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Short note

A proof of the Schwarz theorem on mixed partial derivatives via elementary approximation theory

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Abstract. We show that the equality of the mixed second-order partial derivatives holds for functions that can be well approximated by linear combinations of functions whose variables separate and that this is the case for functions satisfying the hypothesis of Schwarz' theorem.

1 Introduction

A fundamental theorem of multivariate Calculus states that if a function $f(x, y)$ has partial derivatives f_x and f_y, and mixed derivatives f_{xy} and f_{yx} in a neighborhood Ω of a point (x_0, y_0) and if f_{xy} and f_{yx} are continuous at (x_0, y_0), then[1]

$$f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0). \tag{1}$$

According to [5], many outstanding mathematicians, among them Euler, Lagrange and Cauchy, offered proofs of (1), but none of them without a fault. In fact, in 1867, Lindelöf published an article criticizing all those proofs in detail. The first satisfactory proof of (1) appeared only six years later in an article by Schwarz.

All proofs of (1) we found in a collection of Calculus textbooks [1, 2, 7, 8, 9, 10, 11] mainly reproduce Schwarz' original proof [12], which proceeds by showing that the function

$$\phi(h_x, h_y) = \frac{\dfrac{f(x_0 + h_x, y_0 + h_y) - f(x_0, y_0 + h_y)}{h_x} - \dfrac{f(x_0 + h_x, y_0) - f(x_0, y_0)}{h_x}}{h_y} \tag{2}$$

has a limit as $(h_x, h_y) \to (0, 0)$ and that this limit is equal to both $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$. The reader can also find lightly different proofs in [3] and [4]; and [5] also points a proof of (1) based on Lebesgue integration.

What we found curious about this theorem is that none of the proofs presented in the references explores the obvious fact that (1) holds whenever f is a polynomial in the

[1] A few refinements can be found in the literature [6], but we shall not discuss it here.

variables x and y or, more in general, if f is of the form

$$\sum_{i=1}^{n} u_i(x)v_i(y), \tag{3}$$

for some differentiable functions $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$.

Actually this hypothesis can be somewhat relaxed and (1) also holds for functions f that can be well approximated by functions g of the form (3) in the sense that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)-g(x,y)}{(x-x_0)(y-y_0)} = 0. \tag{4}$$

This is true because

Lemma 1. *If f and g satisfy* (4) *and have partial derivatives and mixed partial derivatives of second order in Ω, then both first partial derivatives and mixed partial derivatives of second order of f and g coincide at the point* (x_0, y_0).

In this note we show that, for each function f satisfying the hypothesis of Schwarz' theorem, there is a function g of the form (3) such that Lemma 1 holds for f and g and this proves (1).

Remark 1. We emphasize that, although a little bit longer than the usual argument for proving Schwarz' theorem, our strategy is based on the largely employed technique of *reducing the problem to a simpler one* (just as in the case of the analysis of local extrema by the analysis of quadratic approximations). Therefore, our reasoning is totally coherent and it is somewhat surprising that no one explored it before (at least according to our records).

Proof of (1)

To simplify the notation, let us assume that $(x_0, y_0) = (0, 0)$. We start with the initial approximation $g_1(x, y) := f(0, y) + f_x(0, y)\,x$ obtained by means of the partial derivative f_x:

$$f(x,y) = f(0,y) + f_x(0,y)\,x + R(x,y), \tag{5}$$

with $\lim\limits_{x\to 0} \frac{R(x,y)}{x} = 0$ for every fixed y. We want to check whether f and g_1 satisfy the hypothesis of Lemma 1. In this vein, we note that (5) is not sufficient for our purpose and we must also analyze how

$$R(x,y) = f(x,y) - f(0,y) - f_x(0,y)\,x$$

varies with respect to y.

This expression shows that R is differentiable with respect to y and we have

$$R_y(x,y) = f_y(x,y) - f_y(0,y) - f_{xy}(0,y)\,x,$$

or, by the mean value theorem,

$$R_y(x,y) = [f_{yx}(\tau(x,y),y) - f_{xy}(0,y)]\,x,$$

with $|\tau(x,y)| < |x|$. Once again by the mean value theorem, we obtain

$$\begin{aligned} R(x,y) &= R(x,0) + R_y(x,\xi(x,y))\,y \\ &= R(x,0) + [f_{yx}(\tau(x,\xi(x,y)),\xi(x,y)) - f_{xy}(0,\xi(x,y))]\,x\,y \\ &= R(x,0) + [f_{yx}(0,0) - f_{xy}(0,0)]\,x\,y \\ &\quad + [f_{yx}(\tau(x,\xi(x,y)),\xi(x,y)) - f_{yx}(0,0)]\,x\,y \\ &\quad + [f_{xy}(0,0) - f_{xy}(0,\xi(x,y))]\,x\,y, \end{aligned} \tag{6}$$

with $|\xi(x,y)| < |y|$.

The expression above shows that f and g_1 do not satisfy the hypothesis of Lemma 1 in general, but (5) and (6) and the continuity of f_{xy} and f_{yx} at $(0,0)$ shows that Lemma 1 holds for f and

$$\begin{aligned} g(x,y) &= f(0,y) + f_x(0,y)\,x + R(x,0) + [f_{yx}(0,0) - f_{xy}(0,0)]\,x\,y \\ &= f(0,y) + f_x(0,y)\,x + f(x,0) - f(0,0) - f_x(0,0)\,x \\ &\quad + [f_{yx}(0,0) - f_{xy}(0,0)]\,x\,y. \end{aligned}$$

In fact, we have

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{f(x,y)-g(x,y)}{xy} = {}& \lim_{(x,y)\to(0,0)} [f_{yx}(\tau(x,\xi(x,y)),\xi(x,y)) - f_{yx}(0,0)] \\ &+ \lim_{(x,y)\to(0,0)} [f_{xy}(0,0)) - f_{xy}(0,\xi(x,y))] \\ = {}& 0. \end{aligned}$$

In addition, note that g is of the form (3) for $n = 4$ and

$$\begin{aligned} v_1(y) &= f(0,y); & u_1(x) &= 1; \\ v_2(y) &= f_x(0,y); & u_2(x) &= x; \\ v_3(y) &= 1; & u_3(x) &= f(x,0) - f(0,0) - f_x(0,0)\,x; \\ v_4(y) &= y; & u_4(x) &= [f_{yx}(0,0) - f_{xy}(0,0))]\,x. \end{aligned}$$

Because all these functions are differentiable (some of them by the assumptions on f and other simply because they are polynomials), the proof is complete.

Remark 2. We emphasize that, while $[f_{yx}(0,0) - f_{xy}(0,0)]$ appears explicitly in the definition of g, our argument does not depend on any previous knowledge about its value.

Proof of Lemma 1

Again, assume that $(x_0, y_0) = (0,0)$. Let $h(x,y) = f(x,y) - g(x,y)$. By assumption, given any positive number ϵ there is a number $\delta > 0$ such that

$$|h(x,y)| \le \epsilon |x|\,|y|, \tag{7}$$

for $0 < |x| < \delta$ and $0 < |y| < \delta$. Hence, we obtain

$$h(x, 0) = 0 \text{ for } |x| < \delta \text{ and } h(0, y) = 0 \text{ for } |y| < \delta.$$

This is true because $h(x, .)$ and $h(., y)$ are differentiable (and therefore continuous) for every x and y. In particular,

$$h_x(0, 0) = 0. \tag{8}$$

Moreover, for $0 < |x| < \delta$ and $0 < |y| < \delta$ we have

$$\left|\frac{h(x, y) - h(0, y)}{x}\right| = \left|\frac{h(x, y)}{x}\right| \leq \epsilon|y|$$

and this shows that $|h_x(0, y)| \leq \epsilon|y|$ for $0 < |y| < \delta$. Combining this inequality with (8), we obtain

$$\left|\frac{h_x(0, y) - h_x(0, 0)}{y}\right| = \left|\frac{h_x(0, y)}{y}\right| \leq \epsilon$$

for $0 < |y| < \delta$ and, therefore,

$$\left|h_{xy}(0, 0)\right| \leq \epsilon.$$

Because ϵ can be chosen arbitrarily small, we have that $h_{xy}(0, 0) = 0$.
By symmetry, we also have that $h_{yx}(0, 0) = 0$.

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