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Characteristics of modulo one sequences

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1 Introduction

Let $m_1, m_2, \ldots, m_t$ be a sequence of positive integers, pairwise relatively prime and $m_i \geq 2$ for all $i = 1, 2, \ldots, t$. Set $M = m_1 m_2 \cdots m_t$ and $M_i = \frac{M}{m_i}$ for all $i = 1, 2, \ldots, t$ (notations introduced here will be used throughout the paper).

Ist p eine Primzahl, so folgt aus dem kleinen Satz von Fermat, dass $1^{p-1} + 2^{p-1} + 3^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p}$. Giuseppe Giuga vermutete 1950, dass umgekehrt aus dieser Kongruenz folgt, dass p prim ist. Giugas Vermutung ist bis heute offen. In der vorliegenden Arbeit werden nun Modulo-Eins Folgen studiert: Das sind paarweise teilerfremde Zahlen $m_1, m_2, \ldots, m_t$ mit der Eigenschaft, dass $m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_t \equiv 1 \pmod{m_i}$ für alle $i = 1, 2, \ldots, t$. Es wird gezeigt, dass diese Folgen charakterisiert werden durch die Eigenschaft dass $1/m_1 + 1/m_2 + \cdots + 1/m_t - 1/M$ eine natürliche Zahl ist, wobei $M = m_1 m_2 \cdots m_t$. Somit sind diese Folgen Spezialfälle von Giuga-Folgen, die bei der Untersuchung der Giuga-Vermutung eine zentrale Rolle spielen. Die Autoren klassifizieren die Modulo-Eins Folgen basierend auf deren Länge und stossen auf interessante Eigenschaften. Insbesondere existieren Modulo-Eins Folgen beliebiger Länge.

The notion of working simultaneously with modulo m_i to speed up computer arithmetic is an old idea that goes back to the 1950s and some problems of this type are solved with the use of the Chinese Remainder Theorem and structure theory for finite Abelian groups. For example, to multiply two positive integers A and B, $AB \equiv c_i \pmod{m_i}$ can be computed simultaneously for each i. Then the Chinese Remainder Theorem gives back AB as

$$AB \equiv \sum_{i=1}^{t} c_i M_i \lfloor M_i \rfloor_{m_i}^{-1} \pmod{M} \tag{1}$$

where $\lfloor M_i \rfloor_{m_i}^{-1}$ denotes the inverse, modulo m_i of M_i. If M is sufficiently large, then AB in fact equals the right side of (1) and computation would be quicker if each $\lfloor M_i \rfloor_{m_i}^{-1} = 1$. Based on this idea, we will give the following definition of the modulo one sequence.

Definition. Let $m_1, m_2, \ldots, m_t$ be a sequence of positive integers, pairwise relatively prime, $m_i \geq 2$ for all i and $m_1 < m_2 < \cdots < m_t$. Suppose

$$M_i \equiv 1 \pmod{m_i} \text{ for all } i = 1, 2, \ldots, t. \tag{2}$$

Then $m_1, m_2, \ldots, m_t$ is called a modulo one sequence of length t.

For example, it can be easily seen that the sequences 2, 3, 5 and 2, 3, 11, 13 satisfy the conditions in the definition and hence are modulo one sequences of length 3 and 4 respectively.

Several studies were done on the solutions of the congruence system (2) ([1, 5, 10]) but more attention was received on the solutions to the system satisfying $M_i \equiv -1 \pmod{m_i}$ for all $i = 1, 2, \ldots, t$ ([2–4, 6, 7, 9]). In Section 2, we give the necessary and sufficient conditions for the existence of a modulo one sequence. Based on the work done by Giuga [8], Borwein [1] introduced the Giuga sequences. From our necessary and sufficient condition, it can be concluded that Giuga sequences overlap with the modulo one sequences. Theorem 2 in [1] gives a similar necessary and sufficient condition for the existence of a Giuga sequence but it is proved only for integer sequences consisting of prime numbers whereas our proof is for sequences with any positive integers. Section 3 discusses the classification of modulo one sequences based on its length. Brenton and Joo in 1993 [5] and Borwein et al. [1] gave a complete list of modulo one sequences of length $n \leq 7$ determined by computer algorithms. The propositions given in this section give analytical proofs for the classification of length 3, 4 and 5 modulo one sequences not only confirming Brenton's discoveries but also proving that these are the only modulo one sequences of the given lengths. Brenton & Hill [4] and Janak & Skula [9] discussed some characteristics of sequences based on the congruence system $M_i \equiv -1 \pmod{m_i}$. In Section 4, we prove similar interesting characteristics for modulo one sequences. In Section 5, we will discuss the category of negative sequences based on the congruence system $M_i \equiv -1 \pmod{m_i}$, giving the necessary and sufficient condition for their existence. Besides explaining the relationship between sequences following the two congruent systems, we will also use the negative sequences in proving that there are modulo one sequences of arbitrary length t.

2 Necessary and sufficient condition

Theorem 1. *Let $m_1, m_2, \ldots, m_t$ be a sequence of positive integers such that $(m_i, m_j) = 1$ for $i \neq j$ and $m_i \geq 2$ for all i. The sequence is a modulo one sequence iff there exists a positive integer N such that*

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_t} = N + \frac{1}{m_1 m_2 \ldots m_t}.$$

Proof. Suppose the sequence $m_1, m_2, \ldots, m_t$ satisfies

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_t} = N + \frac{1}{m_1 m_2 \cdots m_t} \quad \text{for some } N \in \mathbb{N}. \tag{3}$$

Then by multiplying both sides of (3) by M gives,

$$M_1 + M_2 + \cdots + M_t = NM + 1. \tag{4}$$

For $1 \leq i \leq t$ and $i \neq j$, $M_j = m_i k_j$ for some $k_j \in \mathbb{N}$. Then by (4),

$$m_i k_1 + m_i k_2 + \cdots + m_i k_{i-1} + M_i + m_i k_{i+1} + \cdots + m_i k_t = NM + 1.$$

This implies $M_i \equiv 1 \pmod{m_i}$ and hence the sequence is a modulo one sequence. Conversely, assume $m_1, m_2, \ldots, m_t$ is a modulo one sequence. Hence $M_i \equiv 1 \pmod{m_i}$ for each i implies that $(1 - M_i) = m_i l_i$ for some $l_i \in \mathbb{Z}^-$. Hence

$$\begin{aligned}\prod_{i=1}^{t}(1 - M_i) &= \prod_{i=1}^{t} m_i l_i \\ &= \prod_{i=1}^{t} m_i K \quad \text{where } K < 0 \\ &= MK.\end{aligned}$$

This implies,

$$\prod_{i=1}^{t}(1 - M_i) \equiv 0 \pmod{M}. \tag{5}$$

Since $M \mid M_i M_j$ for $i \neq j$, expansion of the left of (5) gives $1 - \sum_{i=1}^{t} M_i \equiv 0 \pmod{M}$ and hence $1 - \sum_{i=1}^{t} M_i = MK$ for some negative integer K.

$$\begin{aligned}1 - MK &= \sum_{i=1}^{t} M_i \\ 1 + MN &= \sum_{i=1}^{t} M_i \quad \text{where } N = -K.\end{aligned}$$

Therefore, $\frac{1}{M} + N = \sum_{i=1}^{t} \frac{M_i}{M} = \sum_{i=1}^{t} \frac{1}{m_j}$ which proves the result. □

Remark. Borwein et al. in [1] defined the Giuga sequences (a finite increasing sequence of integers $[n_1, n_2, \ldots, n_m]$ is a Giuga sequence if $\sum_{i=1}^{m} \frac{1}{n_i} - \prod_{i=1}^{m} \frac{1}{n_i} \in \mathbb{N}$) related to a conjecture of Giuga [8] on primality. Combining this definition with Theorem 1 here shows that a modulo one sequence is a Giuga sequence. Theorem 2 in [1] gives a necessary and sufficient condition for the Giuga sequence which is only proved when the terms of the sequence are primes. Looking at the different modulo one sequences of length $t \leq 7$ given in Appendix 1 [9], it can be noticed that the N in Theorem 1 is always equal to 1. In Section 3, we will show that this $N \in \mathbb{N}$ is always 1 for modulo one sequences of length $t < 59$.

3 Classification of modulo one sequences based on its length

Brenton and Joo gave a complete list of modulo one sequences of length $n \leq 7$ in Appendix 1 in [5] that were determined using computer algorithms. The following propositions will confirm some of their results using analytical methods.

Proposition 2. *The only length* 3 *modulo one sequence is* 2, 3, 5.

Proof. Suppose m_1, m_2, m_3. If $m_1 \geq 3$, then since $m_1 < m_2 < m_3$ by definition, $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} < 1$ which contradicts Theorem 1. Hence $m_1 = 2$. If $m_2 \geq 4$, then $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} < 1$ contrary to Theorem 1. Hence $m_2 = 3$. From a similar argument, it can be proved that $m_3 = 5$ and therefore this proves that the only length 3 modulo one sequence is 2, 3, 5. □

In classifying the length 4 modulo one sequences, an approach similar to Proposition 2 can be used but the following proof of Proposition 4 is less tedious. We will first prove the following lemma:

Lemma 3. *Let $m_1, m_2, \ldots, m_t$ be a modulo one sequence and $t < 59$. Then* $\sum_{i=1}^{t} \frac{1}{m_i} = 1 + \frac{1}{\prod_{i=1}^{t} m_i}$.

Proof. Let $m_1, m_2, \ldots, m_t$ be a modulo one sequence of length t. Then by Theorem 1, there exists an $N \in \mathbb{N}$ such that $\sum_{i=1}^{t} \frac{1}{m_i} - \frac{1}{\prod_{i=1}^{t} m_i} = N$. Let p_i be the ith prime number. Since m_i's are pairwise relatively prime, $m_i \geq p_i$ for all i. Hence $\sum_{i=1}^{t} \frac{1}{m_i} \leq \sum_{i=1}^{t} \frac{1}{p_i}$. Therefore, $N = \sum_{i=1}^{t} \frac{1}{m_i} - \frac{1}{\prod_{i=1}^{t} m_i} \leq \sum_{i=1}^{t} \frac{1}{m_i} \leq \sum_{i=1}^{t} \frac{1}{p_i} < 2$ for $t < 59$ as $t = 58$ is the largest for which $\sum_{i=1}^{t} \frac{1}{p_i}$ is less than 2. Hence the result is obtained. □

Proposition 4. *The only length* 4 *modulo one sequences are* 2, 3, 7, 41 *and* 2, 3, 11, 13.

Proof. Suppose m_1, m_2, m_3, m_4 is a mod one sequence. From a similar argument to the proof of Proposition 2, it can be proved that $m_1 = 2$ and $m_2 = 3$. Considering the congruence system (2),

$$m_1 m_2 m_3 - 1 = m_4 x \tag{6}$$

$$m_1 m_2 m_4 - 1 = m_3 y \tag{7}$$

for some positive integers x, y where $x < y$ (since by definition of a modulo one sequence, $m_1 < m_2 < m_3 < m_4$). Set

$$A = m_1^2 m_2^2 - xy. \tag{8}$$

Then $m_3 = \frac{m_1 m_2 + x}{A}$ and $m_4 = \frac{m_1 m_2 + y}{A}$. Multiplying (6) and (7),

$$\begin{aligned} m_1^2 m_2^2 m_3 m_4 - m_1 m_2 m_3 - m_1 m_2 m_4 + 1 &= m_4 m_3 x y \\ m_1 m_2 m_3 + m_1 m_2 m_4 - 1 &= m_3 m_4 A \\ m_1 m_2 \left(\frac{1}{m_4} + \frac{1}{m_3} - \frac{1}{m_1 m_2 m_3 m_4} \right) &= A. \end{aligned}$$

Using Lemma 3, $m_1 m_2 (1 - \frac{1}{m_1} - \frac{1}{m_2}) = A$ and hence $A = m_1 m_2 - m_1 - m_2$. As m_1 and m_2 are 2, 3 respectively, $A = 1$ and hence by (8), $xy = 35$. When $x = 1$ and $y = 35$, using the definitions for m_3 and m_4 above in terms of A give $m_3 = 7$ and $m_4 = 41$. Similarly when $x = 5$ and $y = 7$, m_3 and m_4 take 11, 13 respectively. This concludes that the only length 4 modulo one sequences are 2, 3, 7, 41 and 2, 3, 11, 13. □

If m_1, m_2, m_3, m_4, m_5 is a mod one sequence of length 5, using Theorem 1 with a similar argument to the proof of Proposition 2, it can be proved that m_1, m_2 are 2, 3 respectively. In a similar approach to the proof of Proposition 4 by considering $m_1 m_2 m_3 m_4 - 1 = m_5 x$, $m_1 m_2 m_3 m_5 - 1 = m_4 y$ and $A = m_1^2 m_2^2 m_3^2 - xy$, the following proposition on the classification of length 5 sequences can be proven.

Proposition 5. 2, 3, 7, 43, 1805; 2, 3, 7, 83, 85 *and* 2, 3, 11, 17, 59 *are the only length* 5 *modulo one sequences.*

Remark. A similar method can be applied to find the exact modulo sequences of length 6 and upwards but the computations get very tedious as the length gets higher.

4 Other characteristics

Proposition 6. *Let* $m_1, m_2, \ldots, m_t$ *be a mod one sequence. Suppose* $M_t = 1 + m_t$. *Then* $m_1, m_2, \ldots, m_{t-1}, 2M_t - 1, 2M_t + 1$ *is a mod one sequence.*

Proof. Let $1 \leq i, j \leq t - 1$. Since $m_1, m_2, \ldots, m_t$ is a mod one sequence, m_i and m_j are relatively prime for $i \neq j$. From the definition of M_t, m_i is relatively prime to $2M_t - 1$ and $2M_t + 1$. Also $2M_t - 1$ and $2M_t + 1$ are relatively prime and hence all terms of the sequence $m_1, m_2, \ldots, m_{t-1}, 2M_t - 1, 2M_t + 1$ are pairwise relatively prime.

Since $1 \equiv -m_t \pmod{m_i}$ and $M_i \equiv 1 \pmod{m_i}$,

$$\begin{aligned} m_1 m_2 \cdots m_{i-1} m_{i+1}, \cdots m_{t-1} &\equiv -m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1} m_t \pmod{m_i} \\ &\equiv -M_i \pmod{m_i} \\ &\equiv -1 \pmod{m_i}. \end{aligned}$$

We also know that $2M_t - 1 \equiv -1 \pmod{m_i}$ and $2M_t + 1 \equiv 1 \pmod{m_i}$, for all $1 \leq i \leq t-1$. Therefore,

$$m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1}(2M_t - 1)(2M_t + 1) \equiv 1 \pmod{m_i}. \tag{9}$$

As $2M_t^2 + M_t \equiv 1 \pmod{2M_t - 1}$,

$$m_1 m_2 \cdots m_{t-1}(2M_t + 1) = M_t(2M_t + 1) \equiv 1 \pmod{2M_t - 1}. \tag{10}$$

Similarly as $2M_t^2 - M_t \equiv 1 \pmod{2M_t + 1}$,

$$m_1 m_2 \cdots m_{t-1}(2M_t - 1) = M_t(2M_t - 1) \equiv 1 \pmod{2M_t + 1}. \tag{11}$$

Therefore by equations (9), (10) and (11), the terms of the sequence $m_1, m_2, \ldots, m_{t-1}$, $2M_t - 1, 2M_t + 1$ satisfy the congruence system (2) required for a modulo one sequence. □

Example 4.1. The sequence 2, 3, 7, 47, 395, 779729 is a modulo one sequence of length 6 (Appendix 1, [5]). Here $M_6 = 779730 = 1 + m_6$. Hence using Proposition 6, we can construct 2, 3, 7, 47, 395, 1559459, 1559461 which is one of length 7 modulo one sequences given in Appendix 1, [5].

Proposition 7. *There are no modulo one sequences with all composite terms.*

Proof. Suppose there exists a modulo one sequence $m_1, m_2, \ldots, m_t$ such that $m_i = r_i s_i$ for all $1 \leq i \leq t$ where $1 < r_i \leq s_i < m_i$. Hence,

$$\sum_{i=1}^{t} \frac{1}{m_i} = \sum_{i=1}^{t} \frac{1}{r_i s_i} \leq \sum_{i=1}^{t} \frac{1}{r_i^2} < \sum_{n=2}^{\infty} \frac{1}{n^2}$$

and $\sum_{n=2}^{\infty} \frac{1}{n^2} = \varsigma(2) - 1 = \frac{\pi^2}{6} - 1 < 1$ where ς is the Riemann Zeta function. Therefore $\sum_{i=1}^{t} \frac{1}{m_i} < 1$, contradicting Theorem 1. Hence there is at least one prime term in every modulo one sequence. □

The next proposition characterizes the number of even and odd terms in a modulo one sequence of length $t \leq 58$.

Proposition 8. *Let $m_1, m_2, \ldots, m_t$ be a modulo one sequence where $t < 59$. Let X_j be the number of terms in the sequence which are congruent to $j \pmod 4$, for $j = 0, 1, 2, 3$. Then*

(i) *If $X_0 + X_2 = 0$ then $t > 8$ and $t \equiv 2 \pmod 4$.*

(ii) *If $X_2 = 1$ then X_1 is odd (i.e., there is at least one term congruent to $1 \pmod 4$).*

(iii) *If $X_4 = 1$ then X_3 is even and X_1 is even iff the length of the sequence is odd.*

Proof. Let $X_1 = r$. By Theorem 1 and Lemma 3,

$$\sum_{i=1}^{t} \frac{1}{m_i} = 1 + \frac{1}{M} \Leftrightarrow \sum_{i=1}^{t} M_i = M + 1. \tag{12}$$

Case (i): If $X_0 + X_2 = 0$ (i.e., no even terms in the sequence).

If all terms of the sequence are odd then $m_i \geq p_{i+1}$ where p_i denotes the ith prime for $i = 1, 2, \ldots$. Then $\sum_{i=1}^{t} \frac{1}{m_i} \leq \sum_{i=1}^{t} \frac{1}{p_{i+1}}$ and the largest value of t for which $\sum_{i=1}^{t} \frac{1}{p_{i+1}} < 1$ is 8. Hence by Theorem 1, $t > 8$. That is, there are no modulo one sequences of length less than or equal to 8 with all odd terms. If $X_1 = t$ (i.e., all terms are congruent to $1 \pmod 4$), then M and M_i for each i is congruent to $1 \pmod 4$ and hence by equation (12), $t \equiv 2 \pmod 4$. If $X_3 = t$ (i.e., all terms are congruent to $3 \pmod 4$), then $M \equiv 3^t \pmod 4$ and $M_i \equiv 3^{t-1} \pmod 4$ for each i. Hence by equation (12),

$$\begin{aligned} &\Leftrightarrow t3^{t-1} \equiv 3^t + 1 \pmod 4 \\ &\Leftrightarrow 3^{t-1}(t-3) \equiv 1 \pmod 4 \\ &\Leftrightarrow 3^{t-1}(t+1) \equiv 1 \pmod 4. \end{aligned}$$

As 1 and 3 are the only units in $\mathbb{Z}_4$, t should be even and $t \equiv 2 \pmod 4$.

Now assume that the first r terms of the sequence are congruent to $1 \pmod 4$. Then

$$M_i \equiv \begin{cases} 3^{t-r} \pmod 4, & \text{for } 1 \leq i \leq r \\ 3^{t-r-1} \pmod 4, & \text{for } r+1 \leq i \leq t \end{cases}$$

and $M \equiv 3^{t-r} \pmod 4$. Hence by (12),

$$\begin{aligned} &\Leftrightarrow r3^{t-r} + (t-r)3^{t-r-1} - 3^{t-r} \equiv 1 \pmod 4 \\ &\Leftrightarrow 3^{t-r-1}(2r + t + 1) \equiv 1 \pmod 4. \end{aligned} \tag{13}$$

Since the only units in $\mathbb{Z}_4$ are 1 and 3, by equation (14), $t \equiv 2 \pmod 4$ when r is even or odd. Therefore, if all terms of a modulo one sequence are odd then $t > 8$ and $t \equiv 2 \pmod 4$.

Case (ii): If $X_2 = 1$.

Assuming the terms of the sequence $m_1, m_2, \ldots, m_t$ are of the form

$$m_i \equiv \begin{cases} 2 \pmod 4, & \text{for } i = 1 \\ 1 \pmod 4, & \text{for } 2 \leq i \leq r+1 \\ 3 \pmod 4, & \text{for } r+2 \leq i \leq t \end{cases}$$

and hence,

$$M_i \equiv \begin{cases} 3^{t-r-1} \pmod 4, & \text{for } i = 1 \\ 2 \cdot 3^{t-r-1} \pmod 4, & \text{for } 2 \leq i \leq r+1 \\ 2 \cdot 3^{t-r-2} \pmod 4, & \text{for } r+2 \leq i \leq t \end{cases}$$

and $M \equiv 2 \cdot 3^{t-r-1} \pmod 4$. Then by (12),

$$\begin{aligned} 3^{t-r-1} + 2 \cdot r \cdot 3^{t-r-1} + 2(t-r-1)3^{t-r-2} &\equiv 2 \cdot 3^{t-r-1} + 1 \pmod 4 \\ 3^{t-r-2}(4r + 2t - 5) &\equiv 1 \pmod 4 \\ 3^{t-r-2}(2t+3) &\equiv 1 \pmod 4. \end{aligned} \tag{14}$$

If t is even, then by (15), $3^{t-r-1} \equiv 1 \pmod 4$ which implies r is odd. If t is odd, $3+2t \equiv 1 \pmod 4$ and hence by (15), $3^{t-r-2} \equiv 1 \pmod 4$ implying r to be odd. Hence when $X_1 = 1$, there is at least one odd term congruent to $1 \pmod 4$ and the number of such terms would be odd.

Case (iii): $X_4 = 1$.

Assume the terms of the sequence $m_1, m_2, \ldots, m_t$ are of the form

$$m_i \equiv \begin{cases} 0 \pmod 4, & \text{for } i = 1 \\ 1 \pmod 4, & \text{for } 2 \le i \le r+1 \\ 3 \pmod 4, & \text{for } r+2 \le i \le t \end{cases}$$

where $X_1 = r$ and $X_3 = t - r - 1$. Hence,

$$M_i \equiv \begin{cases} 3^{t-r-1} \pmod 4, & \text{for } i = 1 \\ 0 \pmod 4, & \text{for } 2 \le i \le t \end{cases}$$

and $M \equiv 0 \pmod 4$. Then by (12), $3^{t-r-1} \equiv 1 \pmod 4$ which implies $t - r - 1$ is even and hence the number of terms congruent to $3 \pmod 4$ should be even. Also $t - r - 1$ being even results that t and r are of opposite parity. □

5 Negative one sequence

As explained in the introduction, several studies discussed about the solutions for the congruence system $M_i \equiv -1 \pmod{m_i}$ for $i = 1, 2, \ldots, t$. The following definition explains that the sequence of positive terms that are solutions to this congruence system forms a negative one sequence. Janak et al. in [9], Brenton et al. in [4] and Zhenfu et al. in [7] gave the lists of all such sequences of length $t \le 9$ and discussed some characteristics of the collection of negative one sequences. We will not go into details about the characteristics but explain the connection between modulo one sequences and negative one sequences and hence prove that there exists a modulo one sequence of any arbitrary length t.

Definition. Let $m_1, m_2, \ldots, m_t$ be a sequence of positive integers, pairwise relatively prime and $m_1 \ge 2$. Suppose

$$M_i \equiv -1 \pmod{m_i} \text{ for all } i = 1, 2, \ldots, t. \tag{15}$$

Then $m_1, m_2, \ldots, m_t$ is called a negative one sequence of length t.

Using an approach similar to the proof of Theorem 2, the following theorem which gives a necessary and sufficient condition for the existence of a negative one sequence can be given.

Theorem 9. *Let $m_1, m_2, \ldots, m_t$ be a sequence of positive integers such that $(m_i, m_j) = 1$ for $i \neq j$. The sequence is a negative one sequence iff there exists a positive integer N such that*

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_t} = N - \frac{1}{m_1 m_2 \cdots m_t}.$$

This necessary and sufficient condition helps to explain several characteristics of the negative one sequences using the previous studies ([3, 4] and [7]) that discuss solutions for the equation $\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_t} + \frac{1}{m_1 m_2 \cdots m_t} = 1$.

Lemma 10. *If $m_1, m_2, \ldots, m_t$ is a modulo one sequence with $m_t = M_t - 1$ then m_1, m_2, …, m_{t-1}, $m_t + 2$ is a negative one sequence.*

Proof. Consider m_i where $1 \leq i \leq t-1$.

Claim 1: m_i and $m_t + 2$ are relatively prime.

If $(m_i, m_t + 2) = d$ then $d \mid m_t + 2$ which implies $d \mid M_t + 1$. Also since $d \mid M_t$, $d = 1$.

Claim 2: $m_1 \cdots m_{i-1} m_{i+1} \cdots m_{t-1}(m_t + 2) \equiv -1 \pmod{m_i}$ for $1 \leq i \leq t-1$.

By the definition of M_i and since $m_t = M_t - 1$, if $M_t \equiv 0 \pmod{m_i}$ then $1 + m_t \equiv 0 \pmod{m_i}$ and $M_i \equiv 1 \pmod{m_i}$ for $1 \leq i \leq t-1$. Therefore,

$$\begin{aligned}
&\Leftrightarrow M_i + m_t \equiv 0 \pmod{m_i}\\
&\Leftrightarrow m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_t + m_t \equiv 0 \pmod{m_i}\\
&\Leftrightarrow m_t(m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1} + 1) \equiv 0 \pmod{m_i}.
\end{aligned}$$

Since $(m_i, m_t) = 1$,

$$m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1} \equiv -1 \pmod{m_i}. \tag{16}$$

Hence,

$$\begin{aligned}
&m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1}(m_t + 2) + 1\\
&\quad = m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1}(M_t + 1) + 1\\
&\quad = m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1} M_t + m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1} + 1\\
&\quad \equiv 0 \pmod{m_i} \text{ by (17).}
\end{aligned}$$

This implies $m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_{t-1}(m_t + 2) \equiv -1 \pmod{m_i}$ that proves Claim 2.

Claim 3: $m_1 m_2 \cdots m_{t-1} \equiv -1 \pmod{m_t + 2}$.

Proving Claim 3 is quite straightforward as

$$M_t + 1 = m_t + 2 \quad \text{implies} \quad M_t \equiv -1 \pmod{m_t + 2}.$$

Therefore by Claims 1, 2 and 3, $m_1, m_2, \ldots, m_{t-1}, m_t + 2$ is a negative one sequence. □

Lemma 11. *If $m_1, m_2, \ldots, m_t$ is a negative one sequence then $m_1, m_2, \ldots, m_t, M - 1$ is a modulo one sequence.*

Proof. Since $m_1, m_2, \ldots, m_t$ is a negative one sequence and by the definition of M, for $i \neq j$ and $1 \leq i, j \leq t$, $(m_i, m_j) = 1$ and $(m_i, M-1) = 1$ and also $m_1, m_2, \ldots, m_t \equiv -1 \pmod{M-1}$. Let $1 \leq i \leq t$. Then

$$\begin{aligned}
&m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_t (M-1) \\
&\quad = m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_t M - m_1 m_2 \cdots m_{i-1} m_{i+1} \cdots m_t \\
&\quad \equiv 1 \pmod{m_i},
\end{aligned}$$

proving the result. □

Theorem 12. *For each $t \geq 3$, there exists a modulo one sequence $m_1, m_2, \ldots, m_t$ of length t where $m_t = M_t - 1$.*

Proof. In proving the result, we will use mathematical induction on t. For $t = 3$, the sequence 2, 3, 5 meets the requirements. Suppose the result is true for $k \geq 3$. Hence there is a modulo one sequence $m_1, m_2 \ldots, m_k$ where $m_k = M_k - 1$ and then by Lemma 10, there exists a negative one sequence $m_1, m_2 \ldots, m_{k-1}, m_k+2$. Hence by Lemma 11, $m_1, m_2 \ldots$, $m_{k-1}, m_k+2, M-1$ is a modulo one sequence where $M = m_1 m_2 \cdots m_{k-1}(m_k+2)$ proving the existence of a modulo one sequence of length $k+1$ satisfying the given condition. Therefore the result is proved for any $t \geq 3$. □

The following proposition explains that the well-known Sylvester sequence which is a positive integer sequence that begins with 2 and 3 where each term afterwards is determined by adding 1 to the product of the previous terms, is also a negative one sequence.

Proposition 13. *Let m_1, m_2, …, m_t be the first t terms of the Sylvester sequence. That is m_1, m_2 are 2 and 3 respectively and*

$$m_i = \left(\prod_{k=1}^{i-1} m_k\right) + 1$$

for $3 \leq i \leq t$. Then the sequence forms a negative one sequence.

Proof. From the definition of the Sylvester sequence, all terms are pairwise relatively prime. Now we will show that the terms satisfy the congruence system in equation (16). For all $r > i$, $m_r = m_1 m_2 \cdots m_i \cdots m_{r-1} + 1$ and hence $m_r \equiv 1 \pmod{m_i}$. Therefore for all i, if $M_i + 1 \equiv m_1 m_2 \cdots m_{i-1} + 1 \pmod{m_i}$ then $M_i + 1 \equiv m_i \pmod{m_i}$, which implies $M_i \equiv -1 \pmod{m_i}$, proving the result. □

Remark. Proposition 13 together with Lemma 11 will construct a new modulo one sequence $m_1, m_2, \ldots, m_{t-1}, M-1$ where the first $t-1$ terms gives the Sylvester sequence of length $t-1$ and $M = m_1 m_2 \cdots m_{t-1}$. For example, the sequence 2, 3, 7, 43, 1807, 3262443, 10650056950805 which is a modulo one sequence of length 7 (see Appendix 1, [5]) can be obtained using this idea as the first 6 terms gives the Sylvester sequence of length 6.

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