Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 75 (2020)
Heft: 2

Artikel: Constructions of isospectral circulant graphs
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DOI: https://doi.org/10.5169/seals-880882

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Elem. Math. 75 (2020) 45 – 57
0013-6018/20/020045-13
DOI 10.4171/EM/404

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Elemente der Mathematik

Constructions of isospectral circulant graphs

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Katja Mönius studied mathematics at the University of Würzburg, where she presently is also doing her PhD. She is mainly interested in algebraic graph theory and number theory, particularly in spectra of graphs and their algebraic and number-theoretical properties.

1 Introduction

We consider circulant graphs. A graph is said to be *circulant* if it has a circulant adjacency matrix, that is a matrix of the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

with entries $c_j \in \{0, 1\}$. Since each row is a cyclic shift of the first row, such a matrix is completely determined by specifying its first row. Therefore, with every circulant graph

Konstruktionen isospektraler Graphen, d.h. Graphen, deren Adjazenzmatrizen das gleiche charakteristische Polynom besitzen, gehen auf die berühmte Frage "Can you hear the shape of a drum?" von Mark Kac von 1966 zurück. Tatsächlich finden isospektrale Graphen heutzutage Anwendung in der Chemie und sind seitdem auch Bestandteil mathematischer Forschung. Herkömmliche Konstruktionsmethoden sind jedoch nicht auf sogenannte zirkuläre Graphen anwendbar. Zirkuläre Graphen sind Graphen mit einer zirkulären Adjazenzmatrix und lassen sich eindeutig über Teilmengen von Restklassenringen definieren. Da die Eigenwerte zirkulärer Graphen Summen von Einheitswurzeln sind, hängt das Problem der Konstruktion isospektraler Graphen eng mit der Frage zusammen, wann Summen über unterschiedliche Einheitswurzeln gleich sind. In der vorliegenden Arbeit werden aus der Literatur bekannte Beispiele zirkulärer Graphen verallgemeinert und liefern damit auch Konstruktionen neuer solcher Beispiele.

we can associate a set $S \subseteq \mathbb{Z}_n$ (where $\mathbb{Z}_n$ denotes the ring of integers modulo n) of the positions of non-zero entries of the first row of the adjacency matrix of the graph (i.e., the set of all indices i with $c_i = 1$). Respectively, we denote by $\langle S \rangle_n$ the corresponding graph and call S the *connection set* of $\langle S \rangle_n$. Two vertices $x, y \in \mathbb{Z}_n$ are adjacent in $\langle S \rangle_n$ if and only if $x - y \in S$. All graphs in this paper are assumed to be without loops or multi-edges. Note that a circulant graph $\langle S \rangle_n$ is undirected if and only if $S \equiv -S \bmod n$. These notations are adopted from Mans et al. [10, 11].

Circulant graphs are Cayley graphs on cyclic groups and play an important role, for example, in computer science [3] and physics [16].

The isomorphism problem for general graphs is known to be in NP, which means that, to now, there is no efficient algorithm running in polynomial time which decides whether two graphs are isomorphic or not. However, in 1967 Ádám [1] conjectured that this problem is *easy* for circulant graphs. We can easily see that if there is an $m \in \mathbb{Z}_n^*$ (where $\mathbb{Z}_n^*$ denotes the group of units in $\mathbb{Z}_n$) such that $S \equiv mT = \{m \cdot t \mid t \in T\} \bmod n$, then $\langle S \rangle_n$ and $\langle T \rangle_n$ are isomorphic. If such m exists, we say that S and T are proportional, and write $S \sim T$. Ádám conjectured that the converse is also true, i.e., that for all isomorphic circulant graphs $\langle S \rangle_n$, $\langle T \rangle_n$ with connection sets $S, T \subseteq \mathbb{Z}_n$ we already have that $S \sim T$. But in 1970 Elspas and Turner [5] found a quite simple counterexample. Ever since, further counterexamples were given, for example, by Alspach and Parsons [2] or Mans, Pappalardi and Shparlinski [11]. However, in 2004 Muzychuk [12] proved that the isomorphism problem for circulant graphs is *not* NP-hard. He constructed an algorithm which recognizes isomorphisms between circulant graphs in polynomial time.

In this paper, we study a somehow weaker property of circulant graphs, namely isospectrality rather than isomorphicity. Two graphs are called *isospectral* if they have the same spectrum, i.e., their corresponding adjacency matrices have the same multi-set of eigenvalues. Isospectral graphs are also of great interest, especially in chemistry for the construction of isospectral molecules [9], [14], [15]. Clearly, every pair of isomorphic circulant graphs provides a pair of isospectral circulant graphs. But, as shown by Brown in [4], there exist infinitely many isospectral non-isomorphic circulant graphs as well. The rather general methods for constructing isospectral graphs presented by Godsil and McKay [8], however, do not apply to circulant graphs. Therefore, the goal of this paper is to investigate isospectrality of circulant graphs. Given connection sets $S, T \subseteq \mathbb{Z}_n$, we introduce some techniques to decide whether the corresponding graphs $\langle S \rangle_n$ and $\langle T \rangle_n$ are isospectral or not without determining their spectra explicitly. These techniques arise from generalizing known examples of isospectral circulant graphs and provide constructions of new examples as well. In Section 2, we investigate some necessary conditions for isospectral circulant graphs. In Section 3, we provide some sufficient conditions or rather give some explicit constructions to find isospectral circulant graphs. Our constructions yield non-isomorphic as well as isomorphic graphs, even though, our aim is to find *non-trivial* examples of isospectral circulant graphs, i.e., circulant graphs with non-proportional connection sets. Thus, every pair of isospectral graphs which arises from one of our constructions is either non-isomorphic or provides a counterexample to Ádám's conjecture. Furthermore, we show that every such pair relates to vanishing sums of roots of unity.

2 General observations and notations

It is well known that the spectrum of a circulant graph $\langle S\rangle_n$ is given by

$$\operatorname{spec}(\langle S\rangle_n) = \left\{ \sum_{j\in S} \mathrm{e}\left(\frac{j}{n}\right)^k \mid 0 \le k \le n-1 \right\},$$

where here and in the following $\mathrm{e}(x)$ denotes $\exp(2\pi i x)$. A proof for this can be found in the book of Zhang [17]. Therefore, two circulant graphs $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral if and only if for every $l \in \mathbb{Z}_n$ there is an $l' \in \mathbb{Z}_n$ such that

$$\sum_{s\in S} \mathrm{e}\left(\frac{s}{n}\right)^l = \sum_{t\in T} \mathrm{e}\left(\frac{t}{n}\right)^{l'}$$

and vice versa, i.e., there exists a bijection (permutation) $\sigma : \mathbb{Z}_n \to \mathbb{Z}_n$ such that

$$\sum_{s\in S} \mathrm{e}\left(\frac{s}{n}\right)^l = \sum_{t\in T} \mathrm{e}\left(\frac{t}{n}\right)^{\sigma(l)}, \quad \text{resp.} \quad \sum_{s\in S} \mathrm{e}\left(\frac{s}{n}\right)^{\sigma^{-1}(l)} = \sum_{t\in T} \mathrm{e}\left(\frac{t}{n}\right)^l$$

for all $l \in \mathbb{Z}_n$. We shall call such a bijection a *spectral bijection* of S and T. Note that for $k = 0$ the corresponding eigenvalue of $\langle S\rangle_n$ equals the number of elements in S, and that two circulant graphs have the same number of edges if and only if their connection sets have the same cardinality. This in combination with the fact that isospectral graphs have the same number of vertices and edges yields $\sigma(0) = 0$ for every spectral bijection σ. Therefore, we neglect this case here and elsewhere.

A further basic observation is stated in the next lemma. Therein, for a set

$$S = \{s_1, \ldots, s_m\} \subseteq \mathbb{Z}_n$$

we write $\gcd(S, n)$ instead of $\gcd(s_1, \ldots, s_m, n)$.

Lemma 1. *Let $\langle S\rangle_n$ and $\langle T\rangle_n$ be isospectral circulant graphs. Then,*

$$\gcd(S, n) = \gcd(T, n).$$

Proof. The graph $\langle S\rangle_n$ consists of $\gcd(S, n)$ isomorphic connected components. Each component is a circulant graph and, therefore, regular. By the Perron–Frobenius theorem (see [7], for example), each component has the eigenvalue $\lambda = \#S = \#T$ and all other eigenvalues are smaller than λ in absolute value. The spectrum of $\langle S\rangle_n$ is the union of the spectra of its connected components. Thus, $\langle S\rangle_n$ contains exactly $\gcd(S, n)$ times the eigenvalue λ. Equivalently, λ is an eigenvalue of $\langle T\rangle_n$ of multiplicity $\gcd(T, n)$. Since $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral, the statement follows. □

Now, we define the polynomial $G_{S,T,l,l'}(x)$ by

$$G_{S,T,l,l'}(x) = \sum_{s\in S} x^{s\cdot l} - \sum_{t\in T} x^{t\cdot l'}$$

(where all exponents are understood to be taken modulo n). Let $\langle S\rangle_n$ and $\langle T\rangle_n$ be isospectral circulant graphs with connection sets $S, T \subseteq \mathbb{Z}_n$ and spectral bijection σ, and let ω be a primitive nth root of unity. Then, we observe that $G_{S,T,l,\sigma(l)}(\omega) = 0$ for all $l \in \mathbb{Z}_n$, since $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral, whereas $G_{S,T,1,\sigma(1)}$ equals the zero polynomial if and only if $S \equiv \sigma(1)T \bmod n$. The following theorem shows that the latter statement is already equivalent to $S \sim T$:

Theorem 2. *Let $\langle S\rangle_n$ and $\langle T\rangle_n$ be isospectral circulant graphs with $S \equiv lT \bmod n$ for any (not necessarily invertible) $l \in \mathbb{Z}_n$. Then, S and T are proportional.*

Proof. By Lemma 1, we already know that $\gcd(S, n) = \gcd(T, n) =: g$. Therefore, $S \equiv lT \bmod n$ is equivalent to

$$S/g \equiv l(T/g) \bmod n/g,$$

where, for example, S/g denotes $\{s/g \mid s \in S\}$. From this equation, we deduce that l is invertible in $\mathbb{Z}_{n/g}$, because otherwise we would have

$$\gcd(l(T/g), n/g) > 1 = \gcd(S/g, n/g).$$

Now, let $k := x(n/g) + l$ for $x \in \mathbb{Z}_n$ with $0 \leq x < g$. Then, it follows that

$$kT = (x(n/g) + l)T = x(n/g)T + lT = xn(T/g) + lT \equiv lT \equiv S \bmod n.$$

Note that $\gcd(n, l)|g$ since $l \in \mathbb{Z}^*_{n/g}$. Therefore, we may write $n = p_1 \cdots p_N$ with some $N \in \mathbb{N}$ and not necessarily distinct prime numbers p_i, and, without loss of generality, $g = p_1 \cdots p_m$ for $m \leq N$ and $l = p_1 \cdots p_r q_1 \cdots q_z$ for $r \leq m$, $z \in \mathbb{N}$ and not necessarily distinct prime numbers $q_i \neq p_j$ for all $j = 1, \ldots, N$. Now, let x be the product of all prime numbers in the set $\{p_{r+1}, \ldots, p_m\} \setminus \{p_1, \ldots, p_r\}$, then

$$k = x(n/g) + l = xp_{m+1} \cdots p_N + p_1 \cdots p_r q_1 \cdots q_z.$$

Since l and n/g are relatively prime, we have $\{p_{m+1}, \ldots, p_N\} \cap \{p_1, \ldots, p_r\} = \emptyset$. Hence, every prime divisor p_i of n either is a divisor of $x(n/g)$ but not a divisor of l, or vice versa. Thus, it follows that $\gcd(k, n) = 1$ and, therefore, $S \sim T$, since $S \equiv kT \bmod n$. □

Note that a similar approach was already undertaken by Litow and Mans [10] within their proof of their main result and this is also stated in the paper of Mans, Pappalardi and Sharplinski [11, Lemma 3]. But neither of them gave a complete proof. Our constructions of non-trivial examples of isospectral circulant graphs rely on this necessary condition.

Since it seems to be difficult to gain further necessary conditions for isospectrality of circulant graphs, in the following we investigate sufficient conditions and present some explicit constructions and examples thereof.

3 Main results

Our basic idea for the construction of non-trivial examples of isospectral circulant graphs is stated in the following theorem. It generalizes an example given by Godsil, Holton and McKay [6] (here presented as Example 4).

Theorem 3. *Assume $n \in \mathbb{N}$ and $S, T \subseteq \mathbb{Z}_n$ such that there exists*

$$l \in \mathbb{Z}_n^* \quad \text{with} \quad G_{S,T,1,l}(\omega) = 0$$

for a primitive nth root of unity ω. If there is an $m \in \mathbb{Z}_n^*$ such that $S \equiv mT \bmod \frac{n}{p}$ for every prime divisor p of n, then $\langle S \rangle_n$ and $\langle T \rangle_n$ are isospectral.

Proof. We define

$$\sigma : \mathbb{Z}_n \to \mathbb{Z}_n, \quad k \mapsto \begin{cases} lk, & \text{if } \gcd(k,n) = 1, \\ mk, & \text{otherwise.} \end{cases}$$

Since $l, m \in \mathbb{Z}_n^*$, the map σ is bijective. We show that σ is a spectral bijection of S and T. Let $\omega := \mathrm{e}\left(\frac{1}{n}\right)$. If $\gcd(k,n) = 1$, then ω^k is still a primitive nth root of unity and, therefore, $G_{S,T,k,\sigma(k)}(\omega) = G_{S,T,1,l}(\omega^k) = 0$. On the other hand, if p is a divisor of k and n, then we may write $k = \kappa p$ and we observe that

$$\sum_{s \in S} \omega^{s \cdot k} = \sum_{s \in S} \mathrm{e}\left(\frac{s\kappa}{\frac{n}{p}}\right) = \sum_{t \in T} \mathrm{e}\left(\frac{mt\kappa}{\frac{n}{p}}\right) = \sum_{t \in T} \omega^{t \cdot \sigma(k)},$$

since $S \equiv mT \bmod \frac{n}{p}$. This yields $G_{S,T,k,\sigma(k)}(\omega) = 0$ for all k with $\gcd(k,n) > 1$. □

This theorem does not require the existence of an $m \in \mathbb{Z}_n^*$ satisfying $S \equiv mT \bmod n$. Therefore, it provides also non-trivial examples of isospectral circulant graphs:

Example 4. Let $n = 20$,

$$S = \{2, 3, 4, 7, 13, 16, 17, 18\} \quad \text{and} \quad T = \{3, 6, 7, 8, 12, 13, 14, 17\}.$$

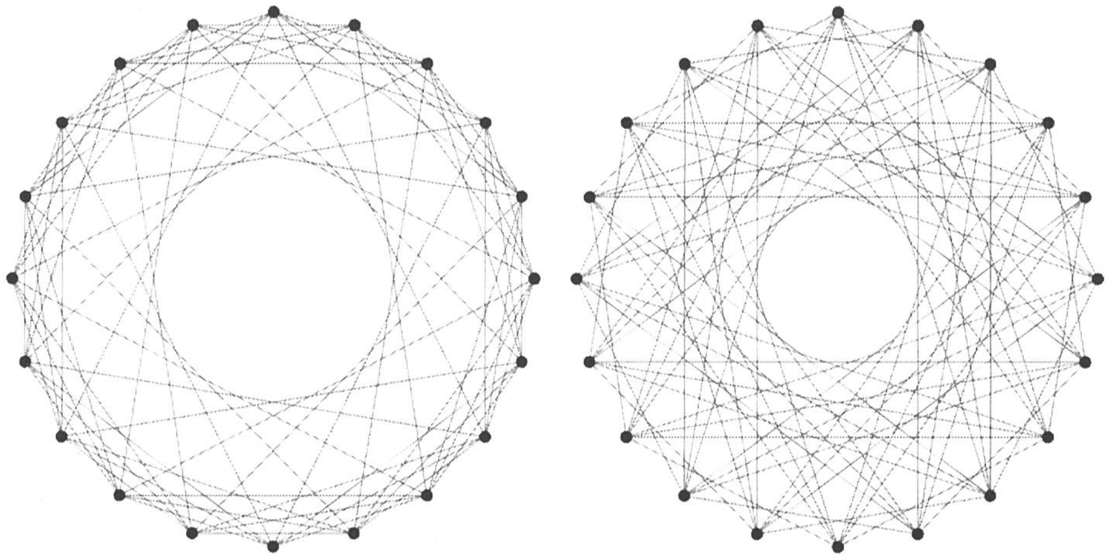
Then, for $l \in \{3, 7, 13, 17\}$, we have that $G_{S,T,1,l}(\omega) = 0$ for every primitive nth root of unity ω, but $S \not\equiv lT \bmod 20$. Since $S \equiv T \bmod 10$ and $S \equiv T \bmod 4$, by Theorem 3 we get that $\langle S \rangle_n$ and $\langle T \rangle_n$ are isospectral but non-proportional.

This example was stated by Godsil, Holton and McKay [6] and seems to be the first published example of isospectral non-isomorphic undirected circulant graphs. The graphs are shown in Figure 1.

To find such an example, first of all, we construct sets $S, T \subseteq \mathbb{Z}_n$ such that there exists $l \in \mathbb{Z}_n$ with

$$G_{S,T,1,l}(\omega) = 0 \quad \text{and} \quad G_{S,T,1,l} \neq 0 \text{ (as a polynomial)} \tag{1}$$

for every primitive nth root of unity ω. Therefore, the remaining part of this paper is an investigation of sets $S, T \subseteq \mathbb{Z}_n$, which satisfy (1). On top of that, we show that Theorem 3 does not provide a necessary condition for the construction of isospectral circulant graphs with non-proportional connection sets.



(a) $S = \{2, 3, 4, 7, 13, 16, 17, 18\}$ (b) $T = \{3, 6, 7, 8, 12, 13, 14, 17\}$

Figure 1 Non-isomorphic isospectral circulant graphs.

The main idea for constructing sets $S, T \subseteq \mathbb{Z}_n$ satisfying (1) is to use subsets $\mathrm{Eq} \subseteq \mathbb{Z}_n$ which have the property that

$$\sum_{j \in \mathrm{Eq}} \mathrm{e}\Big(\frac{j}{n}\Big) = 0. \tag{2}$$

Indeed, we will see that every non-trivial pair of isospectral circulant graphs arises from such sets or rather can be constructed starting from such a set. Motivated by Rédei [13, Theorem 10], we call a set Eq satisfying Equation (2) *equilibrium set*. In his paper, Rédei gave a complete characterization of these sets. In the following, we will use his result, but adapt some notations.

For fixed $n \in \mathbb{N}$, the set $\{0, 1, \ldots, n-1\}$ is an equilibrium set, i.e.,

$$\sum_{j=1}^{n-1} \mathrm{e}\Big(\frac{j}{n}\Big) = 0.$$

Therefore, if d is a divisor of n, the set $\{0, \frac{n}{d}, 2\frac{n}{d}, \ldots, (d-1)\frac{n}{d}\}$ is also an equilibrium set, since

$$\sum_{j \in \{0, \frac{n}{d}, 2\frac{n}{d}, \ldots, (d-1)\frac{n}{d}\}} \mathrm{e}\Big(\frac{j}{n}\Big) = \sum_{j=0}^{d-1} \mathrm{e}\Big(\frac{j}{d}\Big) = 0.$$

Finally, multiplying both sides by $\mathrm{e}(\frac{a}{n})$, for any $a \in \mathbb{Z}_n$, yields the equilibrium set

$$\Big\{a, a + \frac{n}{d}, a + 2\frac{n}{d}, \ldots, a + (d-1)\frac{n}{d}\Big\} =: [a, n, d].$$

Rédei called such sets *trivial* equilibrium sets. We observe that every trivial equilibrium set is an arithmetic progression in $\mathbb{Z}_n$ of length d with common difference $\frac{n}{d}$ for some divisor

d of n. In the following, we denote an arithmetic progression of length d with common difference $\frac{n}{d}$ and initial value a by $[a, n, d] \subseteq \mathbb{Z}_n$. Note that if d is not a prime number, then, for each prime divisor p of d, the arithmetic progression $[a, n, d]$ equals the union $\bigcup_{i=1}^{d/p}[a_i, n, p]$ for suitable values of a_i. Therefore, we call $[a, n, d]$ *indecomposable* if d is a prime number and *decomposable* else. Furthermore, it is clear that, for every $k \in \mathbb{Z}_n$ with $\gcd(k, d) = 1$, the set $k \cdot [a, n, d]$ is still an equilibrium set. If $\gcd(k, d) > 1$ but d does not divide k, then we may write

$$k \cdot [a, n, d] = \bigcup_{i=1}^{\gcd(k,d)} \left[a_i, n, \frac{d}{\gcd(k, d)}\right]$$

for suitable values of a_i, i.e., $k \cdot [a, n, d]$ remains an equilibrium set as well.

Now, we construct non-trivial examples of isospectral circulant graphs by using such trivial equilibrium sets. We use the idea of Theorem 3 as a foundation, but can weaken the second condition of this theorem by exploiting the properties of equilibrium sets. The following theorem provides a sufficient criterion for two circulant graphs to be isospectral:

Theorem 5. *Let $n \in \mathbb{N}$ and d be a divisor of n. Furthermore, define*

$$\mathrm{Eq}_S := \bigcup_{i=1}^{r}[a_i, n, d] \quad \text{and} \quad \mathrm{Eq}_T := \bigcup_{i=1}^{r}[b_i, n, d],$$

for some $a_i, b_i \in \mathbb{Z}_n$ and pairwise disjoint sets $[a_i, n, d]$ resp. $[b_i, n, d]$. Finally, let $S := S' \cup \mathrm{Eq}_S$ and $T := T' \cup \mathrm{Eq}_T$ for $S', T' \subseteq \mathbb{Z}_n$ with $S' \cap \mathrm{Eq}_S = \emptyset = T' \cap \mathrm{Eq}_T$ such that there exists $l \in \mathbb{Z}_n^*$ with $S' \equiv lT' \bmod n$. If there is some m with $\gcd(m, \frac{n}{d}) = 1$ such that $S \equiv mT \bmod \frac{n}{d}$, then the circulant graphs $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral.
In particular, if $\mathrm{Eq}_S \not\equiv l\,\mathrm{Eq}_T \bmod n$ for every $l \in \mathbb{Z}_n^*$ with $S' \equiv lT' \bmod n$, then S and T are non-proportional.

Proof. We define

$$\sigma : \mathbb{Z}_n \to \mathbb{Z}_n, \quad k \mapsto \begin{cases} lk, & \text{if } d \nmid k, \\ mk, & \text{if } d \mid k. \end{cases}$$

Since $l \in \mathbb{Z}_n^*$ and $m \in \mathbb{Z}_{\frac{n}{d}}^*$, the map σ is bijective. We show that σ is a spectral bijection of S and T. Let $\omega := \mathrm{e}(\frac{1}{n})$. If d is not a divisor of k, then $k \cdot \mathrm{Eq}_S$ and $lk \cdot \mathrm{Eq}_T$ are still equilibrium sets (as mentioned above Theorem 5) and, therefore,

$$G_{S,T,k,\sigma(k)}(\omega) = \sum_{s\in S} \omega^{s\cdot k} - \sum_{t\in T} \omega^{t\cdot\sigma(k)} = \sum_{s\in S'} \omega^{s\cdot k} - \sum_{t\in T'} \omega^{t\cdot lk} = 0.$$

On the other hand, if d is a divisor of k, we may write $k = \kappa d$. Since, by assumption, $S \equiv mT \bmod \frac{n}{d}$, we observe that

$$\sum_{s\in S} \omega^{s\cdot k} = \sum_{s\in S} \mathrm{e}\left(\frac{s\kappa}{\frac{n}{d}}\right) = \sum_{t\in T} \mathrm{e}\left(\frac{mt\kappa}{\frac{n}{d}}\right) = \sum_{t\in T} \omega^{t\cdot\sigma(k)}.$$

Thus, we get that $G_{S,T,k,\sigma(k)}(\omega) = 0$ also in this case. □

Note that if $\langle S\rangle_n$ is undirected, then $[a, n, d] \subseteq S$ if and only if $[-a, n, d] \subseteq S$. In the following we write $[\pm a, n, d]$ instead of $[a, n, d] \cup [-a, n, d]$.

Example 4 arises from both, Theorem 3 and Theorem 5. But there are also examples of non-trivial isospectral circulant graphs which satisfy Theorem 5 only:

Example 6. The first known counterexample to Ádám's conjecture, published by Elspas and Turner [5], suffices Theorem 5: Let $n = 16$, $S = \{1, 2, 7, 9, 14, 15\}$ and $T = \{2, 3, 5, 11, 13, 14\}$. Then, S and T are non-proportional and we may write $S = \{2, 14\} \cup [\pm 1, 16, 2]$ and $T = \{2, 14\} \cup [\pm 3, 16, 2]$. Since we easily observe that $S \equiv 3 \cdot T \bmod 8$, the graphs $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral by Theorem 5.

Unfortunately, as Examples 4 and 6 show, the theorem neither guarantees that the isospectral graphs are non-isomorphic nor that they are isomorphic.

We can easily generalize Theorem 5 by considering distinct divisors of n. In the following, let $\mathcal{P}(M)$ denote the power set of a set M.

Theorem 7. *Let $n \in \mathbb{N}$ and $d_1, \ldots, d_z$ be divisors of n which are pairwise relatively prime. Furthermore, let*

$$\mathrm{Eq}_S := \bigcup_{j=1}^{z} \bigcup_{i=1}^{r_j} [a_{ij}, n, d_j] \quad \text{and} \quad \mathrm{Eq}_T := \bigcup_{j=1}^{z} \bigcup_{i=1}^{r_j} [b_{ij}, n, d_j],$$

for some $a_{ij}, b_{ij} \in \mathbb{Z}_n$ and pairwise disjoint sets $[a_{ij}, n, d_j]$ resp. $[b_{ij}, n, d_j]$. Finally, let $S := S' \cup \mathrm{Eq}_S$ and $T := T' \cup \mathrm{Eq}_T$ for $S', T' \subseteq \mathbb{Z}_n$ with $S' \cap \mathrm{Eq}_S = \emptyset = T' \cap \mathrm{Eq}_T$ such that there exists $l \in \mathbb{Z}_n^*$ with $S' \equiv lT' \bmod n$. If for all $\pi \in \mathcal{P}(\{1, \ldots, z\}) \backslash \{\emptyset\}$ there is some m_π with

$$\gcd\left(m_\pi, \frac{n}{\prod_{j\in\pi} d_j}\right) = 1$$

such that

$$S' \equiv m_\pi T' \bmod \frac{n}{\prod_{j\in\pi} d_j}$$

and

$$\bigcup_{j\in\pi} \bigcup_{i=1}^{r_j} [a_{ij}, n, d_j] \equiv m_\pi \bigcup_{j\in\pi} \bigcup_{i=1}^{r_j} [b_{ij}, n, d_j] \bmod \frac{n}{\prod_{j\in\pi} d_j},$$

then the circulant graphs $\langle S\rangle_n$ and $\langle T\rangle_n$ are isospectral.

In particular, if $\mathrm{Eq}_S \not\equiv l\,\mathrm{Eq}_T \bmod n$ for every $l \in \mathbb{Z}_n^*$ with $S' \equiv lT' \bmod n$, then S and T are non-proportional.

Proof. We define

$$\sigma : \mathbb{Z}_n \to \mathbb{Z}_n, \quad k \mapsto \begin{cases} lk, & \text{if } d_j \nmid k \text{ for all } j = 1, \ldots, z, \\ m_\pi k, & \text{if } \prod_{j\in\pi} d_j \mid k \text{ and } d_j \nmid k \text{ for all } j \notin \pi, \end{cases}$$

for all $\pi \in \mathcal{P}(\{1, \ldots, z\}) \backslash \{\emptyset\}$. By the same argument as in the proof of Theorem 5, we can see that σ is bijective. We show that σ is a spectral bijection of S and T. Let $\omega := \mathrm{e}(\frac{1}{n})$. If no d_j divides k, then $k \cdot \mathrm{Eq}_S$ and $lk \cdot \mathrm{Eq}_T$ are still equilibrium sets and, therefore,

$$G_{S,T,k,\sigma(k)}(\omega) = \sum_{s \in S} \omega^{s \cdot k} - \sum_{t \in T} \omega^{t \cdot \sigma(k)} = \sum_{s \in S'} \omega^{s \cdot k} - \sum_{t \in T'} \omega^{t \cdot lk} = 0.$$

Now, let $\pi \in \mathcal{P}(\{1, \ldots, z\}) \backslash \{\emptyset\}$ and define $Eq_{S_\pi} := \bigcup_{j \in \pi} \bigcup_{i=1}^{r_j} [a_{ij}, n, d_j]$ and $Eq_{T_\pi} := \bigcup_{j \in \pi} \bigcup_{i=1}^{r_j} [b_{ij}, n, d_j]$. If $\prod_{j \in \pi} d_j \mid k$ and $d_j \nmid k$ for all $j \notin \pi$, then $k \cdot (\mathrm{Eq}_S \backslash Eq_{S_\pi})$ and $m_\pi k \cdot (\mathrm{Eq}_T \backslash Eq_{T_\pi})$ are still equilibrium sets due to the fact that the d_j's are relatively prime. Since, by assumption, we have

$$Eq_{S_\pi} \equiv m_\pi Eq_{T_\pi} \bmod \frac{n}{\prod_{j \in \pi} d_j} \quad \text{and} \quad S' \equiv m_\pi T' \bmod \frac{n}{\prod_{j \in \pi} d_j},$$

it follows that

$$\sum_{s \in S' \cup Eq_{S_\pi}} \omega^{s \cdot k} = \sum_{s \in S' \cup Eq_{S_\pi}} \mathrm{e}\left(\frac{s\kappa_\pi}{\frac{n}{d_\pi}}\right) = \sum_{t \in T' \cup Eq_{T_\pi}} \mathrm{e}\left(\frac{m_\pi t \kappa_\pi}{\frac{n}{d_\pi}}\right) = \sum_{t \in T' \cup Eq_{T_\pi}} \omega^{t \cdot \sigma(k)}$$

for $d_\pi = \prod_{j \in \pi} d_j$ and $\kappa_\pi = k / d_\pi$. Thus, we get $G_{S,T,k,\sigma(k)}(\omega) = 0$. □

Example 8. Let $n = 120$ and let $\mathrm{Eq}_S := [\pm 5, 120, 5] \cup [\pm 9, 120, 2]$ and $\mathrm{Eq}_T := [\pm 1, 120, 5] \cup [\pm 27, 120, 2]$. Furthermore, let $S := \{34, 86\} \cup \mathrm{Eq}_S$ and $T := \{2, 118\} \cup \mathrm{Eq}_T$. Since

$$\begin{aligned}
[\pm 5, 120, 5] &\equiv 5 \cdot [\pm 1, 120, 5] \bmod \frac{120}{5} && \text{and } \{34, 86\} \equiv 5 \cdot \{2, 118\} \bmod \frac{120}{5}, \\
[\pm 9, 120, 2] &\equiv 13 \cdot [\pm 27, 120, 2] \bmod \frac{120}{2} && \text{and } \{34, 86\} \equiv 13 \cdot \{2, 118\} \bmod \frac{120}{2}, \\
\mathrm{Eq}_S &\equiv 5 \cdot \mathrm{Eq}_T \bmod \frac{120}{10} && \text{and } \{34, 86\} \equiv 5 \cdot \{2, 118\} \bmod \frac{120}{10},
\end{aligned}$$

the undirected circulant graphs $\langle S \rangle_n$ and $\langle T \rangle_n$ are isospectral by Theorem 7. In particular, S and T are non-isomorphic (i.e., also non-proportional).

So far, we only used trivial equilibrium sets in order to construct isospectral circulant graphs. Now we also want to consider so-called *non-trivial* equilibrium sets. Rédei [13, Theorem 10] proved that every non-trivial equilibrium set arises from the trivial ones. We reformulate his theorem in the following way:

Lemma 9. *Let $p_1, \ldots, p_r, \tilde{p}_1, \ldots, \tilde{p}_s$ be (not necessarily distinct) prime divisors of n and let*

$$A := \bigcup_{i=1}^{r} [a_i, n, p_i], \qquad B := \bigcup_{j=1}^{s} [b_j, n, \tilde{p}_j]$$

for some values $a_i, b_j \in \mathbb{Z}_n$, $i = 1, \ldots, r$, $j = 1, \ldots, s$. If $B \subseteq A$, then $A \backslash B$ is an equilibrium set. In particular, all equilibrium sets are of this form.

This lemma not only includes every equilibrium set, but, on top of that, also yields a construction of sets $S, T \subseteq \mathbb{Z}_n$ satisfying (1): we observe that for every prime divisor p of n we have

$$-\mathrm{e}\Big(\frac{a}{n}\Big) = \sum_{j\in[a,n,p]\setminus\{a\}} \mathrm{e}\Big(\frac{j}{n}\Big). \tag{3}$$

Therefore, every equation of the form $G_{S,T,1,l}(\omega) = 0$ with a primitive nth root of unity ω is equivalent to an equation of the form

$$\sum_{j\in \mathrm{Eq}(S,T,l,p)} \omega^j = 0,$$

with an equilibrium set $\mathrm{Eq}(S, T, l, p)$ which depends on S, T, p and l only. Thus, in particular, we can construct every pair of sets $S, T \subseteq \mathbb{Z}_n$ satisfying (1) from an equilibrium set. This yields a new way to construct non-trivial isospectral circulant graphs:

Theorem 10. *Let $n \in 4\mathbb{N}$ and $\mathrm{Eq} := \bigcup_{i=1}^{r}[a_i, n, d]$ be an equilibrium set for pairwise disjoint sets $[a_i, n, d]$ and for d being an even divisor of n such that all elements of* Eq *are odd. Let S be a set containing exactly half of the elements of each set $[a_i, n, d]$ for $i = 1, \dots, r$ and let either $T := \mathrm{Eq}\setminus S$ or $T := \mathrm{Eq}\setminus S + \frac{n}{2} = \{t + \frac{n}{2} \mid t \in \mathrm{Eq}\setminus S\}$. Then, for every set $Z \subseteq 2\mathbb{Z}_n$, the circulant graphs $\langle S \cup Z\rangle_n$ and $\langle T \cup Z\rangle_n$ are isospectral.*

Proof. Let $T = \mathrm{Eq}\setminus S$ and define

$$\sigma_1 : \mathbb{Z}_n \to \mathbb{Z}_n, \quad k \mapsto \begin{cases} k + \frac{n}{2}, & \text{if } d \nmid k, \\ k, & \text{if } d \mid k. \end{cases}$$

Obviously, the map σ_1 is bijective. We show that σ_1 is a spectral bijection of $S \cup Z$ and $T \cup Z$.

We observe that for every $k \in \mathbb{Z}_n$ with $d \nmid k$ we have that

$$0 = \sum_{j\in \mathrm{Eq}} \mathrm{e}\Big(\frac{j}{n}\Big)^k = \sum_{s\in S} \mathrm{e}\Big(\frac{s}{n}\Big)^k + \sum_{t\in T} \mathrm{e}\Big(\frac{t}{n}\Big)^k,$$

or, equivalently,

$$\sum_{s\in S} \mathrm{e}\Big(\frac{s}{n}\Big)^k = -\sum_{t\in T} \mathrm{e}\Big(\frac{t}{n}\Big)^k = \sum_{t\in T} \mathrm{e}\Big(\frac{t}{n}\Big)^{k+\frac{n}{2}},$$

since every $t \in T$ is odd. On top of that, we get

$$\sum_{z\in Z} \mathrm{e}\Big(\frac{z}{n}\Big)^{k+\frac{n}{2}} = \sum_{z\in Z} \mathrm{e}\Big(\frac{zk}{n} + \frac{z}{2}\Big) = \sum_{z\in Z} \mathrm{e}\Big(\frac{z}{n}\Big)^k,$$

since $2 \mid z$ for every $z \in Z$. Therefore, we have $G_{S\cup Z,T\cup Z,k,\sigma_1(k)} = 0$ for all k with $d \nmid k$.

If $d \mid k$ for every $i = 1, \dots, r$, we observe that $k \cdot [a_i, n, d] \equiv \{ka_i, \dots, ka_i\} \bmod n$. Since S and T contain the same number of elements of each set $[a_i, n, d]$, it follows that

$k \cdot S \equiv k \cdot T \bmod n$ and, therefore, we get $G_{S\cup Z, T\cup Z, k, \sigma_1(k)} = 0$. Thus, σ_1 is a spectral bijection of $S \cup Z$ and $T \cup Z$.

Now, let $T = \mathrm{Eq} \setminus S + \frac{n}{2}$ and define

$$\sigma_2 : \mathbb{Z}_n \to \mathbb{Z}_n, \quad k \mapsto \begin{cases} k + \frac{n}{2}, & \text{if } d \nmid k \text{ and } 2 \mid k, \\ k, & \text{otherwise.} \end{cases}$$

Then, for all $k \in \mathbb{Z}_n$ with $d \nmid k$, we get

$$0 = \sum_{j \in \mathrm{Eq}} \mathrm{e}\Big(\frac{j}{n}\Big)^k = \sum_{s \in S} \mathrm{e}\Big(\frac{s}{n}\Big)^k + \sum_{t \in T} \mathrm{e}\Big(\frac{t - \frac{n}{2}}{n}\Big)^k,$$

resp.

$$\sum_{s \in S} \mathrm{e}\Big(\frac{s}{n}\Big)^k = -\sum_{t \in T} \mathrm{e}\Big(\frac{t - \frac{n}{2}}{n}\Big)^k = \begin{cases} \sum_{t \in T} \mathrm{e}\left(\frac{t}{n}\right)^k, & \text{if } 2 \nmid k, \\ -\sum_{t \in T} \mathrm{e}\left(\frac{t}{n}\right)^k, & \text{if } 2 \mid k. \end{cases}$$

If k is odd, then $G_{S\cup Z, T\cup Z, k, \sigma_2(k)} = 0$ obviously holds true, and since

$$-\sum_{t \in T} \mathrm{e}\Big(\frac{t}{n}\Big)^k = \sum_{t \in T} \mathrm{e}\Big(\frac{t}{n}\Big)^{k + \frac{n}{2}} \quad \text{and} \quad \sum_{z \in Z} \mathrm{e}\Big(\frac{z}{n}\Big)^k = \sum_{z \in Z} \mathrm{e}\Big(\frac{z}{n}\Big)^{k + \frac{n}{2}},$$

the equation also holds true for all even k.

By the same argument as in the first case, we also have $G_{S\cup Z, T\cup Z, k, \sigma_2(k)} = 0$ for all k with $d \mid k$. Therefore, σ_2 is a spectral bijection of $S \cup Z$ and $T \cup Z$. □

Note that the condition that all elements of Eq are odd is fulfilled if and only if a_i is odd for every $i = 1, \ldots, r$ and $\frac{n}{d}$ is even. Therefore, we assume $n \in 4\mathbb{N}$. We remark that this theorem provides an explicit construction of isospectral circulant graphs. Unfortunately, we cannot generalize this result simply by exploiting Equation (3) with $p > 2$.

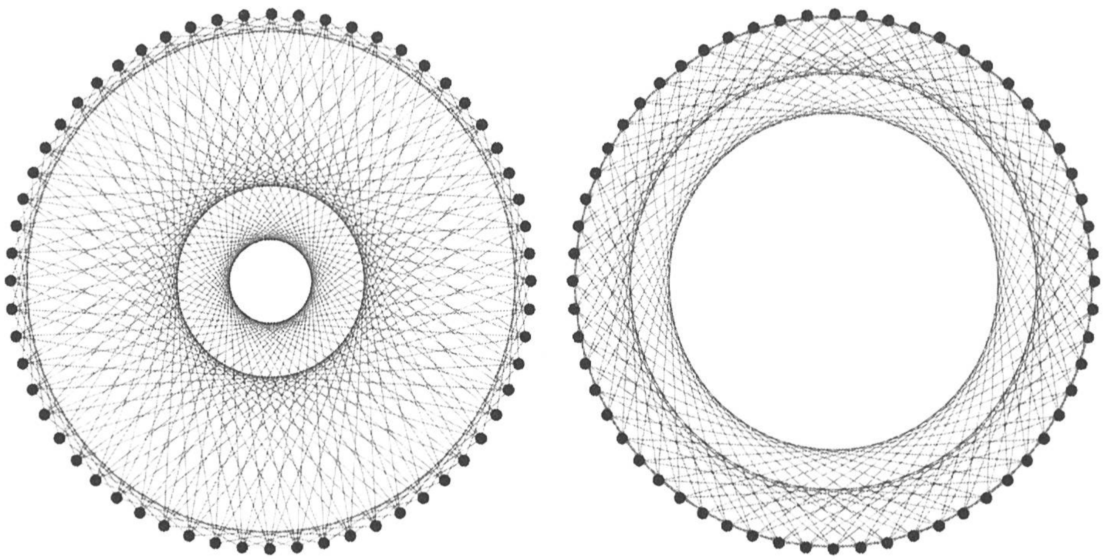
Example 11. Let $n = 60$ and $\mathrm{Eq} := [\pm 3, 60, 6]$. Furthermore, define

$$S := \{\pm 23, \pm 33, \pm 53\} \subseteq \mathrm{Eq}$$

and

$$T_1 := \mathrm{Eq} \setminus S = \{\pm 3, \pm 13, \pm 43\} \quad \text{resp.} \quad T_2 := \mathrm{Eq} \setminus S + 30 = \{\pm 13, \pm 33, \pm 43\}.$$

By Theorem 10, we get that for every set Z with $2 \mid z$ for all $z \in Z$ the circulant graphs $\langle S \cup Z \rangle_n$, $\langle T_1 \cup Z \rangle_n$ and $\langle T_2 \cup Z \rangle_n$ are isospectral. In particular, these graphs are undirected if and only if $Z \equiv -Z \bmod n$. If, for example, $Z = \{2, 58\}$, then the circulant graphs $\langle S \cup Z \rangle_n$, $\langle T_1 \cup Z \rangle_n$ and $\langle T_2 \cup Z \rangle_n$ are isospectral with pairwise non-proportional connection sets. Moreover, we observe that $\langle S \rangle_n$ is neither isomorphic to $\langle T_1 \rangle_n$ nor $\langle T_2 \rangle_n$, whereas $\langle T_1 \rangle_n$ and $\langle T_2 \rangle_n$ are isomorphic. Figure 2 shows the non-isomorphic isospectral circulant graphs $\langle S \rangle_n$ and $\langle T_1 \rangle_n$.



(a) $S = \{\pm 23, \pm 33, \pm 53\}$ (b) $T_1 = \{\pm 3, \pm 13, \pm 43\}$

Figure 2 Non-isomorphic isospectral circulant graphs.

4 Final remark

Apart from the theorems we proved in this paper, it seems that there are still several other ways to construct isospectral circulant graphs. For example, we can think about combining some of these results or exploiting the equilibrium sets in more complex ways as it is done in the following example:

Example 12. We consider an example of a pair of non-isomorphic isospectral circulant graphs with directed edges stated in the paper of Elspas and Turner [5]. Let $n = 32$ and $S = \{1, 2, 6, 18, 22, 25\}$, $T = \{2, 6, 7, 18, 22, 31\}$. We observe that the subset $\{2, 6, 18, 22\} \subseteq S, T$ is an equilibrium set which may be written as $\{2, 6, 18, 22\} = [2, 32, 2] \cup [6, 32, 2]$, i.e., the union of two arithmetic progressions with common difference $\frac{32}{2} = 16$. One has that $\{1, 25\} \equiv 31 \cdot \{7, 31\} \bmod 32$, but there is no m with $\gcd(m, \frac{32}{2}) = 1$ satisfying $\{2, 6, 18, 22\} \equiv m \cdot \{2, 6, 18, 22\} \bmod 16$ and $\{1, 25\} \equiv m \cdot \{7, 31\} \bmod 16$. Therefore, Theorem 5 does not hold for this example. But the point here is that $2 \cdot \{1, 25\} \equiv \{2, 18\} \bmod 32$ and $2 \cdot \{7, 31\} \equiv \{14, 30\} \bmod 32$ are equilibrium sets of common difference 16 as well. Thus, it suffices to find m with $\gcd(m, \frac{32}{2^2}) = 1$ such that $\{2, 6, 18, 22\} \equiv m \cdot \{2, 6, 18, 22\} \bmod 8$ and $\{1, 25\} \equiv m \cdot \{7, 31\} \bmod 8$ hold true. Because then, the map

$$\sigma : \mathbb{Z}_{32} \to \mathbb{Z}_{32}, \quad k \mapsto \begin{cases} 31k, & \text{if } 4 \nmid k, \\ mk, & \text{if } 4 \mid k \end{cases}$$

provides a spectral bijection of S and T. Indeed, $m = 7$ fulfills these requirements.

It remains an open problem to give a complete characterization of isospectral circulant graphs.

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