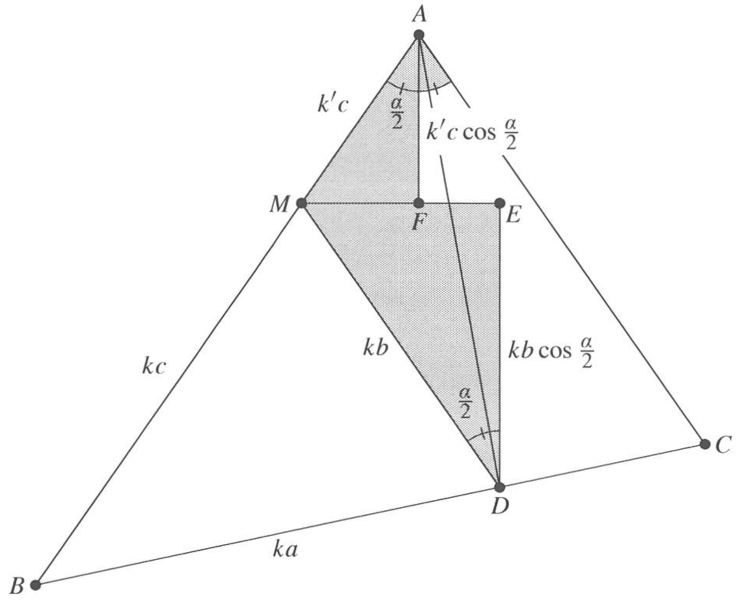

Figure 1

Then the lengths of AF and DE are $k'c\cos\frac{\alpha}{2}$ and $kb\cos\frac{\alpha}{2}$ respectively. It is evident that the length of the cevian AD is greater than or equal to their sum. □

The inequality is sharp, with equality when the cevian AD is the bisector w_a of α. In that case $\lambda = c/b$, Euclid, Book VI. 3. On the other hand by Pappus $w_a = \frac{2bc}{b+c}\cos\frac{\alpha}{2}$, hence equality holds in (1).

We could have proceeded applying the elementary inequality for a triangle ABC

$$\sin\frac{\alpha}{2} \leqslant \frac{a}{b+c} \tag{3}$$

to the triangle AMD, where $\angle M = \angle DMA = \pi - \alpha$, to obtain

$$AD \geqslant (MD + MA)\sin\frac{M}{2} = \left(kb + k'c\right)\cos\frac{\alpha}{2}.$$

The inequality (3) is a direct byproduct of Mollweide's formula [5], [7]

$$\sin\frac{\alpha}{2}\Big/\cos\frac{\beta-\gamma}{2} = \frac{a}{b+c}.$$

It can also be derived geometrically by drawing perpendiculars from vertices B and C to the bisector of α.

2 Applications

Next we apply the inequality (1) to some special cevians. For the median m_a, we have $\lambda = 1$, so (1) gives

$$m_a \geqslant \frac{b+c}{2}\cos\frac{\alpha}{2}. \tag{4}$$

By the AM-HM and AM-GM inequalities $\frac{b+c}{2} \geqslant \frac{2bc}{b+c}$, $\frac{b+c}{2} \geqslant \sqrt{bc}$ and $\cos(\alpha/2) = \sqrt{s(s-a)/(bc)}$, $w_a = \frac{2bc}{b+c}\cos\frac{\alpha}{2}$, (4) implies the known useful inequalities

$$m_a \geqslant \sqrt{s(s-a)} \geqslant w_a.$$

From (4) and $m_a = \sqrt{2(b^2+c^2)-a^2}/2$, follows an inequality for cosines similar to inequality (3)

$$\cos\frac{\alpha}{2} \leqslant \frac{\sqrt{2(b^2+c^2)-a^2}}{b+c}. \tag{5}$$

The next inequalities for symmedians, Gergonne and Nagel Cevians are possibly new.

The *symmedian* is a cevian which is a reflection of the median in the corresponding angle bisector. The three symmedians are concurrent in the *symmedian point*, sometimes referred to as the Lemoine point or Grebe point.

Corollary 1. *For the symmedian s_a, the following inequality holds*

$$s_a \geqslant \frac{bc(b+c)}{b^2+c^2}\cos\frac{\alpha}{2}. \tag{6}$$

Proof. It is well known [3] that the symmedian $s_a = AD$ divides the side BC in the ratio of the squares of the adjacent sides, that is $\lambda = c^2/b^2$. Substituting in (1), gives (6). □

From the equation of the symmedian $s_a = bc\sqrt{2(b^2+c^2)-a^2}/(b^2+c^2)$, which can be derived from Stewart's theorem, and (6), follows again the inequality (5).

The three lines connecting each vertex of a triangle to the point of contact of the incircle and the opposite side are concurrent at the *Gergonne point* and are called *Gergonne cevians*, named after the French geometer Joseph Diaz Gergonne (1771–1859). *Nagel cevians*, named after the German geometer Christian Heinrich von Nagel (1803–1882), are the three lines concurrent at the *Nagel point*, connecting each vertex to the point of contact of the corresponding excircle and the opposite side. We denote the length of the Gergonne cevian AG_a by g_a and the length of the Nagel cevian AN_a by n_a. All the notation appears in Figure 2.

For Gergonne cevians

$$\lambda = \frac{BG_a}{G_aC} = \frac{s-b}{s-c},$$

hence by (1)

$$g_a = AG_a \geqslant \frac{b(s-b)+c(s-c)}{a}\cos\frac{\alpha}{2}.$$

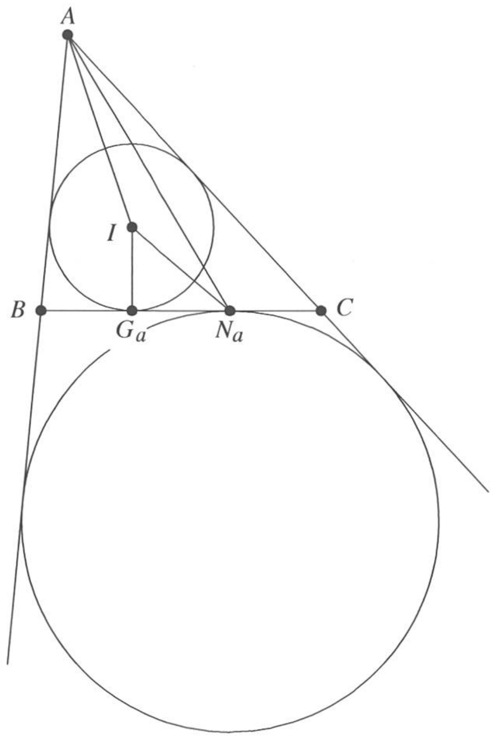

Figure 2

For Nagel cevians

$$\lambda = \frac{BN_a}{N_aC} = \frac{s-c}{s-b},$$

hence by (1)

$$n_a = AN_a \geqslant \frac{b(s-c)+c(s-b)}{a}\cos\frac{\alpha}{2}.$$

We derive as a consequence of (1) an inequality for the product of cevians. Let AD, BE, CF be three cevians such that

$$\frac{BD}{DC} = \lambda, \quad \frac{CE}{EA} = \mu, \quad \frac{AF}{FB} = \nu.$$

Corollary 2. *For the product of cevians AD, BE, CF, we have the inequality*

$$AD \cdot BE \cdot CF \geqslant \frac{8rs^2\sqrt{\lambda\mu\nu}}{(\lambda+1)(\mu+1)(\nu+1)} \tag{7}$$

Proof. By (1) and the AM-GM inequality $\frac{\lambda}{\lambda+1}b + \frac{1}{\lambda+1}c \geqslant 2\sqrt{bc\lambda}/(\lambda+1)$, we get

$$AD \geqslant \frac{2\sqrt{bc\lambda}}{\lambda+1}\cos\frac{\alpha}{2}. \tag{8}$$

Multiplying (8) by the analogous inequalities for the cevians BE, CF

$$BE \geqslant \frac{2\sqrt{ca\mu}}{\mu+1}\cos\frac{\beta}{2}, \quad CF \geqslant \frac{2\sqrt{ab\nu}}{\nu+1}\cos\frac{\gamma}{2},$$

and using the well-known identities $abc = 4Rrs$ and

$$\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} = \frac{s}{4R},$$

we obtain (7). □

For concurrent cevians, Ceva's theorem [1], [3], [8] says that $\lambda\mu\nu = 1$, so the inequality (7) in that case simplifies to

Corollary 3. *Let the cevians AD, BE, CF be concurrent. Then the following inequality holds*

$$AD \cdot BE \cdot CF \geqslant \frac{8rs^2}{(\lambda+1)(\mu+1)(\nu+1)}. \tag{9}$$

Let us consider now the product of cevians of some special triangle centers.

For the centroid G and the corresponding medians holds $\lambda = \mu = \nu = 1$. Hence by (9) we have

$$m_a m_b m_c \geqslant rs^2.$$

Since for the exradii of the triangle, the identity $r_a r_b r_c = rs^2$ holds, this last inequality is actually the known inequality $m_a m_b m_c \geqslant r_a r_b r_c$ [2, 8.21].

The bisectors are the cevians of the incenter I and they divide the sides in the ratio of the corresponding sides: $\lambda = c/b$, $\mu = a/c$, $\nu = b/a$. Thus by (9), we have the elegant inequality for the product of bisectors

$$w_a w_b w_c \geqslant \frac{8rs^2abc}{(a+b)(b+c)(c+a)}, \tag{10}$$

which complements $rs^2 \geqslant w_a w_b w_c$ [2, 8.14]. We remark that the inequality (10) is a refinement of $w_a w_b w_c \geqslant 8Rr^2s^2/(2R^2+3Rr+2r^2)$ given in [6, p. 217]. Indeed, by $abc = 4Rrs$ and the identity [6, p. 53]

$$\prod(b+c) = 2s(s^2+2Rr+r^2),$$

the inequality

$$\frac{8rs^2abc}{(a+b)(b+c)(c+a)} \geqslant \frac{8Rr^2s^2}{2R^2+3Rr+2r^2}$$

is equivalent to Gerretsen's inequality [4] $s^2 \leqslant 4R^2+4Rr+3r^2$.

Next we give an inequality for the product of symmedians. Since for symmedians $\lambda = c^2/b^2$, $\mu = a^2/c^2$, $\nu = b^2/a^2$, by (9), we have

$$s_a s_b s_c \geqslant \frac{8rs^2a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

We end this article by asking for a geometric proof of inequality (5), similar to the proof of inequality (3) or the one given in Theorem 1.

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