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**Autor:** Blinkiewicz, Dorota / Rzonsowski, Piotr / Szydło, Bogdan  
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## Parallelograms inscribed in a pair of confocal ellipses

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Dorota Blinkiewicz, Piotr Rzonsowski and Bogdan Szydło

Dorota Blinkiewicz obtained her Ph.D. from Adam Mickiewicz University (Poznań) in 2017. Her research interests include i.a. number theory and arithmetic algebraic geometry.

Piotr Rzonsowski obtained his Ph.D. from Adam Mickiewicz University (Poznań) in 2010. The area of his research is mainly arithmetic geometry.

Bogdan Szydło obtained his Ph.D. from Adam Mickiewicz University (Poznań) in 1988. His main interests are in geometry and number theory.

### 1 Introduction

The following particular but interesting maximal property of confocal ellipses was proved by A. Connes and D. Zagier [4, Theorem 2]; see Figure 1.

**Theorem.** *Let  $\mathcal{E}$  and  $\mathcal{E}'$  be a pair of confocal ellipses with foci  $F$  and  $G$ . Let  $O$  be a centre of the ellipses,  $a$  and  $b$  semiaxes of  $\mathcal{E}$ ,  $a \geq b > 0$ ,  $a'$  and  $b'$  semiaxes of  $\mathcal{E}'$ ,  $a' \geq b' > 0$ ,  $c = OF = OG \geq 0$ . Then from the confocality of  $\mathcal{E}$  and  $\mathcal{E}'$  it follows that  $a^2 - b^2 = a'^2 - b'^2 = c^2$  and also  $a^2 + b'^2 = a'^2 + b^2 = c^2 + b^2 + b'^2$ . Let  $AC$  be any diameter of the ellipse  $\mathcal{E}$  ( $A, C \in \mathcal{E}$ ). Then there exists a unique diameter  $BD$  of the ellipse  $\mathcal{E}'$  ( $B, D \in \mathcal{E}'$ ) such that the perimeter  $p(AC, BD) = AD + DC + CB + BA = 2(AD + DC) = 2(AB + BC)$  of the parallelogram  $ABCD$  reaches its maximum value*

Alain Connes und Don Zagier machten 2007 bei einem Paar konfokaler Ellipsen  $\mathcal{E}$  und  $\mathcal{E}'$  folgende bemerkenswerte Beobachtung: Wählt man einen beliebigen Durchmesser  $AC$  von  $\mathcal{E}$ , so existiert ein eindeutiger Durchmesser  $BD$  von  $\mathcal{E}'$ , so dass das Parallelogramm  $ABCD$  maximalen Umfang besitzt, und der Wert dieses Umfangs ist unabhängig von  $AC$ . Den Fall  $\mathcal{E} = \mathcal{E}'$  hat Michel Chasles bereits 1843 betrachtet. Die Problemstellung hat Bezüge zur Theorie der Billardbahnen in Ellipsen und zum Schliessungssatz von Poncelet. In der vorliegenden Arbeit wird der oben genannte Satz auf elementare Weise bewiesen, indem nur die Grundeigenschaften der Ellipse verwendet werden. Dabei ergeben sich weitere interessante Resultate, wie das Coxeter–Greitzer Lemma und eine Verallgemeinerung eines klassischen Satzes von Monge.

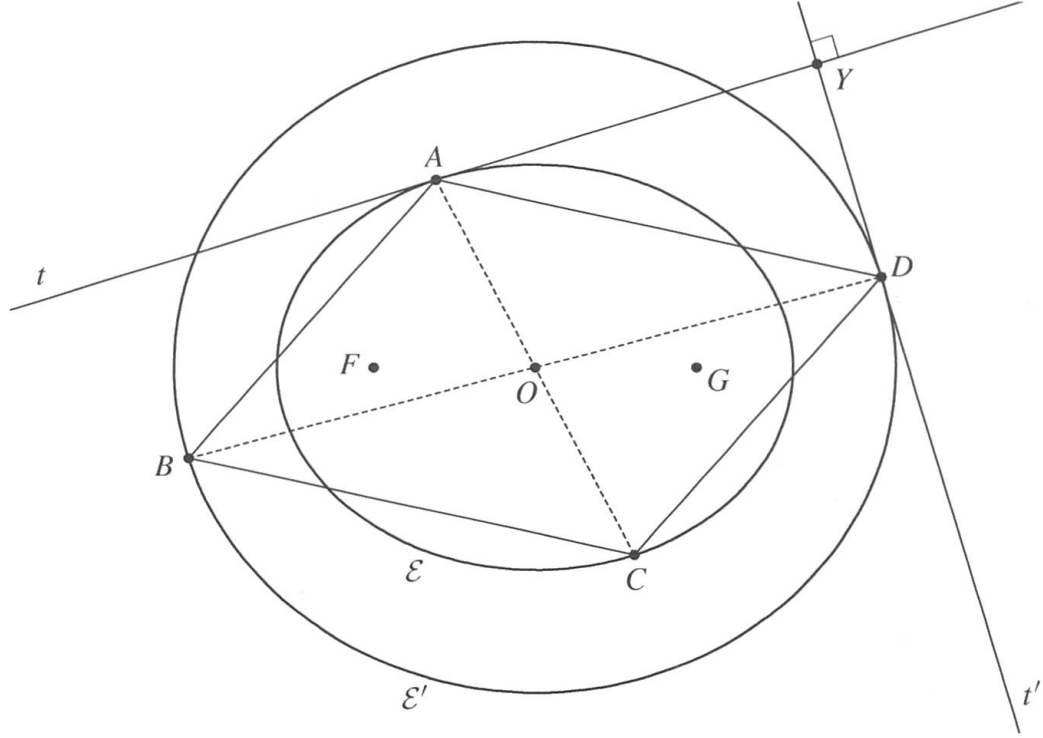


Figure 1 The parallelogram  $ABCD$  of maximal perimeter inscribed in a pair of confocal ellipses.

$p_{AC}$ . Moreover, the value of the maximal perimeter  $p_{AC}$  is independent of  $AC$  and is equal to  $p$  with

$$p = 4\sqrt{a^2 + b'^2} = 4\sqrt{a'^2 + b^2} = 4\sqrt{c^2 + b^2 + b'^2}.$$

Their proof of the above theorem is analytic and elementary, but in fact they establish the following clear *Geometric Criterion* for the maximal-perimeter property of a parallelogram  $ABCD$  (see Figure 1):

*The tangent  $t$  of  $\mathcal{E}$  at  $A$  (or  $C$ ) is perpendicular to the tangent  $t'$  of  $\mathcal{E}'$  at  $B$  (or  $D$ ).*

Compare the condition (3) in [4, p. 911] and (14) in Section 5. Incidentally, let us notice that the formula for  $\|P \pm P'\|$  at the bottom of p. 911 of [4] is incorrect. In the notation of [4] it should be replaced by

$$\|P \pm P'\| = \sqrt{C} \left( 1 \pm \frac{\lambda - \mu}{\lambda \lambda'} x x' \right).$$

For a single ellipse  $\mathcal{E} = \mathcal{E}'$  A. Connes and D. Zagier presented also a quite different and geometric proof of the result; see [4, Theorem 1]. Their argument involves Pascal's theorem from projective geometry which combined with basic metrical properties of an ellipse (i.e., the focal and optical ones, cf. Section 2) results in that the above-stated *Geometric Criterion* holds; see [4, Lemma]. The perpendicularity of tangents in the *Geometric Cri-*

*terion* is in turn related to the Monge circle of an ellipse; cf. Section 4. Another proof of the Theorem for a single ellipse, using projective arguments, was given by M. Berger [2] and analytic ones by J.-M. Richard [8]. As it is sketched in [4] by A. Connes and D. Zagier, the Theorem can be generalized and proved by considering  $2n$ -gons inscribed in a proper manner in  $n$  given confocal ellipses. This kind of generalization is connected with the theory of billiards and the Great Theorem of Poncelet; cf. [9].

Let us, however, observe that the Theorem describes the fine metric property of a pair of confocal ellipses. It then seems to be not out of interest to give a purely metric (i.e., using only Euclidean plane geometry) proof of it. The aim of this note is to give such a proof. The proof will be based only on the direct and straightforward applications of the standard tools of metric Euclidean plane geometry such as the laws of cosines and sines, etc., and the two main metric properties of an ellipse, namely the focal and optical ones.

## 2 Properties of an ellipse

Let  $\mathcal{E}$  be an ellipse with foci  $F$  and  $G$ ,  $O$  its centre,  $c = OF = OG$ ,  $2a$  its major axis,  $a > c \geq 0$ ,  $2b$  its minor axis. Let  $M \in \mathcal{E}$  and  $s$  be the tangent of  $\mathcal{E}$  at  $M$ , i.e.,  $s \cap \mathcal{E} = \{M\}$ . Denote by  $F'$  the point symmetric to  $F$  with respect to  $s$  and by  $\dot{F}$  the intersection of the straight lines  $FF'$  and  $s$ , so  $\dot{F}$  is the orthogonal projection of the focus  $F$  on the tangent  $s$ . Let  $AA'$  be the greatest diameter of  $\mathcal{E}$ . Under these assumptions we state the following Propositions 1–5; see Figure 2. We find all of them in *konika* by Apollonius from Perga; see [1, III.52, III.48, III.52, III.50, III.42], respectively.

**Proposition 1 (Focal property).**  $MF + MG = 2a$ .

**Proposition 2 (Optical property).** The radii  $FM$  and  $GM$  will make equal angles ( $= \alpha$ ,  $0 < \alpha \leq \pi/2$ ) with the tangent  $s$ .

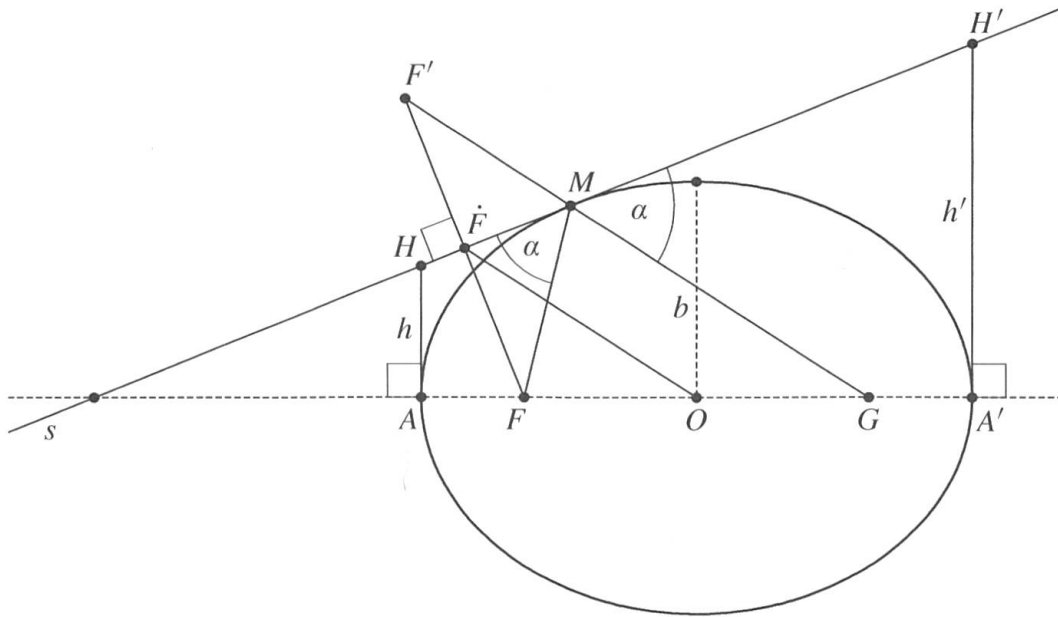


Figure 2 Properties of an ellipse.

**Proposition 3.**  $GF' = 2a$ .

**Proposition 4.**  $O\dot{F} = a$ .

**Proposition 5.** Let  $M \neq A, A'$ . Draw tangents of  $\mathcal{E}$  at  $A$  and  $A'$ . (They are perpendicular to the diameter  $AA'$ .) Denote by  $H$  and  $H'$  respective intersection points of these tangents with the tangent  $s$ . Let  $h = AH$ ,  $h' = A'H'$ . Then  $hh' = b^2$ .

**Remark 1.** The lines  $O\dot{F}$  and  $GF'$  in Figure 2 are parallel. This observation will be used later on in Section 5 in our geometric proof of the theorem.

### 3 Coxeter–Greitzer Lemma

**Proposition 6.** Let us suppose that a point  $Q$  lies outside a parallelogram  $P_1P_2P_3P_4$  ( $P_1P_2 > 0$ ,  $P_2P_3 > 0$ ) and the convex angle  $P_1QP_3$  is included in the convex angle  $P_2QP_4$ . We do not exclude a possibility that  $P_1P_2P_3P_4$  degenerates as a point set to a segment. Denote

$$\alpha := \angle P_1QP_4, \quad \beta := \angle P_2QP_3, \quad \gamma := \angle P_1P_2Q, \quad \delta := \angle P_1P_4Q.$$

Let us assume that

$$\alpha = \beta > 0.$$

Then we have

$$\gamma = \delta > 0.$$

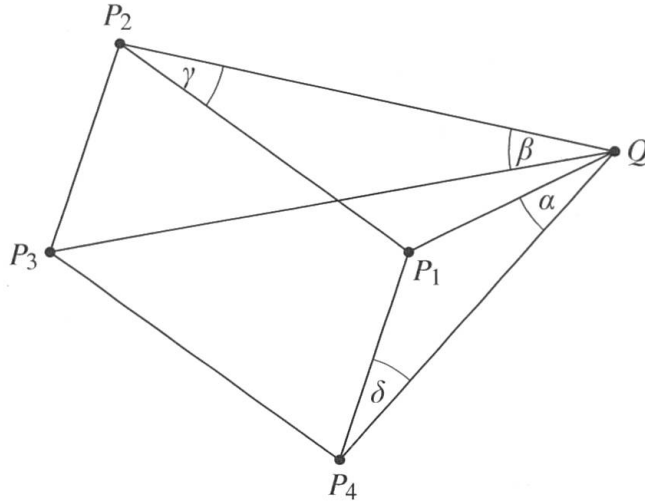


Figure 3 The Coxeter–Greitzer Lemma:  $\alpha = \beta > 0 \Rightarrow \gamma = \delta > 0$ .

**Remarks 2.** Proposition 6 states that the implication

$$\alpha = \beta > 0 \Rightarrow \gamma = \delta > 0 \tag{1}$$

is true. In their classic book [5] H.S.M. Coxeter and S.L. Greitzer presented the converse implication

$$\gamma = \delta > 0 \Rightarrow \alpha = \beta > 0 \tag{2}$$

as an exercise to be proved; see [5, pp. 25–26]. Interesting enough, they described (2) and a few other exercises as “*well-known posers*” and “*hardy perennials*”; see [5, p. 25]. A method of proof of (2) given in their book [5, pp. 158–159], attributed to D. Sokolowski, can also be used to prove (1), i.e., Proposition 6.

#### 4 Monge circle theorem for a pair of confocal ellipses

**Proposition 7.** *Let  $\mathcal{E}$  and  $\mathcal{E}'$  be a pair of confocal ellipses with foci  $F$  and  $G$ . Let  $O$  be a centre of the ellipses,  $a$  and  $b$  semiaxes of  $\mathcal{E}$ ,  $a \geq b > 0$ ,  $a'$  and  $b'$  semiaxes of  $\mathcal{E}'$ ,  $a' \geq b' > 0$ ,  $c = OF = OG \geq 0$ . Let  $t$  be a tangent of  $\mathcal{E}$  at  $T \in \mathcal{E}$  and  $t'$  be a tangent of  $\mathcal{E}'$  at  $T' \in \mathcal{E}$ . Suppose that  $t$  is perpendicular to  $t'$ . Denote by  $Y$  their intersection point. Then*

$$OY^2 = b^2 + b'^2 + c^2.$$

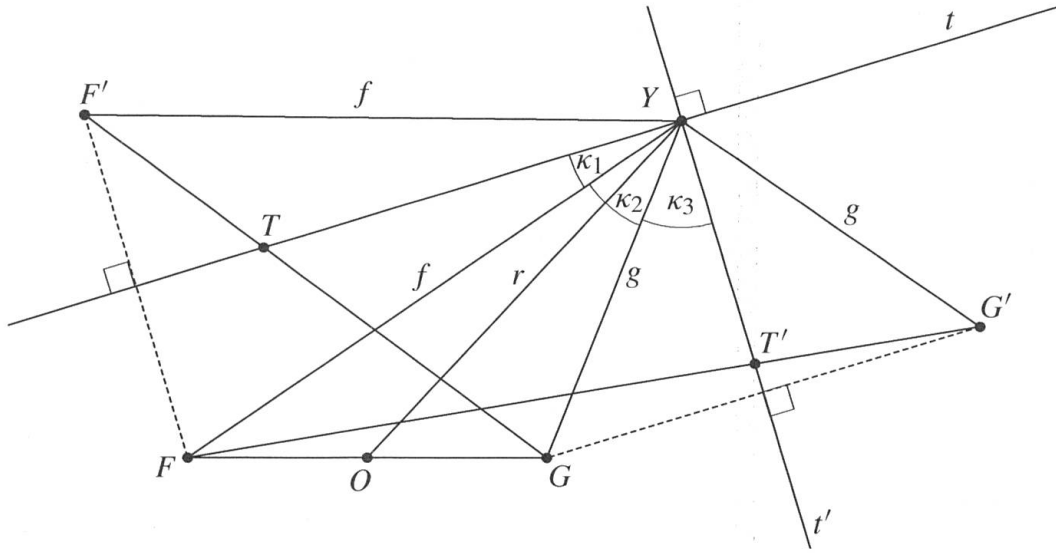


Figure 4 Proving the Monge circle theorem for a pair of confocal ellipses via the law of cosines.

*First Proof.* Denote

$$r := OY, f := FY, g := GY;$$

see Figure 4. The segment  $OY$  is a median of the triangle  $FYG$ . We have the parallelogram equation [7, VII.122]

$$r^2 = \frac{1}{2}(f^2 + g^2) - c^2. \quad (3)$$

Denote

$$\kappa_1 := \angle TYF, \kappa_2 := \angle FYG, \kappa_3 := \angle GYT'.$$

If necessary, make the change  $F \leftrightarrow G$  of the notations of the foci to rewrite our assumption about the perpendicularity of  $t$  and  $t'$  in the form

$$\kappa_1 + \kappa_2 + \kappa_3 = \frac{\pi}{2}; \quad (4)$$

see Figure 4.

Reflect the focus  $F$  of  $\mathcal{E}$  at  $t$  to the point  $F'$  and the focus  $G$  of  $\mathcal{E}'$  at  $t'$  to  $G'$ . We have constructed the triangles  $F'YG$  and  $FYG'$ . In the triangle  $F'YG$  we have by construction

$$F'Y = FY = f, YG = g, \angle F'YG = 2\kappa_1 + \kappa_2$$

and also  $F'G = 2a$  by Proposition 3 applied to the ellipse  $\mathcal{E}$ . In the triangle  $FYG'$  we have by construction

$$FY = f, YG' = YG = g, \angle FYG' = \kappa_2 + 2\kappa_3$$

and  $FG' = 2a'$  by Proposition 3 applied to the ellipse  $\mathcal{E}'$ .

The law of cosines [6, II.12,13] applied to the triangles  $F'YG$  and  $FYG'$  gives

$$4a^2 = f^2 + g^2 - 2fg \cos(2\kappa_1 + \kappa_2) \quad (5)$$

and

$$4a'^2 = f^2 + g^2 - 2fg \cos(\kappa_2 + 2\kappa_3). \quad (6)$$

By (4),

$$\cos(2\kappa_1 + \kappa_2) + \cos(\kappa_2 + 2\kappa_3) = \cos\left(\frac{\pi}{2} + \kappa_1 - \kappa_3\right) + \cos\left(\frac{\pi}{2} - \kappa_1 + \kappa_3\right) = 0. \quad (7)$$

From (3), (5), (6) and (7) we get

$$OY^2 = r^2 = a^2 + a'^2 - c^2 = b^2 + b'^2 + c^2,$$

for the confocality of  $\mathcal{E}$  and  $\mathcal{E}'$  means in particular that  $a^2 = b^2 + c^2$ ,  $a'^2 = b'^2 + c^2$ .  $\square$

*Second Proof (Gerhard Wanner).* Let us resort to Cartesian analytic geometry and equip the plane with a rectangular coordinate system to the effect that the ellipses  $\mathcal{E}$  and  $\mathcal{E}'$  are defined by the equations  $x^2/a^2 + y^2/b^2 = 1$  and  $x^2/a'^2 + y^2/b'^2 = 1$  respectively, i.e., the major and minor axes of  $\mathcal{E}$  and  $\mathcal{E}'$  become the  $x$ -axis and the  $y$ -axis of the introduced coordinate system. Let  $x_0$  and  $y_0$  be the coordinates of  $Y$ .

Consider a generic case of the location of tangents  $t$  and  $t'$  when neither  $t$  nor  $t'$  is parallel to the  $y$ -axis; cf. Figure 5. If  $p \neq 0$  is the slope of the line  $t$ , then the slope of the line  $t'$  is  $-1/p$ . Recalling that  $Y \in t \cap t'$ ,  $Y = (x_0, y_0)$  we can write down the equations of  $t$  and  $t'$  in the form

$$y = y_0 + p(x - x_0) \quad (8)$$

and

$$y = y_0 - \frac{1}{p}(x - x_0) \quad (9)$$

respectively.

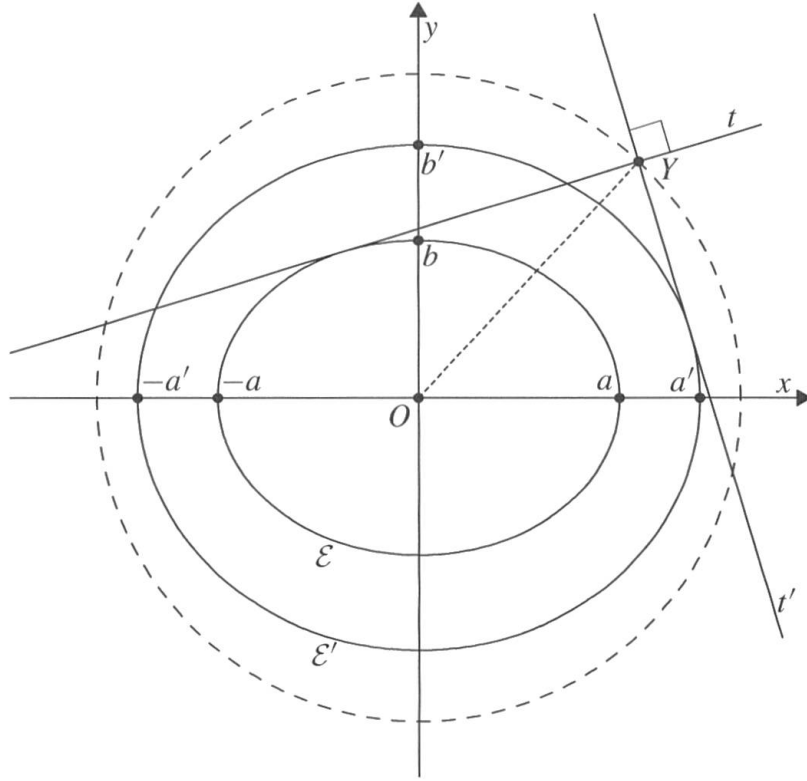


Figure 5 Monge circle theorem for a pair of confocal ellipses via Apollonius III.42.

Let us apply Proposition 5 to the ellipses  $\mathcal{E}$  and  $\mathcal{E}'$  and their tangents  $t$  and  $t'$ . By (8) and (9) we conclude that

$$(y_0 + p(-a - x_0))(y_0 + p(a - x_0)) = b^2 \quad (10)$$

and

$$\left(y_0 - \frac{1}{p}(-a' - x_0)\right)\left(y_0 - \frac{1}{p}(a' - x_0)\right) = b'^2. \quad (11)$$

On adding the equation (10) and the equation (11) premultiplied by  $p^2$  and arranging terms of the resulting equation we get

$$(1 + p^2)(x_0^2 + y_0^2) = a'^2 + b^2 + p^2(a^2 + b'^2).$$

Consequently,

$$\begin{aligned} OY^2 &= x_0^2 + y_0^2 \\ &= a^2 + b'^2 + ((a'^2 + b^2) - (a^2 + b'^2))/(1 + p^2). \end{aligned} \quad (12)$$

By straightforward verification we can check that the nice formula (12), just established under the assumption  $p \in \mathbb{R}$ ,  $p \neq 0$ , is also true for  $p = 0$  and  $p = \pm\infty$  (on the understanding that  $1/(1 + (\pm\infty)^2) := 0$ ). The conclusion is that in all cases of the possible



location of  $t$  and  $t'$  (12) holds. Finally making use of the confocality condition  $a^2 + b'^2 = a'^2 + b^2$  we get from (12)

$$OY^2 = a^2 + b'^2 = a'^2 + b^2. \quad \square$$

**Remarks 3.** By Proposition 6 the locus of points from which one sees a pair of confocal ellipses at right angles is a circle. In the case of a single ellipse  $\mathcal{E} = \mathcal{E}'$  this statement reduces to the classical theorem of G. Monge (1746–1818).

## 5 Geometric proof of the theorem

We assume that  $a' \geq a > 0$ . The case  $0 < a' < a$  can be treated similarly. Let us fix a diameter  $AC$  of  $\mathcal{E}$  and consider any parallelogram  $AKCL$ ,  $KL$  being a diameter of  $\mathcal{E}'$ ; see Figure 6. Then the segment  $KL$  is also a diameter of the ellipse  $\mathcal{E}''(KL)$  with foci  $A$  and  $C$  and, by the focal property (Proposition 1), the major axis equal to  $AL + LC = \frac{1}{2}p(AC, KL)$ , where  $p(AC, KL)$  is the perimeter of  $AKCL$ . By rotating the diameter  $KL$  of  $\mathcal{E}'$  about  $O$  we obtain a whole family of ellipses  $\mathcal{E}''(KL)$ . A typical ellipse  $\mathcal{E}''(KL)$  intersects  $\mathcal{E}'$  in four different points [1, IV]:  $K, L$  and  $M, N$ , say; see Figure 6.

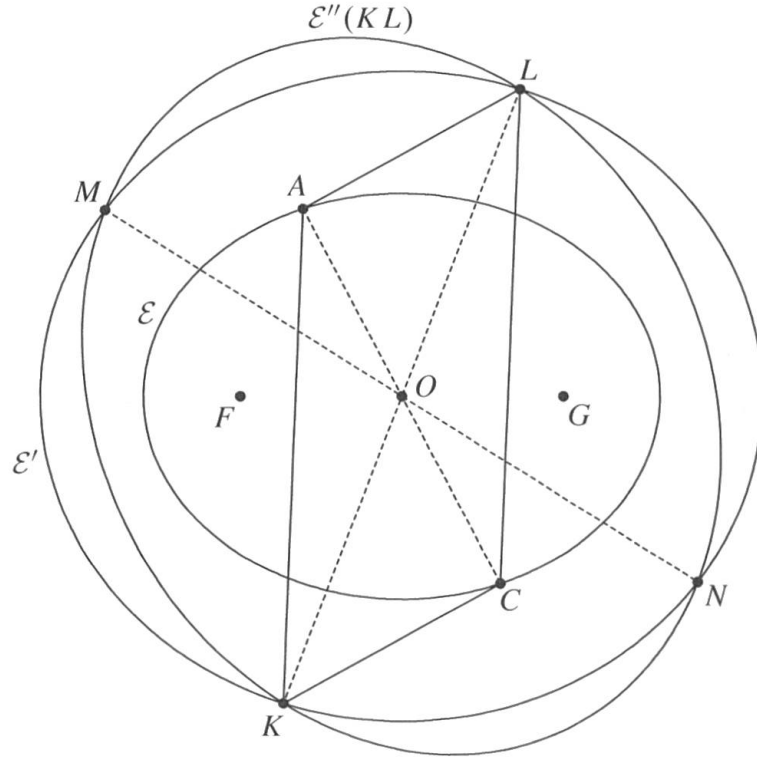


Figure 6 A parallelogram  $AKCL$  inscribed in  $\mathcal{E}$  and  $\mathcal{E}'$ .

The largest ellipse of this family is produced when the diameter  $KL$  of  $\mathcal{E}'$  is rotated to such a position  $KL := BD$  that  $p(AC, KL)$  reaches its maximum value  $p_{AC} = p(AC, BD) = 2(AD + DC)$ . Denote  $\tilde{\mathcal{E}} := \mathcal{E}''(BD)$ . The major axis of  $\tilde{\mathcal{E}}$  is equal to  $2\tilde{a} := (AD + DC) = \frac{1}{2}p_{AC}$ ; see Figure 7.

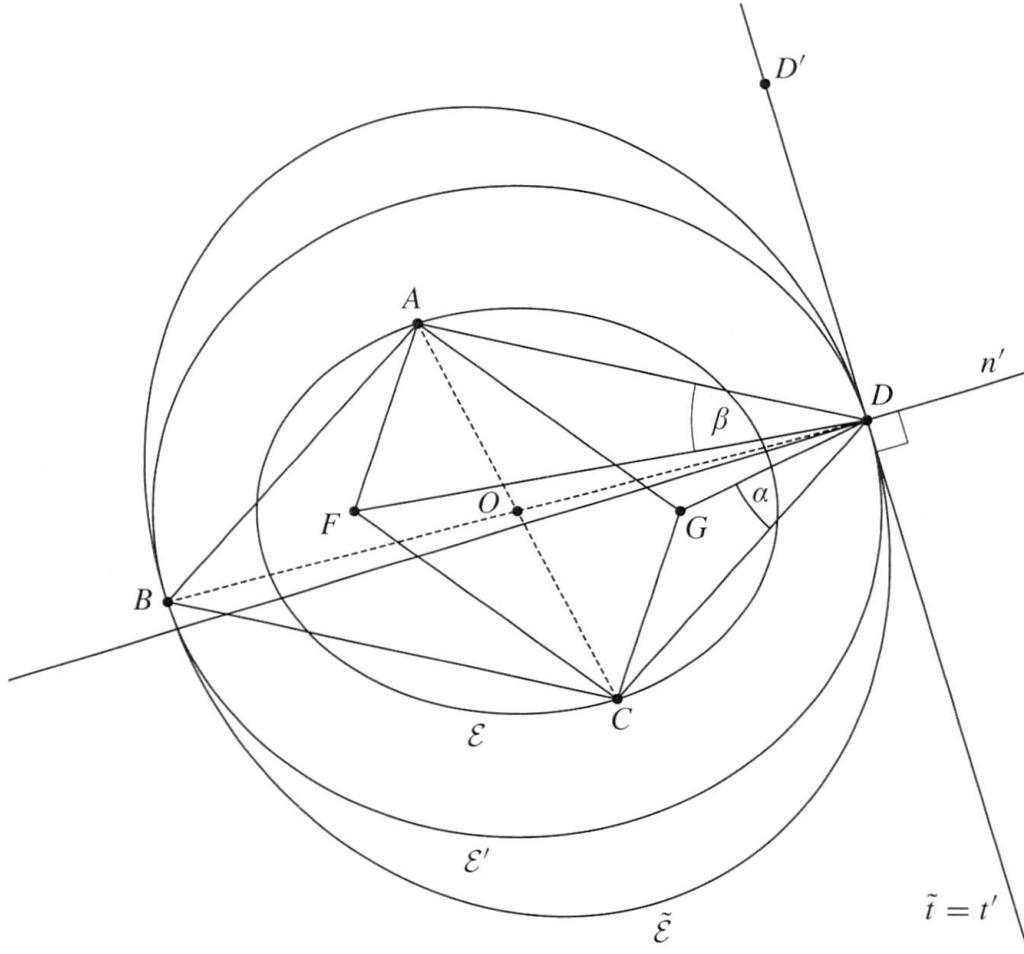


Figure 7 The parallelogram  $ABCD$  of the maximal perimeter  $p_{AC} = 2(AD + DC)$  and  $\alpha = \beta$ .

Now consider the tangent line  $\tilde{t}$  of  $\tilde{\mathcal{E}}$  at  $D \in \mathcal{E}' \cap \tilde{\mathcal{E}} : \tilde{t} \cap \tilde{\mathcal{E}} = \{D\}$ . We claim that  $\tilde{t}$  is also tangent to the ellipse  $\mathcal{E}'$ . To prove this claim suppose the contrary. Then there exists  $D' \in \tilde{t}$ ,  $D' \neq D$  such that  $D' \in \mathcal{E}'$ ; see Figure 7. From the focal property of  $\tilde{\mathcal{E}}$  (Proposition 1) we get  $AD' + D'C > AD + DC$ . But our special choice of  $D \in \mathcal{E}'$  implies  $AD + DC = \frac{1}{2}p_{AC} \geq AD' + D'C$ . We get a contradiction, and this proves the claim. From now on we use the notation  $t'$  for the line  $\tilde{t}$  as well:  $t' = \tilde{t}$ . It can be observed that the two diameters  $KL$  and  $MN$  of  $\mathcal{E}'$  have merged to form the single diameter  $BD$ .

Let us notice that we have just shown that the points  $B$  and  $D$  are two double common points of the two different ellipses  $\mathcal{E}'$  and  $\tilde{\mathcal{E}}$ . So it is possible to resort to elementary Cartesian algebraic geometry to see that a diameter  $BD$  of  $\mathcal{E}'$  with the property that  $p(AC, BD) = p_{AC}$  is unique; see [3, Section 16.4]. But we stress that the uniqueness of the diameter  $BD$  will also follow from the important geometric step in our proof to be established below (see (14)) that the tangent  $t'$  of  $\mathcal{E}'$  at  $D$  is perpendicular to the tangent  $t$  of  $\mathcal{E}$  at  $A$ ; cf. the *Geometric Criterion* in the introduction.

From the optical property (Proposition 2) it follows that the radii  $AD$  and  $CD$  of  $\tilde{\mathcal{E}}$  make equal angles with the tangent  $\tilde{t} = t'$  at  $D$  and also the radii  $FD$  and  $DG$  of  $\mathcal{E}'$  do the same.

So we conclude that

$$\alpha := \angle GDC = \beta := \angle FDA;$$

see Figure 7.

Our next claim is that a configuration depicted in Figure 7 of the convex angles  $FDG$  and  $ADC$ , both symmetric with respect to the common normal  $n'$  of  $\mathcal{E}'$  and  $\tilde{\mathcal{E}}$  at  $D$ , is generic, i.e., the angle  $FDG$  is included properly in the angle  $ADC$ , so  $\alpha = \beta > 0$ .

First, we have the sharp inequality  $\tilde{a} = \frac{1}{4}p_{AC} > a'$ , because if  $RS$  is the greatest diameter of  $\mathcal{E}'$  and the parallelogram  $ARCS$  does not degenerate to a segment, then  $\tilde{a} \geq \frac{1}{4}p(AC, RS) = \frac{1}{2}(AR + AS) > \frac{1}{2}RS = a'$ , and if  $ARCS$  degenerates to a segment, then  $AC$  is the greatest diameter  $AC$  of  $\mathcal{E}$  and we choose the smallest diameter  $XZ$  of  $\mathcal{E}'$  (perpendicular to  $AC$ ) instead of  $RS$  to get by Pythagoras' theorem [6, I. 47] that  $\tilde{a} = \frac{1}{4}p_{AC} \geq \frac{1}{4}p(AC, XZ) = \sqrt{a^2 + b'^2} = \sqrt{b^2 + b'^2 + c^2} > \sqrt{b'^2 + c^2} = a'$ , as  $b > 0$ .

Next project orthogonally the points  $O, F, A$  on the line  $t' = \tilde{t}$  to the points  $\dot{O}, \dot{F}, \dot{A}$ , respectively; see Figure 8. Apply Proposition 4 to the ellipses  $\mathcal{E}'$  and  $\tilde{\mathcal{E}}$  to get  $O\dot{F} = a'$ ,  $O\dot{A} = \tilde{a} > a'$ .

In Figure 8 denote

$$\varphi := \angle O\dot{A}\dot{O}, \quad \psi := \angle O\dot{F}\dot{O}.$$

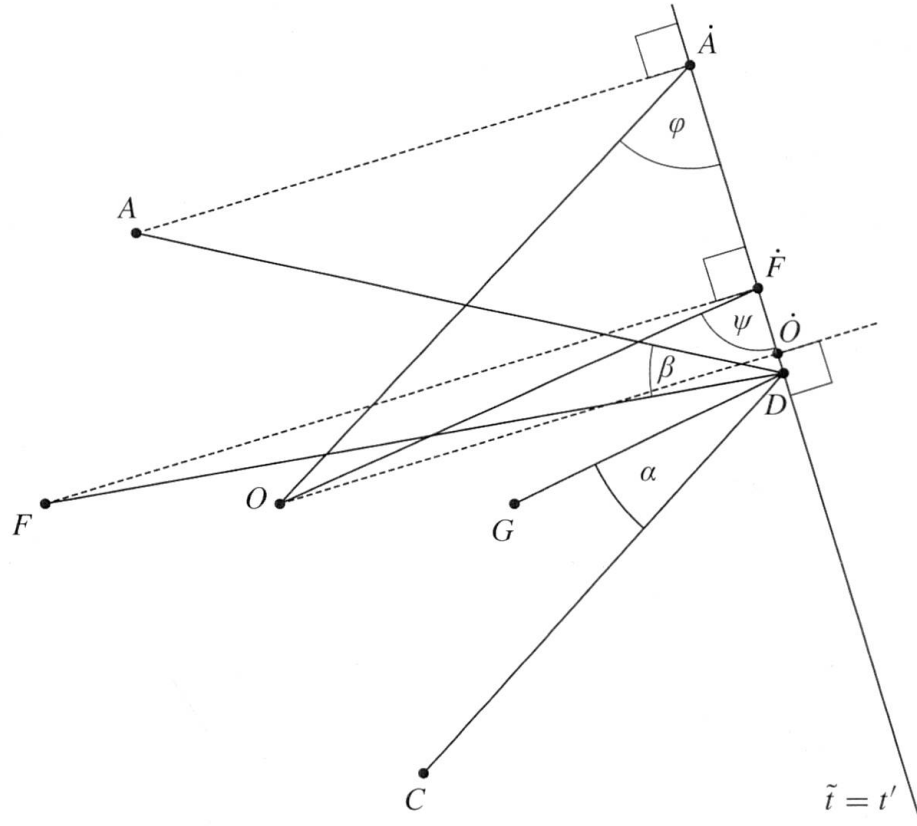


Figure 8 Generic configuration of the angles  $FDG$  and  $ADC$ ,  $\alpha = \beta = \psi - \varphi > 0$ . Details from Figure 7.

We see that

$$\sin \varphi = \frac{O\dot{O}}{O\dot{A}} = \frac{O\dot{O}}{\tilde{a}}, \quad \sin \psi = \frac{O\dot{O}}{O\dot{F}} = \frac{O\dot{O}}{a'} \quad \left(0 < \varphi, \psi \leq \frac{\pi}{2}\right).$$

Because  $\tilde{a} > a'$ , it follows that

$$0 < \varphi < \psi \leq \frac{\pi}{2}.$$

Now observe that in Figure 8 we have

$$O\dot{A} \parallel CD, \quad O\dot{F} \parallel GD;$$

cf. Remark 1 in Section 2. Therefore

$$\alpha = \beta = \psi - \varphi > 0.$$

This ends the verification of our claim that a configuration of the convex angles  $FDG$  and  $ADC$  in Figure 7 is generic.

Now we are prepared to apply the Coxeter–Greitzer Lemma (Proposition 6) to the parallelogram  $AGCF$  and the point  $D$  outside it; see Figure 7. We get

$$\gamma := \angle GAD = \angle GCD > 0;$$

see Figure 9.

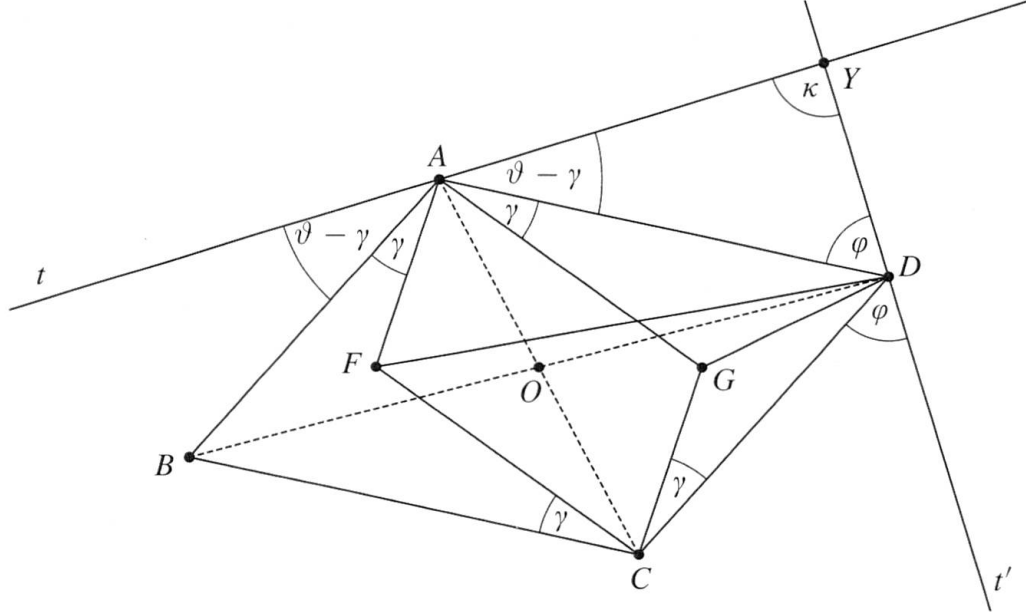


Figure 9 Proving  $\kappa = \frac{\pi}{2}$ .

Both the parallelograms  $AFCG$  and  $ABCD$  are symmetric with respect to  $O$ . So the triangles  $BAF$  and  $DCG$  are also symmetric with respect to  $O$ . Therefore

$$\angle BAF = \angle GAD = \gamma > 0.$$

Also, for symmetry reasons, the configuration of the radii  $FA$  and  $GA$  of  $\mathcal{E}$  and of the segments  $BA$  and  $AD$  in Figure 9 is generic, i.e., the convex angle  $FAG$  lies inside the convex angle  $BAD$  properly.

Let  $t$  be a tangent of  $\mathcal{E}$  at  $A$ . By the optical property of  $\mathcal{E}$  (Proposition 2) the radii  $FA$  and  $GA$  make equal angles  $= \vartheta$  with  $t$ . It follows that the segments  $BA$  and  $AD$  make equal angles  $= \vartheta - \gamma$  with  $t$ . In fact  $0 < \vartheta - \gamma < \frac{\pi}{2}$ , because in the parallelogram  $ABCD$  we have  $\angle BAD = \pi - \angle ADC = 2\varphi$ , so  $2(\vartheta - \gamma) = \pi - \angle BAD = \pi - 2\varphi$  and

$$0 < \vartheta - \gamma = \frac{\pi}{2} - \varphi < \frac{\pi}{2}, \quad (13)$$

for we proved before that  $0 < \varphi < \psi \leq \frac{\pi}{2}$ . The relation (13) means that the tangents  $t$  and  $t'$  intersect at the right angle, so the *Geometric Criterion* from the introduction for the maximality of the perimeter of  $ABCD$  holds; see Figures 9 and 1. In these figures the intersection point of  $t$  and  $t'$  is denoted by  $Y$ . Let us formulate the perpendicularity relation (13) as

$$\kappa := \angle(t, t') = \angle AYD = \frac{\pi}{2}. \quad (14)$$

As the diameter  $AC$  of  $\mathcal{E}$  determines a direction of the tangent  $t$  of  $\mathcal{E}$  ([1, II.49]) and for any ellipse there exists only one pair of tangents of the ellipse having a given direction ([1, II.50]), we conclude from (14) that a diameter  $BD$  of  $\mathcal{E}'$  with the property that  $p(AC, BD) = p_{AC}$  is unique. We mentioned in the beginning of proof that the uniqueness of  $BD$  (just proved geometrically) is an algebraic consequence of the coincidence of the tangents  $t'$  of  $\mathcal{E}'$  and  $\tilde{t}$  of  $\tilde{\mathcal{E}}$  at  $D$  as well.

Due to (14) the assumptions of the Monge circle theorem for a pair of confocal ellipses (Proposition 7) are satisfied. So we get

$$OY = \sqrt{b^2 + b'^2 + c^2}.$$

Recall that the ellipse  $\tilde{\mathcal{E}}$  with foci  $A$  and  $C$  touches the straight line  $t' = \tilde{t}$  at  $D$ ; see Figures 7 and 9. Proposition 4 applied to  $\tilde{\mathcal{E}}$  and  $M := D$ ,  $\tilde{F} := Y$  gives

$$OY = \tilde{a}.$$

It follows that

$$p_{AC} = 2(AD + DC) = 4\tilde{a} = 4 OY = 4\sqrt{b^2 + b'^2 + c^2}.$$

The theorem is proved.

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Dorota Blinkiewicz, Piotr Rzonsowski and Bogdan Szydło

Faculty of Mathematics and Computer Science

Adam Mickiewicz University

Umultowska 87

PL-61-614 Poznań, Poland

e-mails: `dorota.blinkiewicz@amu.edu.pl`

`rzonsol@amu.edu.pl`

`bszydlo@amu.edu.pl`