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Ratios, roots, and means: A quartet of convergence tests

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1 Introduction

When students are first learning about the Root Test, they often encounter examples such as

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{2n+1}{3n+5} \right)^n .$$

While it is perfectly reasonable to apply the Root Test to this series, it is not technically necessary. The Ratio Test yields the same result, although the computation is slightly more complicated:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(1 + \frac{2}{2n+1} \right)^n \left(1 + \frac{3}{3n+5} \right)^{-n} \left(\frac{2n+3}{3n+8} \right),$$

which converges to $e \cdot e^{-1} \cdot (2/3) = 2/3$. To find an example where the Root Test is conclusive but the Ratio Test is not, we need to consider a series, such as

$$\sum_{n=1}^{\infty} \frac{4 + (-1)^n \cdot 3}{5^{n-1}} = 1 + \frac{7}{5} + \frac{1}{25} + \frac{7}{125} + \cdots, \tag{1}$$

Konvergenztests für Reihen gehören zum mathematischen Handwerkszeug und sind Teil jeder Analysis-Grundvorlesung. Am gebräuchlichsten sind das Quotientenkriterium von d'Alembert und das Wurzelkriterium von Cauchy. Der Autor der vorliegenden Arbeit stellt auch die weniger bekannten Kriterien von Joseph Ludwig Raabe und von Victor Jamet vor und zeigt in welcher Weise diese vier zusammenhängen. Den Schlüssel hierzu liefert das Stolz–Cesàro Theorem: Auf diesem beruht nämlich die Verbindung zwischen den wurzelbasierten Kriterien und ihren quotientenbasierten Gegenstücken.

where the ratios of consecutive terms do not converge to a single value. Again, it is perfectly reasonable to apply the standard form of the Root Test in this case, although there is an alternate perspective that more clearly illustrates the relationship between the Ratio Test and the Root Test.

Both the Ratio Test and the Root Test can be viewed as constituting an implicit comparison with a geometric series. If one applies either test to a series $\sum_{n=1}^{\infty} ar^{n-1}$ with $a \neq 0$, one obtains the value r. For an arbitrary series $\sum_{n=1}^{\infty} a_n$, both the Ratio Test and the Root Test, if they yield a value $r \neq 1$, tell us that $\sum_{n=1}^{\infty} |a_n|$ "behaves like" the corresponding geometric series.

While it has not received as much attention, Raabe's Test (see Theorem 3 below) serves as the analogue of the Ratio Test in the context of p-series. In other words, if one applies Raabe's Test to $\sum_{n=1}^{\infty} 1/n^p$, one obtains the value p. Moreover, if a series $\sum_{n=1}^{\infty} a_n$ has value p with respect to Raabe's Test, $\sum_{n=1}^{\infty} |a_n|$ "behaves like" the corresponding p-series (unless $p = 0$ or $p = 1$). We will refer to both the Ratio Test and Raabe's Test as "ratio-based" tests, as they both relate to the ratios of consecutive terms in the series.

It is reasonable to wonder whether there might also be an analogue of the Root Test relative to p-series. Such a test does exist, although it has gone almost entirely unnoticed. We will reintroduce this test, which we call *Jamet's Test*, and examine its relationship with Raabe's Test.

At this point, we will mention a few facts that may seem unrelated to the principal topic of this article. The following result, however, turns out to be fundamental to our later work.

Theorem 1 (Stolz–Cesàro Theorem). *Let (a_n) be any sequence of real numbers and (b_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} b_n$ diverges. If a_n/b_n converges to a finite value, then*

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{b_1 + b_2 + b_3 + \cdots + b_n}$$

converges to the same value.

See either [1, pp. 149–150] or [6, pp. 76–77] for a proof of Theorem 1. Taking $b_n = 1$, we make another important observation.

Corollary 1. *Let (a_n) be any sequence of real numbers. If a_n converges to a finite value, then*

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{n} \tag{2}$$

converges to the same value.

Both of these facts will be critical in understanding the relationships among the various convergence tests.

Expression (2) is often referred to as the nth *Cesàro mean* of the sequence (a_n). Cesàro means provide a straightforward way of extending the notion of convergence to sequences which, due to some sort of oscillation, do not converge in the standard sense. If (a_n) converges according to the conventional definition, Corollary 1 shows that its Cesàro means converge to the same limit. On the other hand, it is not difficult to show that the Cesàro means of the divergent sequences $(1, 0, 1, 0, \ldots)$ and $(0, 1, 0, 1, \ldots)$ both converge to $1/2$. In other words, we say that these two sequences are *Cesàro convergent* to $1/2$.

While there are many benefits associated with this extended notion of convergence, there are also a few drawbacks. Some fundamental facts relating to convergence are invalid within this new context. For example, the familiar statement

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

does not hold for Cesàro convergence. (Take, for example, the two sequences mentioned in the previous paragraph.) Working with Cesàro means inherently involves a compromise: extending the notion of convergence to a larger collection of sequences while curtailing some of the general results relating to convergence.

A theme that appears throughout this article is that the "root-based" tests (namely the Root Test and Jamet's Test) have both a literal and an analogical connection to Cesàro means. Both tests depend in some sense on the Stolz–Cesàro Theorem. Both tests represent a way of extending a simpler test (either the Ratio Test or Raabe's Test) to series whose terms oscillate in some fashion. Furthermore, the extension from Raabe's Test to Jamet's Test eliminates at least one of the distinctive benefits of Raabe's Test.

2 The Ratio Test and the Root Test

Throughout this article, unless otherwise noted, we will assume that every series consists of nonzero terms. Although we will not restate the Ratio Test and the Root Test explicitly, we will be using the most restrictive version of both tests. In particular, we will take

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \qquad \text{and} \qquad r = \lim_{n\to\infty} |a_n|^{1/n}$$

for the Ratio Test and the Root Test respectively, where r is a finite value. The reason for this choice is that we are interested in the significance of the specific value of r, rather than just the question of convergence or divergence.

The following result is well known. See [8, Theorem 3.37] for a standard version of the proof. We will present a slightly different explanation, based on an observation made by Cruz-Uribe [3].

Theorem 2. *If $\sum_{n=1}^{\infty} a_n$ has value r with respect to the Ratio Test, then $\sum_{n=1}^{\infty} a_n$ also has value r with respect to the Root Test.*

Proof. First of all, observe that

$$|a_n|^{1/n} = \left(\left|\frac{a_1}{a_0}\right| \cdot \left|\frac{a_2}{a_1}\right| \cdot \left|\frac{a_3}{a_2}\right| \cdots \left|\frac{a_n}{a_{n-1}}\right|\right)^{1/n},$$

where we are taking a_0 to be 1. In other words, $|a_n|^{1/n}$ is the geometric mean of the first n ratios. We simply need to show that this expression converges to r whenever $|a_{n+1}/a_n|$ converges to r.

If $r > 0$, then Corollary 1 shows that both the arithmetic means

$$A_n = \frac{\left|\frac{a_1}{a_0}\right| + \left|\frac{a_2}{a_1}\right| + \left|\frac{a_3}{a_2}\right| + \cdots + \left|\frac{a_n}{a_{n-1}}\right|}{n}$$

and the harmonic means

$$H_n = \frac{n}{\left|\frac{a_0}{a_1}\right| + \left|\frac{a_1}{a_2}\right| + \left|\frac{a_2}{a_3}\right| + \cdots + \left|\frac{a_{n-1}}{a_n}\right|}$$

are converging to r. Thus the Arithmetic Mean-Geometric Mean Inequality (see [10, Problem 2.1]) shows that

$$H_n \le |a_n|^{1/n} \le A_n$$

for all n, so $|a_n|^{1/n}$ converges to r. A similar argument holds when $r = 0$, although we omit the lower bound corresponding to the harmonic mean. □

As we have already mentioned, the connection between Cesàro means and the Root Test goes beyond the technicalities of this proof. If a sequence is convergent in the standard sense, its Cesàro means converge to the limit of the sequence. Cesàro means are only useful when a sequence is diverging due to some sort of oscillatory behavior. Likewise, the Root Test yields the same information as the Ratio Test whenever $|a_{n+1}/a_n|$ converges. The Root Test is only necessary when these ratios diverge but their averages (i.e., their geometric means) approach a fixed value. The next two results illustrate this perspective.

Proposition 1. *If*

$$\lim_{k\to\infty}\left|\frac{a_{2k+1}}{a_{2k-1}}\right| \qquad \textit{and} \qquad \lim_{k\to\infty}\left|\frac{a_{2k+2}}{a_{2k}}\right|$$

both exist and have the same value s, then $\sum_{n=1}^{\infty} a_n$ has value $\sqrt{s}$ with respect to the Root Test.

Proof. Take $b_k = a_{2k-1}$ and $c_k = a_{2k}$. Theorem 2 dictates that $|b_k|^{1/k}$ and $|c_k|^{1/k}$ are both converging to s, so

$$|a_{2k-1}|^{1/(2k-1)} = \left(|b_k|^{1/k}\right)^{k/(2k-1)} \qquad \text{and} \qquad |a_{2k}|^{1/(2k)} = \left(|c_k|^{1/k}\right)^{1/2}$$

both converge to $\sqrt{s}$. Therefore $|a_n|^{1/n}$ converges to $\sqrt{s}$. □

While the following result is more restrictive, it clearly demonstrates the connection between the Root Test and geometric means.

Corollary 2. *If*

$$\lim_{k\to\infty}\left|\frac{a_{2k}}{a_{2k-1}}\right| = s_1 \qquad \textit{and} \qquad \lim_{k\to\infty}\left|\frac{a_{2k+1}}{a_{2k}}\right| = s_2,$$

then $\sum_{n=1}^{\infty} a_n$ has value $\sqrt{s_1 s_2}$ with respect to the Root Test.

Proof. Observe that

$$\left|\frac{a_{2k+1}}{a_{2k-1}}\right| = \left|\frac{a_{2k+1}}{a_{2k}}\right| \cdot \left|\frac{a_{2k}}{a_{2k-1}}\right| \quad \text{and} \quad \left|\frac{a_{2k+2}}{a_{2k}}\right| = \left|\frac{a_{2k+2}}{a_{2k+1}}\right| \cdot \left|\frac{a_{2k+1}}{a_{2k}}\right|$$

are both converging to the product $s_1 s_2$. Hence our Proposition 1 dictates that $|a_n|^{1/n}$ converges to $\sqrt{s_1 s_2}$. □

Proposition 1 and Corollary 2 can both be extended to situations where there is a more elaborate pattern. For example, if

$$\lim_{k\to\infty}\left|\frac{a_{3k+1}}{a_{3k-2}}\right| = \lim_{k\to\infty}\left|\frac{a_{3k+2}}{a_{3k-1}}\right| = \lim_{k\to\infty}\left|\frac{a_{3k+3}}{a_{3k}}\right| = s,$$

then $\sum_{n=1}^{\infty} a_n$ has value $\sqrt[3]{s}$ with respect to the Root Test.

Proposition 1 and Corollary 2 both show that series (1) has value $1/5$ with respect to the Root Test. The following example is slightly more substantial.

Example 1. Consider the series

$$\sum_{n=1}^{\infty} \frac{\left((n+(-1)^{n-1})!\right)^2}{(2n+(-1)^n)!} = \frac{(2!)^2}{1!} + \frac{(1!)^2}{5!} + \frac{(4!)^2}{5!} + \frac{(3!)^2}{9!} + \cdots.$$

Note that

$$\left|\frac{a_{2k+1}}{a_{2k-1}}\right| = \frac{\left((2k+2)!\right)^2}{(4k+1)!} \cdot \frac{(4k-3)!}{\left((2k)!\right)^2} = \frac{\left((2k+2)(2k+1)\right)^2}{(4k+1)(4k)(4k-1)(4k-2)}$$

and

$$\left|\frac{a_{2k+2}}{a_{2k}}\right| = \frac{\left((2k+1)!\right)^2}{(4k+5)!} \cdot \frac{(4k+1)!}{\left((2k-1)!\right)^2} = \frac{\left((2k+1)(2k)\right)^2}{(4k+5)(4k+4)(4k+3)(4k+2)}$$

both converge to $1/16$. Thus Proposition 1 shows that $\sum_{n=1}^{\infty} a_n$ has value $1/4$ with respect to the Root Test, so the series converges. One cannot apply Corollary 2 in this instance, since

$$\left|\frac{a_{2k}}{a_{2k-1}}\right| = \frac{\left((2k-1)!\right)^2}{(4k+1)!} \cdot \frac{(4k-3)!}{\left((2k)!\right)^2} = \frac{1}{(4k+1)(4k)(4k-1)(4k-2)(2k)^2}$$

is converging to 0 and

$$\left|\frac{a_{2k+1}}{a_{2k}}\right| = \frac{\left((2k+2)!\right)^2}{(4k+1)!} \cdot \frac{(4k+1)!}{\left((2k-1)!\right)^2} = \left((2k+2)(2k+1)(2k)\right)^2$$

is diverging to ∞.

3 Raabe's Test and Jamet's Test

Raabe's Test, the ratio-based test corresponding to p-series, is thoroughly discussed in [4]. While there are several versions of the test, we will be referring to the following result.

Theorem 3 (Raabe's Test)**.** *Suppose $\sum_{n=1}^{\infty} a_n$ is a series consisting of nonzero terms, for which*

$$p = \lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \tag{3}$$

exists (as a finite value). If $p > 1$, the series converges absolutely. If $p < 0$, the series diverges. If $0 \leq p < 1$, the series is either conditionally convergent or divergent. If $p = 1$, the test provides no information.

For the sake of completeness, we will sketch a proof of Raabe's Test; see [4, Theorem 1] for more details. If $p > 1$, the Monotone Convergence Test shows that the sequence $(n|a_n|)$ converges, from which it follows that the telescoping series

$$\sum_{n=1}^{\infty}\left(n|a_n| - (n+1)|a_{n+1}|\right),$$

converges. Since

$$\left(\frac{p-1}{2}\right)|a_{n+1}| < n|a_n| - (n+1)|a_{n+1}|$$

for n sufficiently large, we conclude that $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $p < 0$, one can show that (a_n) does not converge to 0, so $\sum_{n=1}^{\infty} a_n$ must diverge. If $0 \leq p < 1$, one can find a positive constant M such that

$$\frac{M}{n} \leq |a_n|$$

for n sufficiently large. Thus $\sum_{n=1}^{\infty} |a_n|$ is divergent, so $\sum_{n=1}^{\infty} a_n$ is either conditionally convergent or divergent. Furthermore, there are specific examples with $0 \leq p \leq 1$ for which $\sum_{n=1}^{\infty} a_n$ converges conditionally and examples for which $\sum_{n=1}^{\infty} a_n$ diverges; likewise, there are examples with $p = 1$ for which $\sum_{n=1}^{\infty} a_n$ converges absolutely (see [4, Example 8]).

There is another formulation of Raabe's Test that will be useful in our discussion:

$$p = \lim_{n\to\infty} n\log\left|\frac{a_n}{a_{n+1}}\right|, \tag{4}$$

where log denotes the natural logarithm. Applying the inequalities $(x-1)/x \leq \log x \leq x - 1$, we see that

$$\left|\frac{a_{n+1}}{a_n}\right| n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \leq n\log\left|\frac{a_n}{a_{n+1}}\right| \leq n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$$

and

$$n \log \left| \frac{a_n}{a_{n+1}} \right| \leq n \left(\left| \frac{a_n}{a_{n+1}} \right| - 1 \right) \leq \left| \frac{a_n}{a_{n+1}} \right| n \log \left| \frac{a_n}{a_{n+1}} \right|$$

for all n. For either (3) or (4) to exist, it is necessary that $|a_n/a_{n+1}|$ converges to 1. Consequently (3) and (4) are equivalent: one exists if and only if the other exists, in which case the two values are identical. Theorem 3, restated in terms of (4), is sometimes called *Schlömilch's Test* [7], although we will simply treat it as an alternate form of Raabe's Test.

Our goal at this point is to identify an analogue of the Root Test corresponding to p-series. In other words, we need to find a convergence test that yields the value p for a series $\sum_{n=1}^{\infty} 1/n^p$, but which is based on individual terms rather than the ratios of consecutive terms. One possible choice, which serves our purpose remarkably well, is

$$p = \lim_{n \to \infty} \frac{-\log |a_n|}{\log n}.$$

If such a limit exists, then for any positive ε there is a natural number N such that

$$\left| \frac{-\log |a_n|}{\log n} - p \right| < \varepsilon$$

whenever $n \geq N$. In other words,

$$p_1 < \frac{-\log |a_n|}{\log n} < p_2$$

for $n \geq N$, where $p_1 = p - \varepsilon$ and $p_2 = p + \varepsilon$, from which it follows that

$$\frac{1}{n^{p_2}} < |a_n| < \frac{1}{n^{p_1}}.$$

If $p > 1$, we may choose ε so that $p_1 > 1$. If $p < 0$, we may choose ε so that $p_2 < 0$. If $0 \leq p < 1$, we may choose ε so that $p_2 < 1$. Hence, by comparing $\sum_{n=1}^{\infty} |a_n|$ with either $\sum_{n=1}^{\infty} 1/n^{p_1}$ or $\sum_{n=1}^{\infty} 1/n^{p_2}$, we obtain the formal statement of our test.

Theorem 4 (Jamet's Test). *Suppose $\sum_{n=1}^{\infty} a_n$ is a series consisting of nonzero terms, for which*

$$p = \lim_{n \to \infty} \frac{-\log |a_n|}{\log n} \tag{5}$$

exists (as a finite value). If $p > 1$, the series converges absolutely. If $p < 0$, the series diverges. If $0 \leq p < 1$, the series is either conditionally convergent or divergent. If $p = 1$, the test provides no information.

The inconclusiveness when $p = 1$ follows from the relationship between this test and Raabe's Test, which we will discuss momentarily.

Although this test has not been widely used, it is discussed briefly by Bromwich [1, p. 45]. There are several other ways of presenting the test. Again applying the inequalities $(x-1)/x \leq \log x \leq x - 1$, we see that

$$|a_n|^{-1/n} n\left(|a_n|^{1/n} - 1\right) \leq \log |a_n| \leq n\left(|a_n|^{1/n} - 1\right)$$

and

$$\log|a_n| \leq n\left(|a_n|^{1/n} - 1\right) \leq |a_n|^{1/n}\log|a_n|.$$

If (5) exists as a finite value, then

$$\frac{\log|a_n|}{n} = \frac{\log|a_n|}{\log n} \cdot \frac{\log n}{n}$$

must converge to 0, so $|a_n|^{1/n}$ must be converging to 1. Thus (5) may equivalently be rewritten

$$p = \lim_{n\to\infty} \frac{n\left(1 - |a_n|^{1/n}\right)}{\log n}. \tag{6}$$

A similar argument leads to another equivalent expression:

$$p = \lim_{n\to\infty} \frac{|a_n|^{-1/n} - 1}{n^{1/n} - 1}. \tag{7}$$

A test involving (5) was introduced by Cauchy (see [2, pp. 137–140]), but the name *Cauchy's Test* would not be sufficiently distinctive. Bromwich attributes a test involving (6) to Victor Jamet [5], although Jamet only hinted at such a result. We will refer to Theorem 4, stated in terms of (5), (6), or (7), as *Jamet's Test*, a name that was actually proposed almost a century ago (see [9, pp. 113–114]).

Example 2. For any real number p, one can use (6) to show that

$$\sum_{n=1}^{\infty}\left(1 - \frac{p\log n}{n}\right)^n$$

has value p with respect to Jamet's Test (see [1, p. 45]). Since the terms are eventually positive, we conclude that the series converges for $p > 1$ and diverges for $p < 1$. The series also diverges when $p = 1$, which can be shown by applying the Limit Comparison Test with respect to the harmonic series.

While it may not be especially useful from a computational perspective, (7) has two conceptual benefits. First of all, this form clearly illustrates the similarity between Jamet's Test and the Root Test. Secondly, if one replaces $|a_n|^{1/n}$ with $|a_{n+1}/a_n|$ and $n^{1/n}$ with $(n+1)/n$, one obtains the standard version of Raabe's Test. These observations suggest that Jamet's Test is indeed the analogue of the Root Test relative to p-series. Our next result firmly establishes the relationship between Raabe's Test and Jamet's Test, as well as the connection between Jamet's Test and the Stolz–Cesàro Theorem.

Theorem 5. *If $\sum_{n=1}^{\infty} a_n$ has value p with respect to Raabe's Test, then $\sum_{n=1}^{\infty} a_n$ also has value p with respect to Jamet's Test.*

Proof. Suppose that

$$n\log\left|\frac{a_n}{a_{n+1}}\right| = \frac{\log\left|\dfrac{a_n}{a_{n+1}}\right|}{\dfrac{1}{n}}$$

converges to p. Applying the Stolz–Cesàro Theorem (Theorem 1), we see that

$$\frac{\log\left|\frac{a_1}{a_2}\right| + \log\left|\frac{a_2}{a_3}\right| + \log\left|\frac{a_3}{a_4}\right| + \cdots + \log\left|\frac{a_{n-1}}{a_n}\right|}{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}}$$

also converges to p. The expression above can be rewritten

$$\frac{\log|a_1| - \log|a_n|}{\log(n-1) + \gamma_{n-1}},$$

where γ_n converges to Euler's constant γ. Therefore

$$\frac{-\log|a_n|}{\log n} = \left(\frac{\log|a_1| - \log|a_n|}{\log(n-1) + \gamma_{n-1}}\right) \cdot \left(\frac{\log(n-1) + \gamma_{n-1}}{\log n}\right) - \frac{\log|a_1|}{\log n}$$

converges to p. □

Assuming that we have established the behavior of p-series, we may view Raabe's Test (Theorem 3) as a corollary of Theorem 5 and Jamet's Test (Theorem 4). Moreover, if we know the value of any series with respect to Raabe's Test, that information can automatically be transferred to Jamet's Test. In particular, the inconclusiveness of Jamet's Test when $p = 1$ follows from the inconclusiveness of Raabe's Test (see [4, Example 8]).

Example 3. Even though we already know the value of $\sum_{n=1}^{\infty} 1/n^p$ with respect to Raabe's Test, it is not difficult to calculate its value with respect to Jamet's Test. Applying (5), we see that

$$\frac{-\log|1/n^p|}{\log n} = \frac{p\log n}{\log n} = p$$

for $n \geq 2$. Alternatively, we may use (7) to obtain

$$\frac{|1/n^p|^{-1/n} - 1}{n^{1/n} - 1} = \frac{\left(n^{1/n}\right)^p - 1}{n^{1/n} - 1}.$$

Since $n^{1/n}$ converges to 1, the expression above converges to p.

Similarly, it is not difficult to compute the value of $\sum_{n=1}^{\infty} \log(n+1)$ with respect to Jamet's Test. Since $\log x/x$ converges to 0 as x tends to ∞, the expression

$$\frac{-\log\left|\log(n+1)\right|}{\log n} = \frac{-\log\left(\log(n+1)\right)}{\log(n+1)} \cdot \frac{\log(n+1)}{\log n}$$

converges to 0.

We are now in a position to prove an analogue of Proposition 1. While this result is stated in terms of the Schlömilch form of Raabe's Test, it is equally valid in terms of the standard form.

Proposition 2. *If*

$$\lim_{k\to\infty}(2k-1)\log\left|\frac{a_{2k-1}}{a_{2k+1}}\right| \qquad \textit{and} \qquad \lim_{k\to\infty} 2k\log\left|\frac{a_{2k}}{a_{2k+2}}\right|$$

both exist and have the same value q, then $\sum_{n=1}^{\infty} a_n$ has value $q/2$ with respect to Jamet's Test.

Proof. Take $b_k = a_{2k-1}$ and $c_k = a_{2k}$. Since $|a_{2k-1}/a_{2k+1}|$ must be converging to 1, we see that

$$k\log\left|\frac{b_k}{b_{k+1}}\right| = \frac{1}{2}\left((2k-1)\log\left|\frac{a_{2k-1}}{a_{2k+1}}\right| + \log\left|\frac{a_{2k-1}}{a_{2k+1}}\right|\right)$$

and

$$k\log\left|\frac{c_k}{c_{k+1}}\right| = \frac{1}{2}\left(2k\log\left|\frac{a_{2k}}{a_{2k+1}}\right|\right)$$

are both converging to $q/2$. Therefore Theorem 5 shows that

$$\frac{-\log|b_k|}{\log k} \qquad \text{and} \qquad \frac{-\log|c_k|}{\log k}$$

both converge to $q/2$. Thus

$$\frac{-\log|a_{2k-1}|}{\log(2k-1)} = \frac{-\log|b_k|}{\log k}\cdot\frac{\log k}{\log(2k-1)}$$

and

$$\frac{-\log|a_{2k}|}{\log 2k} = \frac{-\log|c_k|}{\log k}\cdot\frac{\log k}{\log 2k}$$

are both converging to $q/2$, so

$$\frac{-\log|a_n|}{\log n}$$

converges to $q/2$. □

While the Root Test is associated with geometric means, the analogous results for Jamet's Test are stated in terms of arithmetic means.

Corollary 3. *If*

$$\lim_{k\to\infty}(2k-1)\log\left|\frac{a_{2k-1}}{a_{2k}}\right| = q_1 \qquad \textit{and} \qquad \lim_{k\to\infty} 2k\log\left|\frac{a_{2k}}{a_{2k+1}}\right| = q_2,$$

then $\sum_{n=1}^{\infty} a_n$ has value $(q_1+q_2)/2$ with respect to Jamet's Test.

Proof. Since $|a_{2k-1}/a_{2k}|$ and $|a_{2k}/a_{2k+1}|$ must both converge to 1, we see that

$$\begin{aligned}(2k-1)\log\left|\frac{a_{2k-1}}{a_{2k+1}}\right| &= (2k-1)\log\left|\frac{a_{2k-1}}{a_{2k}}\cdot\frac{a_{2k}}{a_{2k+1}}\right| \\ &= (2k-1)\log\left|\frac{a_{2k-1}}{a_{2k}}\right| + 2k\log\left|\frac{a_{2k}}{a_{2k+1}}\right| - \log\left|\frac{a_{2k}}{a_{2k+1}}\right|\end{aligned}$$

and

$$2k \log \left| \frac{a_{2k}}{a_{2k+2}} \right| = 2k \log \left| \frac{a_{2k}}{a_{2k+1}} \cdot \frac{a_{2k+1}}{a_{2k+2}} \right|$$
$$= 2k \log \left| \frac{a_{2k}}{a_{2k+1}} \right| + (2k+1) \log \left| \frac{a_{2k+1}}{a_{2k+2}} \right| - \log \left| \frac{a_{2k+1}}{a_{2k+2}} \right|$$

are both converging to $q_1 + q_2$. Hence Proposition 2 dictates that $\sum_{n=1}^{\infty} a_n$ has value $(q_1 + q_2)/2$ with respect to Jamet's Test. □

As one would expect, Proposition 2 and Corollary 3 can both be extended to a wider array of patterns. For example, if

$$\lim_{k\to\infty} (3k-2) \log \left| \frac{a_{3k-2}}{a_{3k+1}} \right| = \lim_{k\to\infty} (3k-1) \log \left| \frac{a_{3k-1}}{a_{3k+2}} \right| = \lim_{k\to\infty} 3k \log \left| \frac{a_{3k}}{a_{3k+3}} \right| = q,$$

then $\sum_{n=1}^{\infty} a_n$ has value $q/3$ with respect to Jamet's Test.

Example 4. Let $a_n = 1 + (-1)^{n-1}/n$, so that

$$(2k-1)\left(\left| \frac{a_{2k-1}}{a_{2k}} \right| - 1 \right) = (2k-1)\left(\left| \frac{1 + \frac{1}{2k-1}}{1 - \frac{1}{2k}} \right| - 1 \right) = \frac{4k-1}{2k-1}$$

and

$$2k\left(\left| \frac{a_{2k}}{a_{2k+1}} \right| - 1 \right) = 2k\left(\left| \frac{1 - \frac{1}{2k}}{1 + \frac{1}{2k+1}} \right| - 1 \right) = -\frac{4k+1}{2k+2}.$$

The first expression converges to 2 and the second expression converges to -2. Hence the series has no value with respect to Raabe's Test, but has value 0 with respect to Jamet's Test.

4 Comparisons and contrasts

In many cases, Raabe's Test and Jamet's Test provide exactly the same information. As one would expect, Jamet's Test often turns out to be more versatile. There is at least one situation, though, where Raabe's Test is more sensitive than Jamet's Test. This section will explore these relationships in more detail.

Our first two observations are essentially identical to the corresponding results for Raabe's Test (see [4, Propositions 2 and 6]).

Proposition 3. *Suppose $\sum_{n=1}^{\infty} a_n$ has value p with respect to Jamet's Test. If $p > 0$, the terms a_n are converging to* 0.

Proof. As noted in the previous section, we may choose $p_1 = p - \varepsilon$ so that $p_1 > 0$ and

$$|a_n| < \frac{1}{n^{p_1}}$$

for n sufficiently large. Thus the terms a_n are converging to 0. □

Proposition 4. *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have values p and q with respect to Jamet's Test. The series $\sum_{n=1}^{\infty} a_n b_n$ has value $p+q$ with respect to Jamet's Test. For any real number k, the series $\sum_{n=1}^{\infty} a_n^k$ (whenever it is defined) has value kp with respect to Jamet's Test.*

Proof. Observe that

$$\frac{-\log|a_n b_n|}{\log n} = \frac{-\log|a_n|}{\log n} + \frac{-\log|b_n|}{\log n}$$

and

$$\frac{-\log|a_n^k|}{\log n} = \frac{-k\log|a_n|}{\log n},$$

from which our assertions follow directly. □

The next proposition is considerably stronger than the corresponding result for Raabe's Test.

Proposition 5. *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ have values p and q with respect to Jamet's Test. If $p < q$, then $\sum_{n=1}^{\infty} (a_n + b_n)$ has value p with respect to Jamet's Test.*

Proof. Note that

$$\frac{-\log|a_n + b_n|}{\log n} = \frac{-\log|a_n| - \log|1 + b_n/a_n|}{\log n}. \tag{8}$$

Proposition 4 shows that b_n/a_n has value $q - p > 0$ with respect to Jamet's Test, so Proposition 3 shows that b_n/a_n is converging to 0. Therefore $\log|1 + b_n/a_n|$ is converging to 0, which means that (8) converges to p. □

The analogous result for Raabe's Test (see [4, Proposition 9]) is limited to the situation where a_n and b_n either have the same sign for all n or have opposite signs for all n. Otherwise, it is possible that $\sum_{n=1}^{\infty}(a_n + b_n)$ has no value with respect to Raabe's Test. For instance, take $a_n = 1$ and $b_n = (-1)^{n-1}/n$ (see Example 4 above). Proposition 5 illustrates the robustness of Jamet's Test, in that Jamet's Test can deal with a greater degree of oscillation than Raabe's Test. The following observation, whose proof is evident from (5), further exemplifies this principle.

Proposition 6. *If there are positive constants m and M such that $m \le |a_n| \le M$ for all n, the series $\sum_{n=1}^{\infty} a_n$ has value 0 with respect to Jamet's Test.*

While a value of 0 with respect to Jamet's Test is not particularly informative in this case, combining Proposition 6 with other results can be quite useful.

Example 5. Consider the series

$$\sum_{n=1}^{\infty} \sqrt{\frac{2+\cos n}{\left(n^r+(-1)^{n-1}n^{r-1}\right)\left(\log(n+1)\right)^s}}$$

for any real numbers r and s. Applying Propositions 4, 5, and 6, along with the results from Example 3, we see that this series has value $r/2$ with respect to Jamet's Test. Hence the series converges when $r > 2$ and diverges when $r < 2$. The behavior of the series when $r = 2$ depends on the value of s. It is important to note that Raabe's Test is not directly applicable to this series.

The advantage of Jamet's Test with regard to the oscillation of terms gives rise to a complementary disadvantage. Unlike Raabe's Test, Jamet's Test is not especially useful for determining the behavior of an alternating series. For $a_n > 0$, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges whenever the series has value $0 < p \leq 1$ with respect to Raabe's Test (see [4, Theorem 3]). No such result holds for Jamet's Test, as we can see from the following example.

Example 6. Take $0 < p \leq 1$ and let $a_n = \left(2+(-1)^n\right)/n^p$, so that

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n-1}a_n &= 1 - \frac{3}{2^p} + \frac{1}{3^p} - \frac{3}{4^p} + \frac{1}{5^p} - \frac{3}{6^p} + \cdots \\ &= \left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p} - \frac{1}{6^p} + \cdots\right) \\ &\quad - 2^{1-p}\left(0 + 1 + 0 + \frac{1}{2^p} + 0 + \frac{1}{3^p} + \cdots\right).\end{aligned}$$

Since it can be written as the difference of a convergent series and a divergent series, the original alternating series diverges. Nevertheless,

$$\frac{-\log a_n}{\log n} = \frac{-\log\left(2+(-1)^n\right) + p\log n}{\log n}$$

converges to p, so the series has value p with respect to Jamet's Test.

The issue here is that a positive value with respect to Raabe's Test forces a_n to decrease monotonically for n sufficiently large, so that $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ automatically satisfies the hypotheses of the Alternating Series Test. Jamet's Test, on the other hand, imposes no such restriction. This situation should not come as a surprise, since Jamet's Test pertains to the long-term behavior of a series rather than the relationship between consecutive terms.

5 Conclusions

While we strongly advocate featuring Raabe's Test more prominently in the undergraduate curriculum, it may be impractical to include Jamet's Test as well. Nevertheless, it is helpful to know that there is a convergence test that fills this particular niche. The most important observation in this article is that the root-based tests serve as a form of averaging for their ratio-based counterparts. This insight would enrich any presentation of convergence tests, even at the level of a first-year calculus course.

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