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Elemente der Mathematik

Short note

Mixtilinear radii and Finsler–Hadwiger inequality

Martin Lukarevski and Gerhard Wanner

Abstract. In this short note we derive few relations for the mixtilinear radii of a triangle and then as a curious application give an interesting proof of the celebrated Finsler–Hadwiger inequality.

The *mixtilinear incircle* is a circle tangent to two sides of a triangle and internally to the circumcircle of the triangle, [1]. Another, more suggestive name is *inarc circle*, [6]. We de-

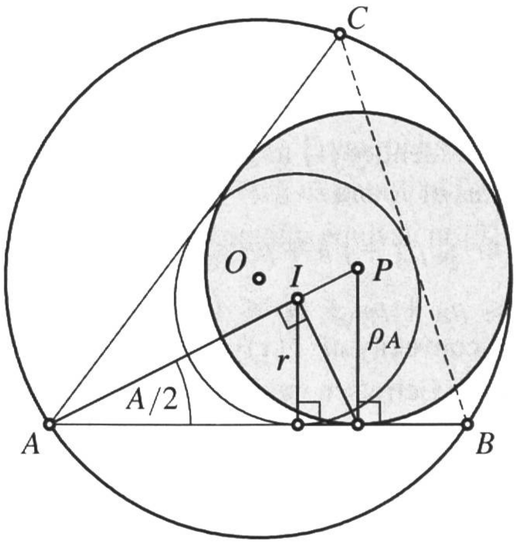

Figure 1 Mixtilinear incircle and proof of $\rho_A = r \sec^2 \frac{A}{2}$.

note by ρ_A, ρ_B, ρ_C the radii of the mixtilinear incircles of a triangle ABC with inradius r, circumradius R and semiperimeter s. It is known, [11], that the center of the mixtilinear incircle is the orthogonal projection on the bisector of the point on the side which is itself the orthogonal projection of the incenter. See Figure 1. Hence one readily gets that

$$\rho_A = r \sec^2 \frac{A}{2}.$$

Theorem 1. *It holds for the radii of the mixtilinear incircles*

$$\rho_A + \rho_B + \rho_C = r\left(1 + \left(\frac{4R + r}{s}\right)^2\right), \tag{1}$$

$$\rho_A\rho_B + \rho_B\rho_C + \rho_C\rho_A = \frac{8Rr^2(4R+r)}{s^2}, \tag{2}$$

$$\rho_A\rho_B\rho_C = \frac{16R^2r^3}{s^2}, \tag{3}$$

$$\frac{1}{\rho_A} + \frac{1}{\rho_B} + \frac{1}{\rho_C} = \frac{4R+r}{2Rr}. \tag{4}$$

Proof. The cosines $\cos A$, $\cos B$, $\cos C$ are the roots of the polynomial [10, p. 56]

$$4R^2x^3 - 4R(R+r)x^2 + (s^2+r^2-4R^2)x + (2R+r)^2 - s^2 = 0.$$

Putting $x = 2y - 1$ and rearranging, we get that the polynomial

$$16R^2y^3 - 8R(4R+r)y^2 + (s^2 + (4R+r)^2)y - s^2 = 0 \tag{5}$$

has $\cos^2\frac{A}{2}$, $\cos^2\frac{B}{2}$, $\cos^2\frac{C}{2}$ as its roots, and since $\cos^2\frac{A}{2} = r/\rho_A$, we obtain that the mixtilinear radii ρ_A, ρ_B, ρ_C are roots of the polynomial

$$s^2t^3 - r(s^2 + (4R+r)^2)t^2 + 8Rr^2(4R+r)t - 16R^2r^3 = 0.$$

By Vieta's formulas the identities (1), (2) and (3) follow. The last identity (4) follows from

$$\frac{1}{\rho_A} + \frac{1}{\rho_B} + \frac{1}{\rho_C} = \frac{1}{r}\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right)$$

and the polynomial (5). □

We note that one can use the identity (1) to give a nice linear bound for the sum of the mixtilinear radii only in terms of R and r

$$4r \leqslant \rho_A + \rho_B + \rho_C \leqslant R + 2r, \tag{6}$$

which is sharper than $4r \leqslant \rho_A + \rho_B + \rho_C \leqslant (5R+6r)/4$, given in [12]. The LHS of (6) follows easily from the convexity of $f(x) = \sec^2\frac{x}{2}$ and Jensen's inequality. For the RHS we shall make use of the Gerretsen inequality $s^2 \geqslant r(16R - 5r)$, [4, 5]. By (1), the RHS of (6)

$$r\left(1 + \left(\frac{4R+r}{s}\right)^2\right) \leqslant R + 2r$$

is equivalent to $s^2(R+r) \geqslant r(4R+r)^2$. By Gerretsen's inequality and Euler's inequality $R \geqslant 2r$,

$$\begin{aligned} s^2(R+r) &\geqslant r(16R-5r)(R+r) \\ &= r\left[(4R+r)^2 + 3(R-2r)\right] \\ &\geqslant r(4R+r)^2, \end{aligned}$$

which proves (6), with equality for the equilateral triangle.

For the end of this note, we show how the relations for the mixtilinear radii can be used to give an interesting proof of the celebrated Finsler–Hadwiger inequality from 1937 [3, 13]

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\mathcal{A} + (a-b)^2 + (b-c)^2 + (c-a)^2, \tag{7}$$

where $\mathcal{A}$ is the area of ABC. It can be rewritten as

$$2\sqrt{3}\mathcal{A} \leqslant a(s-a) + b(s-b) + c(s-c) = 2s^2 - a^2 - b^2 - c^2.$$

By the identities $a^2 + b^2 + c^2 = 2(s^2 - 4Rr - r^2)$ and $\mathcal{A} = sr$, known since Euler's times (see, e.g., [2, §11, §19]), the Finsler–Hadwiger inequality is equivalent to

$$s \leqslant (4R + r)/\sqrt{3}. \tag{8}$$

We can apply the AM-HM inequality to (1) and (4) to obtain

$$(\rho_A + \rho_B + \rho_C)\left(\frac{1}{\rho_A} + \frac{1}{\rho_B} + \frac{1}{\rho_C}\right) \geqslant 9,$$

which after substituting and rearranging is

$$s^2 \leqslant \frac{(4R + r)^3}{14R - r}.$$

It holds $3(4R + r) \leqslant 14R - r$ by Euler's inequality. Hence

$$s^2 \leqslant \frac{(4R + r)^3}{14R - r} \leqslant \frac{(4R + r)^2}{3},$$

and that is the Finsler–Hadwiger inequality (8). Equality holds for $\rho_A = \rho_B = \rho_C$, that is $\sec^2 \frac{A}{2} = \sec^2 \frac{B}{2} = \sec^2 \frac{C}{2}$, and $R = 2r$, which implies that the triangle is equilateral.

The Hadwiger–Finsler inequality (7) is actually equivalent [7] to the seemingly weaker Weitzenböck inequality

$$a^2 + b^2 + c^2 \geqslant 4\sqrt{3}\mathcal{A}.$$

One sharpening of (7) is

$$a^2 + b^2 + c^2 \geqslant 4\mathcal{A}\sqrt{3 + \frac{R - 2r}{R}} + (a-b)^2 + (b-c)^2 + (c-a)^2,$$

which is equivalent to Kooi's inequality [8, 9].

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