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## Short note    A characterization of the focals of hyperbolas

Paris Pamfilos

### 1 Chords through a point

The property which we discuss here relates to the tangents of a hyperbola at the end points of a chord and their intersections with the asymptotes of the hyperbola. It is formulated by the following lemma.

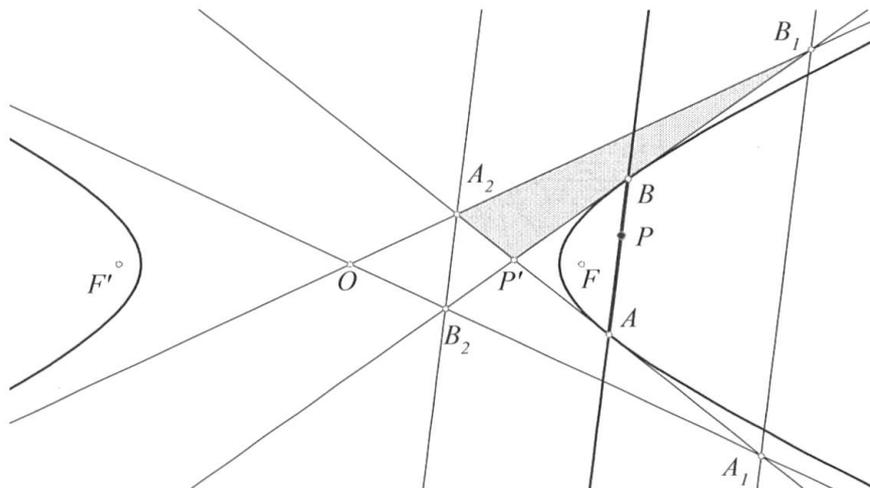


Figure 1 Asymptotic triangles and parallels

**Lemma 1.** *If the tangents to the hyperbola at the end points of a chord  $AB$  intersect the asymptotes respectively at points  $\{A_1, A_2\}$  and points  $\{B_1, B_2\}$ , then  $\{A_1B_1, A_2B_2\}$  are parallels and  $AB$  is their middle-parallel.*

*Proof.* The proof of the lemma, in the case  $AB$  runs in the *inner* domain of the hyperbola (see Figure 1), derives from the equality of the areas of the triangles  $\{A_1A_2B_1, A_1B_2B_1\}$ , which have in common the area of the triangle  $A_1P'B_1$ , and are complemented by the equal areas of the triangles  $\{P'A_2B_1, P'B_2A_1\}$  ([3, III.43, p. 112], [5, p.192]), point  $P'$

being the intersection of the tangents. The claim about the middle-parallel follows from the equally well-known property ([3, II.3, p. 56], [4, Fig. 10.18, p. 315], [5, p. 191]), that  $\{A, B\}$  are respectively the middles of  $\{A_1A_2, B_1B_2\}$ . The proof, when  $AB$  runs in the *outer* domain of the hyperbola is completely analogous<sup>1</sup>.  $\square$

## 2 The property of focal points

The next theorem characterizes the focal points  $\{F, F'\}$  of the hyperbola by measuring the distance of the parallels  $\{A_1B_1, A_2B_2\}$ , as the chord  $AB$  turns about a fixed point  $P$ .

**Theorem 1.** *Under the notation and conventions made above, for chords passing through a fixed point  $P$ , the distance between the parallels  $\{AB, A_1B_1\}$  is variable, depending on their direction, except when  $P$  is a focal point. In the case  $P$  is a focal point, this distance is independent of the direction and equal to the conjugate axis  $b$  of the hyperbola.*

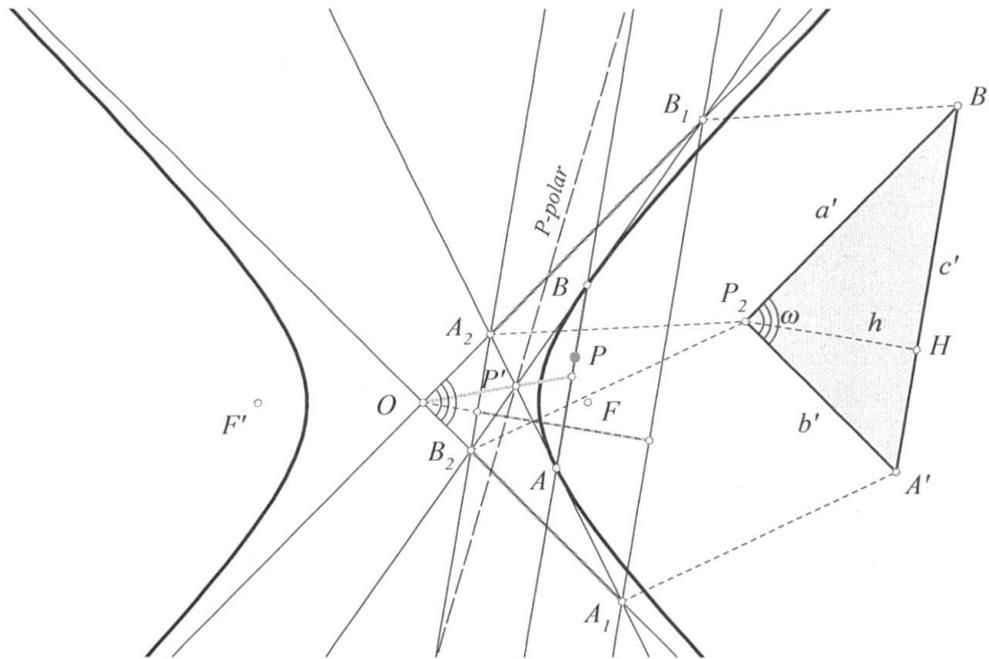


Figure 2 Triangle formed by the segments cut on the asymptotes

*Proof.* To prove this, we represent the hyperbola with its canonical coordinates in the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

We consider also the quadratic equation, giving the product of the tangents from the point  $P'(x_1, y_1)$ . This can be seen to be ([2, p. 251, I])

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 - b^2(x - x_1)^2. \tag{1}$$

<sup>1</sup>At this point I would like to express my gratitude to the referee, who kindly suggested not only the references to the classical literature, but also a complete alternative proof to the main theorem. I hope to see this proof, as well as some other, possibly better or simpler proofs, from interested readers, published in this journal.

The intersection points  $\{A_2, B_1\}$  and  $\{B_2, A_1\}$  of these lines with the asymptotes are found by solving the systems consisting of the previous equation and the equation of each asymptote  $x/a - y/b = 0$  and  $x/a + y/b = 0$  (see Figure 2). These are found to be

$$A_2, B_1 = \frac{-ab \pm g}{ay_1 - bx_1}(a, b) \quad \text{and} \quad B_2, A_1 = \frac{ab \pm g}{ay_1 + bx_1}(a, -b), \quad (2)$$

where,  $g = g(x_1, y_1) = \sqrt{a^2y_1^2 - b^2x_1^2 + a^2b^2}$ . This implies that

$$|A_2B_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 - bx_1)^2} \quad \text{and} \quad |B_2A_1|^2 = \frac{4g^2(a^2 + b^2)}{(ay_1 + bx_1)^2}. \quad (3)$$

The required distance  $h$  of the parallels can be measured from the altitude of the triangle  $P_2A'B'$ , resulting by parallel translating at an arbitrary point  $P_2$  the segments  $\{A_2B_1, B_2A_1\}$ . Since the property under consideration is invariant by similarities, we can assume that  $a^2 + b^2 = 1$ . Thus, using the well-known formula, deriving from the area of a triangle,  $h = \frac{b'c' \sin(\omega)}{a'}$ , we find that

$$h^2 = \frac{b'^2c'^2 \sin(\omega)^2}{a'^2} = \frac{2(a^2y_1^2 - b^2x_1^2 + a^2b^2) \sin(\omega)^2}{a^2y_1^2 + b^2x_1^2 + (a^2y_1^2 - b^2x_1^2) \cos(\omega)}, \quad (4)$$

where  $\omega$  is the angle of the asymptotes. Taking into account that  $\sin(\omega) = 2ab$ , and  $\cos(\omega) = a^2 - b^2$ , we obtain the simplified expression

$$h^2 = 4a^2b^2 \frac{a^2y_1^2 - b^2x_1^2 + a^2b^2}{a^4y_1^2 + b^4x_1^2}. \quad (5)$$

Letting the chord  $AB$  revolve about  $P(x_0, y_0)$ , the corresponding point  $P'(x_1, y_1)$  moves on the polar line  $\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$  of  $P$  ([1, p. 192]), a particular point of which is

$$K_2(x_2, y_2) = (a^2/x_0, 0).$$

A parametric form of the polar is consequently given by

$$x_1 = \frac{a^2}{x_0} + t \frac{y_0}{b^2}, \quad y_1 = t \frac{x_0}{a^2}.$$

Introducing this into equation (5) and simplifying, we obtain

$$h^2 = 4 \frac{p(t)}{q(t)}, \quad \text{with}$$

$$p(t) = t^2[-x_0^2(a^2y_0^2 - b^2x_0^2)] + t[-2a^4b^2x_0y_0] + [a^4b^4(x_0^2 - a^2)],$$

$$q(t) = t^2[x_0^2(x_0^2 + y_0^2)] + t[2a^2b^2x_0y_0] + [a^4b^4].$$

The condition of constancy of  $h^2$  is equivalent with the vanishing of coefficients of the quadratic equation  $p(t) - kq(t)$ , for a constant  $k$ , which implies the equations

$$\begin{aligned}x_0^2(y_0^2(a^2 + k) - x_0^2(b^2 - k)) &= 0 \\(a^2 + k)x_0y_0 &= 0, \\(x_0^2 - a^2) - k &= 0.\end{aligned}$$

The two last equations lead, for  $x_0y_0 \neq 0$ , to a contradiction. The condition  $x_0 = 0$  leads also to the contradiction  $h^2 = -4a^2$ . Thus, if a point  $(x_0, y_0)$  has the stated property, it must satisfy  $y_0 = 0, x_0 \neq 0$ , implying  $k = b^2 = (x_0^2 - a^2)$ , hence  $x_0^2 = 1$ , which determines the position of a focal point  $F(\pm 1, 0)$  and the value for  $h^2 = 4b^2$ , which proves the theorem.  $\square$

## References

- [1] Joseph Carnoy. *Cours de geometrie analytique*. Gauthier-Villars, Paris, 1876.
- [2] S. Loney. *The elements of coordinate geometry I, II*. AITBS publishers, Delhi, 1991.
- [3] Apollonius of Perga. *Treatise on Conic Sections, edited by T.L. Heath*. Cambridge University Press, Cambridge, 1896.
- [4] Alexander Ostermann and Gerhard Wanner. *Geometry by its History*. Springer, Berlin, 2012.
- [5] George Salmon. *A treatise on Conic Sections*. Longmans, Green and Co., London, 1917.

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