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Parallelograms inscribed in a pair of confocal ellipses

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1 Introduction

The following particular but interesting maximal property of confocal ellipses was proved by A. Connes and D. Zagier [4, Theorem 2]; see Figure 1.

Theorem. *Let $\mathcal{E}$ and $\mathcal{E}'$ be a pair of confocal ellipses with foci F and G. Let O be a centre of the ellipses, a and b semiaxes of $\mathcal{E}$, $a \geq b > 0$, a' and b' semiaxes of $\mathcal{E}'$, $a' \geq b' > 0$, $c = OF = OG \geq 0$. Then from the confocality of $\mathcal{E}$ and $\mathcal{E}'$ it follows that $a^2 - b^2 = a'^2 - b'^2 = c^2$ and also $a^2 + b'^2 = a'^2 + b^2 = c^2 + b^2 + b'^2$. Let AC be any diameter of the ellipse $\mathcal{E}$ $(A, C \in \mathcal{E})$. Then there exists a unique diameter BD of the ellipse $\mathcal{E}'$ $(B, D \in \mathcal{E}')$ such that the perimeter $p(AC, BD) = AD + DC + CB + BA = 2(AD + DC) = 2(AB + BC)$ of the parallelogram $ABCD$ reaches its maximum value*

Alain Connes und Don Zagier machten 2007 bei einem Paar konfokaler Ellipsen $\mathcal{E}$ und $\mathcal{E}'$ folgende bemerkenswerte Beobachtung: Wählt man einen beliebigen Durchmesser AC von $\mathcal{E}$, so existiert ein eindeutiger Durchmesser BD von $\mathcal{E}'$, so dass das Parallelogramm $ABCD$ maximalen Umfang besitzt, und der Wert dieses Umfangs ist unabhängig von AC. Den Fall $\mathcal{E} = \mathcal{E}'$ hat Michel Chasles bereits 1843 betrachtet. Die Problemstellung hat Bezüge zur Theorie der Billardbahnen in Ellipsen und zum Schliessungssatz von Poncelet. In der vorliegenden Arbeit wird der oben genannte Satz auf elementare Weise bewiesen, indem nur die Grundeigenschaften der Ellipse verwendet werden. Dabei ergeben sich weitere interessante Resultate, wie das Coxeter–Greitzer Lemma und eine Verallgemeinerung eines klassischen Satzes von Monge.

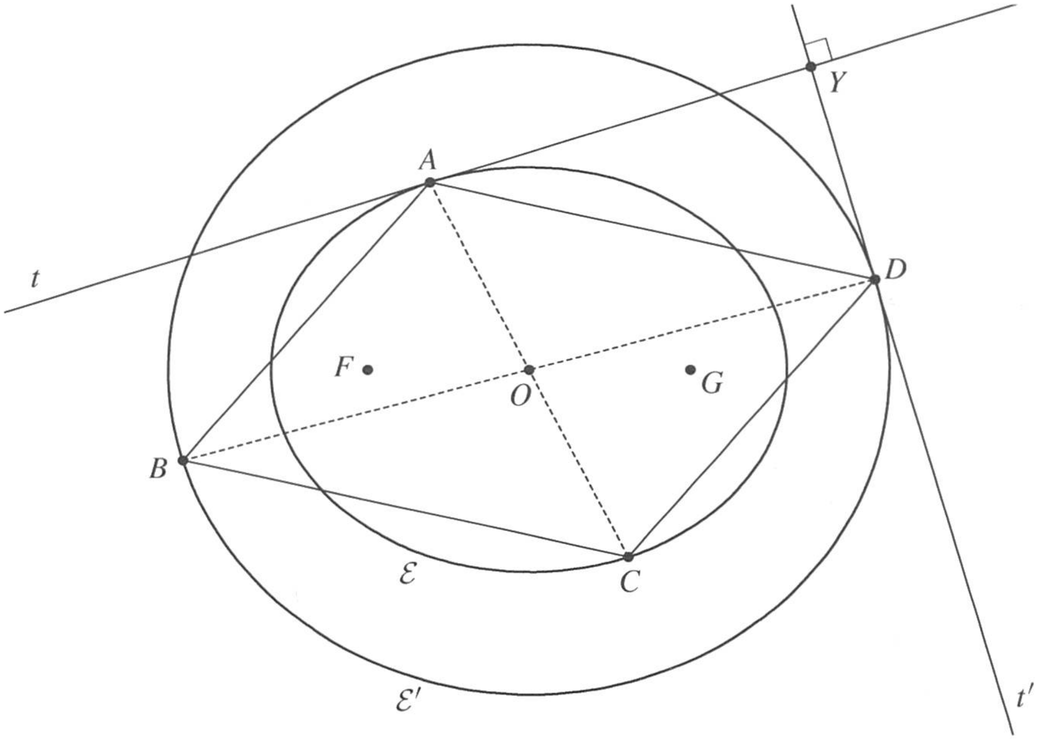

Figure 1 The parallelogram $ABCD$ of maximal perimeter inscribed in a pair of confocal ellipses.

p_{AC}. Moreover, the value of the maximal perimeter p_{AC} is independent of AC and is equal to p with

$$p = 4\sqrt{a^2 + b'^2} = 4\sqrt{a'^2 + b^2} = 4\sqrt{c^2 + b^2 + b'^2}.$$

Their proof of the above theorem is analytic and elementary, but in fact they establish the following clear *Geometric Criterion* for the maximal-perimeter property of a parallelogram $ABCD$ (see Figure 1):

The tangent t of $\mathcal{E}$ at A (or C) is perpendicular
to the tangent t' of $\mathcal{E}'$ at B (or D).

Compare the condition (3) in [4, p. 911] and (14) in Section 5. Incidentally, let us notice that the formula for $\|P \pm P'\|$ at the bottom of p. 911 of [4] is incorrect. In the notation of [4] it should be replaced by

$$\|P \pm P'\| = \sqrt{C}\left(1 \pm \frac{\lambda - \mu}{\lambda\lambda'} x x'\right).$$

For a single ellipse $\mathcal{E} = \mathcal{E}'$ A. Connes and D. Zagier presented also a quite different and geometric proof of the result; see [4, Theorem 1]. Their argument involves Pascal's theorem from projective geometry which combined with basic metrical properties of an ellipse (i.e., the focal and optical ones, cf. Section 2) results in that the above-stated *Geometric Criterion* holds; see [4, Lemma]. The perpendicularity of tangents in the *Geometric Cri-*

terion is in turn related to the Monge circle of an ellipse; cf. Section 4. Another proof of the Theorem for a single ellipse, using projective arguments, was given by M. Berger [2] and analytic ones by J.-M. Richard [8]. As it is sketched in [4] by A. Connes and D. Zagier, the Theorem can be generalized and proved by considering $2n$-gons inscribed in a proper manner in n given confocal ellipses. This kind of generalization is connected with the theory of billiards and the Great Theorem of Poncelet; cf. [9].

Let us, however, observe that the Theorem describes the fine metric property of a pair of confocal ellipses. It then seems to be not out of interest to give a purely metric (i.e., using only Euclidean plane geometry) proof of it. The aim of this note is to give such a proof. The proof will be based only on the direct and straightforward applications of the standard tools of metric Euclidean plane geometry such as the laws of cosines and sines, etc., and the two main metric properties of an ellipse, namely the focal and optical ones.

2 Properties of an ellipse

Let $\mathcal{E}$ be an ellipse with foci F and G, O its centre, $c = OF = OG$, $2a$ its major axis, $a > c \geq 0$, $2b$ its minor axis. Let $M \in \mathcal{E}$ and s be the tangent of $\mathcal{E}$ at M, i.e., $s \cap \mathcal{E} = \{M\}$. Denote by F' the point symmetric to F with respect to s and by $\dot{F}$ the intersection of the straight lines FF' and s, so $\dot{F}$ is the orthogonal projection of the focus F on the tangent s. Let AA' be the greatest diameter of $\mathcal{E}$. Under these assumptions we state the following Propositions 1–5; see Figure 2. We find all of them in *konika* by Apollonius from Perga; see [1, III.52, III.48, III.52, III.50, III.42], respectively.

Proposition 1 (Focal property). $MF + MG = 2a$.

Proposition 2 (Optical property). *The radii FM and GM will make equal angles ($= \alpha$, $0 < \alpha \leq \pi/2$) with the tangent s.*

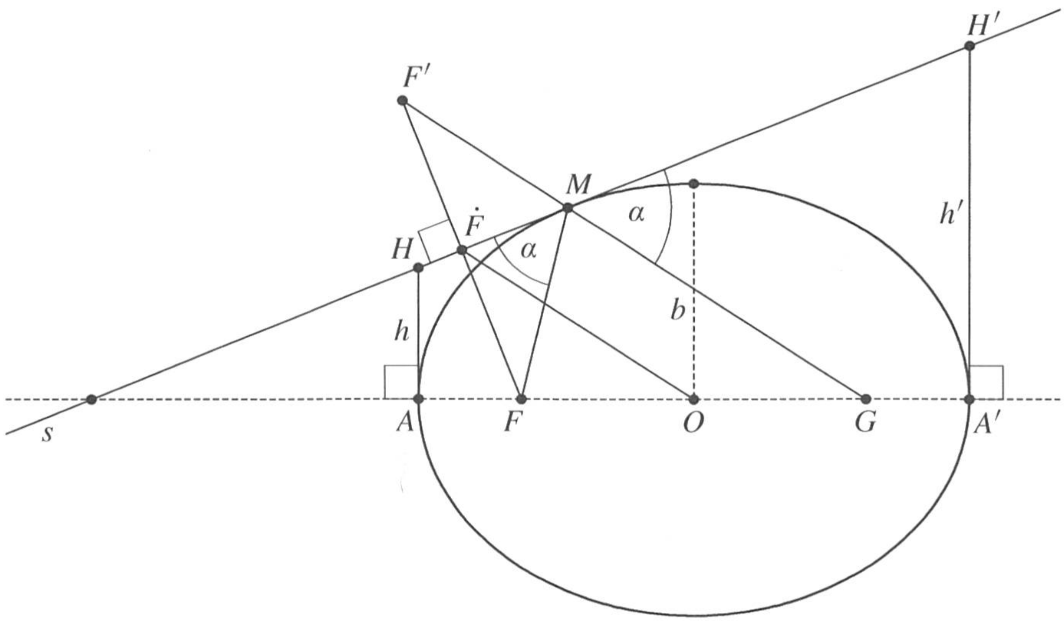

Figure 2 Properties of an ellipse.

Proposition 3. $GF' = 2a$.

Proposition 4. $O\dot{F} = a$.

Proposition 5. *Let $M \neq A, A'$. Draw tangents of $\mathcal{E}$ at A and A'. (They are perpendicular to the diameter AA'.) Denote by H and H' respective intersection points of these tangents with the tangent s. Let $h = AH$, $h' = A'H'$. Then $hh' = b^2$.*

Remark 1. The lines $O\dot{F}$ and GF' in Figure 2 are parallel. This observation will be used later on in Section 5 in our geometric proof of the theorem.

3 Coxeter–Greitzer Lemma

Proposition 6. *Let us suppose that a point Q lies outside a parallelogram $P_1P_2P_3P_4$ ($P_1P_2 > 0$, $P_2P_3 > 0$) and the convex angle P_1QP_3 is included in the convex angle P_2QP_4. We do not exclude a possibility that $P_1P_2P_3P_4$ degenerates as a point set to a segment. Denote*

$$\alpha := \angle P_1QP_4,\ \beta := \angle P_2QP_3,\ \gamma := \angle P_1P_2Q,\ \delta := \angle P_1P_4Q.$$

Let us assume that

$$\alpha = \beta > 0.$$

Then we have

$$\gamma = \delta > 0.$$

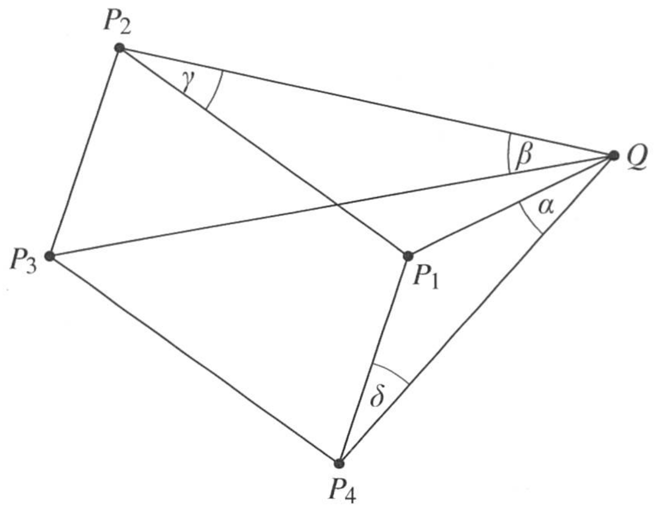

Figure 3 The Coxeter–Greitzer Lemma: $\alpha = \beta > 0 \Rightarrow \gamma = \delta > 0$.

Remarks 2. Proposition 6 states that the implication

$$\alpha = \beta > 0 \ \Rightarrow\ \gamma = \delta > 0 \tag{1}$$

is true. In their classic book [5] H.S.M. Coxeter and S.L. Greitzer presented the converse implication

$$\gamma = \delta > 0 \ \Rightarrow\ \alpha = \beta > 0 \tag{2}$$

as an exercise to be proved; see [5, pp. 25–26]. Interesting enough, they described (2) and a few other exercises as "*well-known posers*" and "*hardy perennials*"; see [5, p. 25]. A method of proof of (2) given in their book [5, pp. 158–159], attributed to D. Sokolowski, can also be used to prove (1), i.e., Proposition 6.

4 Monge circle theorem for a pair of confocal ellipses

Proposition 7. *Let $\mathcal{E}$ and $\mathcal{E}'$ be a pair of confocal ellipses with foci F and G. Let O be a centre of the ellipses, a and b semiaxes of $\mathcal{E}$, $a \geq b > 0$, a' and b' semiaxes of $\mathcal{E}'$, $a' \geq b' > 0$, $c = OF = OG \geq 0$. Let t be a tangent of $\mathcal{E}$ at $T \in \mathcal{E}$ and t' be a tangent of $\mathcal{E}'$ at $T' \in \mathcal{E}$. Suppose that t is perpendicular to t'. Denote by Y their intersection point. Then*

$$OY^2 = b^2 + b'^2 + c^2.$$

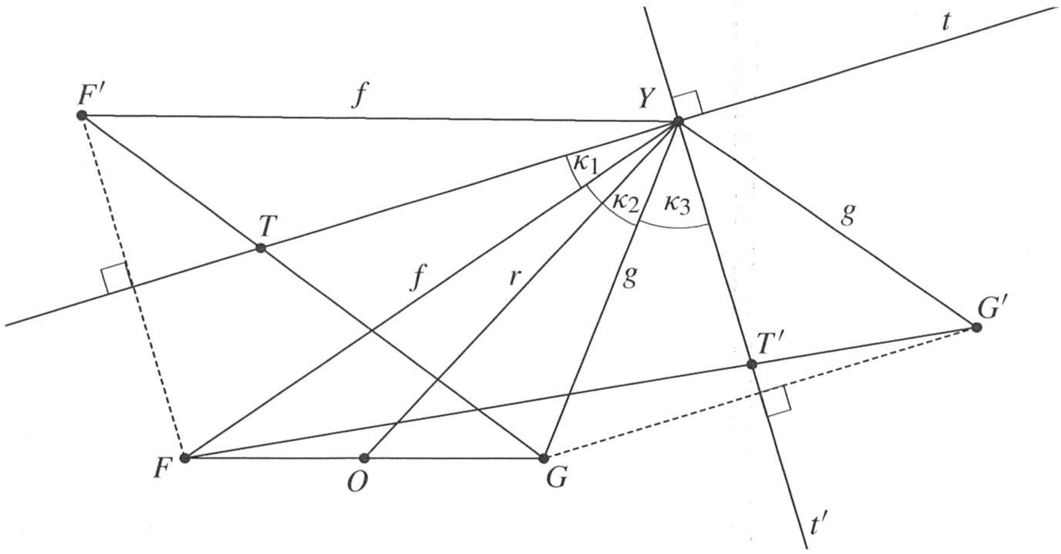

Figure 4 Proving the Monge circle theorem for a pair of confocal ellipses via the law of cosines.

First Proof. Denote

$$r := OY, \; f := FY, \; g := GY;$$

see Figure 4. The segment OY is a median of the triangle FYG. We have the parallelogram equation [7, VII.122]

$$r^2 = \frac{1}{2}(f^2 + g^2) - c^2. \tag{3}$$

Denote

$$\kappa_1 := \angle TYF, \; \kappa_2 := \angle FYG, \; \kappa_3 := \angle GYT'.$$

If necessary, make the change $F \leftrightarrow G$ of the notations of the foci to rewrite our assumption about the perpendicularity of t and t' in the form

$$\kappa_1 + \kappa_2 + \kappa_3 = \frac{\pi}{2}; \tag{4}$$

see Figure 4.

Reflect the focus F of $\mathcal{E}$ at t to the point F' and the focus G of $\mathcal{E}'$ at t' to G'. We have constructed the triangles $F'YG$ and FYG'. In the triangle $F'YG$ we have by construction

$$F'Y = FY = f,\ YG = g,\ \angle F'YG = 2\kappa_1 + \kappa_2$$

and also $F'G = 2a$ by Proposition 3 applied to the ellipse $\mathcal{E}$. In the triangle FYG' we have by construction

$$FY = f,\ YG' = YG = g,\ \angle FYG' = \kappa_2 + 2\kappa_3$$

and $FG' = 2a'$ by Proposition 3 applied to the ellipse $\mathcal{E}'$.

The law of cosines [6, II.12,13] applied to the triangles $F'YG$ and FYG' gives

$$4a^2 = f^2 + g^2 - 2fg\cos(2\kappa_1 + \kappa_2) \tag{5}$$

and

$$4a'^2 = f^2 + g^2 - 2fg\cos(\kappa_2 + 2\kappa_3). \tag{6}$$

By (4),

$$\cos(2\kappa_1 + \kappa_2) + \cos(\kappa_2 + 2\kappa_3) = \cos\left(\frac{\pi}{2} + \kappa_1 - \kappa_3\right) + \cos\left(\frac{\pi}{2} - \kappa_1 + \kappa_3\right) = 0. \tag{7}$$

From (3), (5), (6) and (7) we get

$$OY^2 = r^2 = a^2 + a'^2 - c^2 = b^2 + b'^2 + c^2,$$

for the confocality of $\mathcal{E}$ and $\mathcal{E}'$ means in particular that $a^2 = b^2 + c^2$, $a'^2 = b'^2 + c^2$. □

Second Proof (*Gerhard Wanner*). Let us resort to Cartesian analytic geometry and equip the plane with a rectangular coordinate system to the effect that the ellipses $\mathcal{E}$ and $\mathcal{E}'$ are defined by the equations $x^2/a^2 + y^2/b^2 = 1$ and $x^2/a'^2 + y^2/b'^2 = 1$ respectively, i.e., the major and minor axes of $\mathcal{E}$ and $\mathcal{E}'$ become the x-axis and the y-axis of the introduced coordinate system. Let x_0 and y_0 be the coordinates of Y.

Consider a generic case of the location of tangents t and t' when neither t nor t' is parallel to the y-axis; cf. Figure 5. If $p \neq 0$ is the slope of the line t, then the slope of the line t' is $-1/p$. Recalling that $Y \in t \cap t'$, $Y = (x_0, y_0)$ we can write down the equations of t and t' in the form

$$y = y_0 + p(x - x_0) \tag{8}$$

and

$$y = y_0 - \frac{1}{p}(x - x_0) \tag{9}$$

respectively.

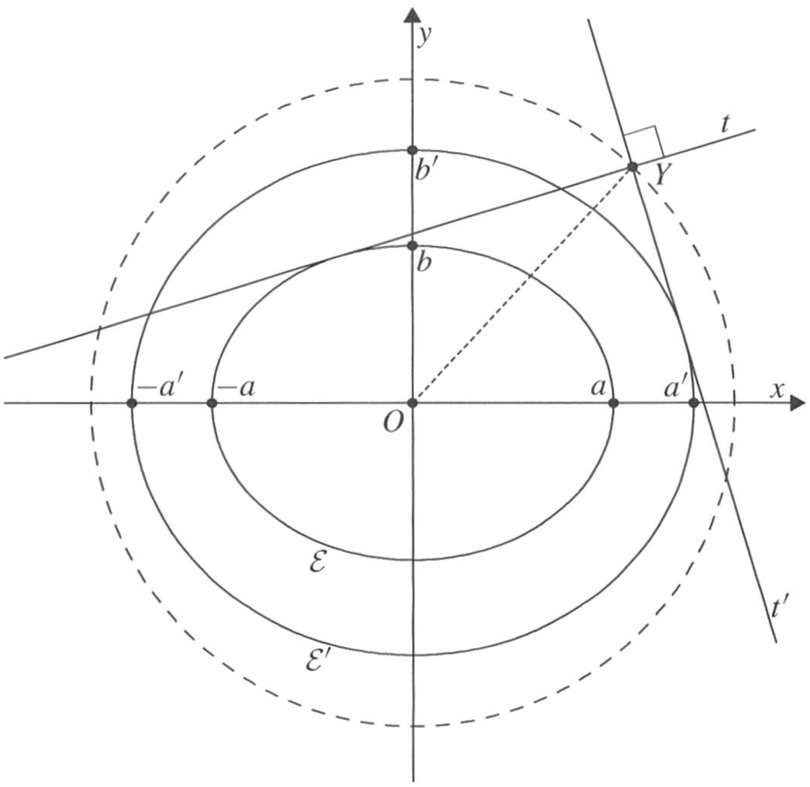

Figure 5 Monge circle theorem for a pair of confocal ellipses via Apollonius III.42.

Let us apply Proposition 5 to the ellipses $\mathcal{E}$ and $\mathcal{E}'$ and their tangents t and t'. By (8) and (9) we conclude that

$$(y_0 + p(-a - x_0))(y_0 + p(a - x_0)) = b^2 \tag{10}$$

and

$$\left(y_0 - \frac{1}{p}\left(-a' - x_0\right)\right)\left(y_0 - \frac{1}{p}\left(a' - x_0\right)\right) = b'^2. \tag{11}$$

On adding the equation (10) and the equation (11) premultiplied by p^2 and arranging terms of the resulting equation we get

$$(1 + p^2)(x_0^2 + y_0^2) = a'^2 + b^2 + p^2(a^2 + b'^2).$$

Consequently,

$$\begin{aligned} OY^2 &= x_0^2 + y_0^2 \\ &= a^2 + b'^2 + ((a'^2 + b^2) - (a^2 + b'^2))/(1 + p^2). \end{aligned} \tag{12}$$

By straightforward verification we can check that the nice formula (12), just established under the assumption $p \in \mathbb{R}$, $p \neq 0$, is also true for $p = 0$ and $p = \pm\infty$ (on the understanding that $1/(1+(\pm\infty)^2) := 0$). The conclusion is that in all cases of the possible location of t and t' (12) holds. Finally making use of the confocality condition $a^2 + b'^2 = a'^2 + b^2$ we get from (12)

$$OY^2 = a^2 + b'^2 = a'^2 + b^2. \qquad \square$$

Remarks 3. By Proposition 6 the locus of points from which one sees a pair of confocal ellipses at right angles is a circle. In the case of a single ellipse $\mathcal{E} = \mathcal{E}'$ this statement reduces to the classical theorem of G. Monge (1746–1818).

5 Geometric proof of the theorem

We assume that $a' \geq a > 0$. The case $0 < a' < a$ can be treated similarly. Let us fix a diameter AC of $\mathcal{E}$ and consider any parallelogram $AKCL$, KL being a diameter of $\mathcal{E}'$; see Figure 6. Then the segment KL is also a diameter of the ellipse $\mathcal{E}''(KL)$ with foci A and C and, by the focal property (Proposition 1), the major axis equal to $AL + LC = \frac{1}{2}p\,(AC, KL)$, where $p\,(AC, KL)$ is the perimeter of $AKCL$. By rotating the diameter KL of $\mathcal{E}'$ about O we obtain a whole family of ellipses $\mathcal{E}''(KL)$. A typical ellipse $\mathcal{E}''(KL)$ intersects $\mathcal{E}'$ in four different points [1, IV]: K, L and M, N, say; see Figure 6.

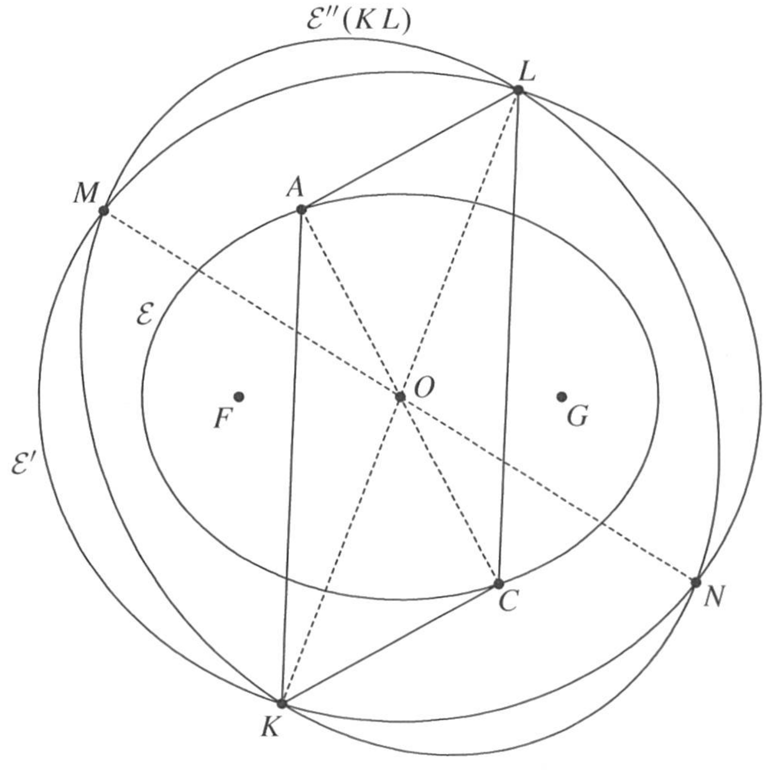

Figure 6 A parallelogram $AKCL$ inscribed in $\mathcal{E}$ and $\mathcal{E}'$.

The largest ellipse of this family is produced when the diameter KL of $\mathcal{E}'$ is rotated to such a position $KL := BD$ that $p\,(AC, KL)$ reaches its maximum value $p_{AC} = p\,(AC, BD) = 2(AD + DC)$. Denote $\tilde{\mathcal{E}} := \mathcal{E}''(BD)$. The major axis of $\tilde{\mathcal{E}}$ is equal to $2\tilde{a} := (AD + DC) = \frac{1}{2}p_{AC}$; see Figure 7.

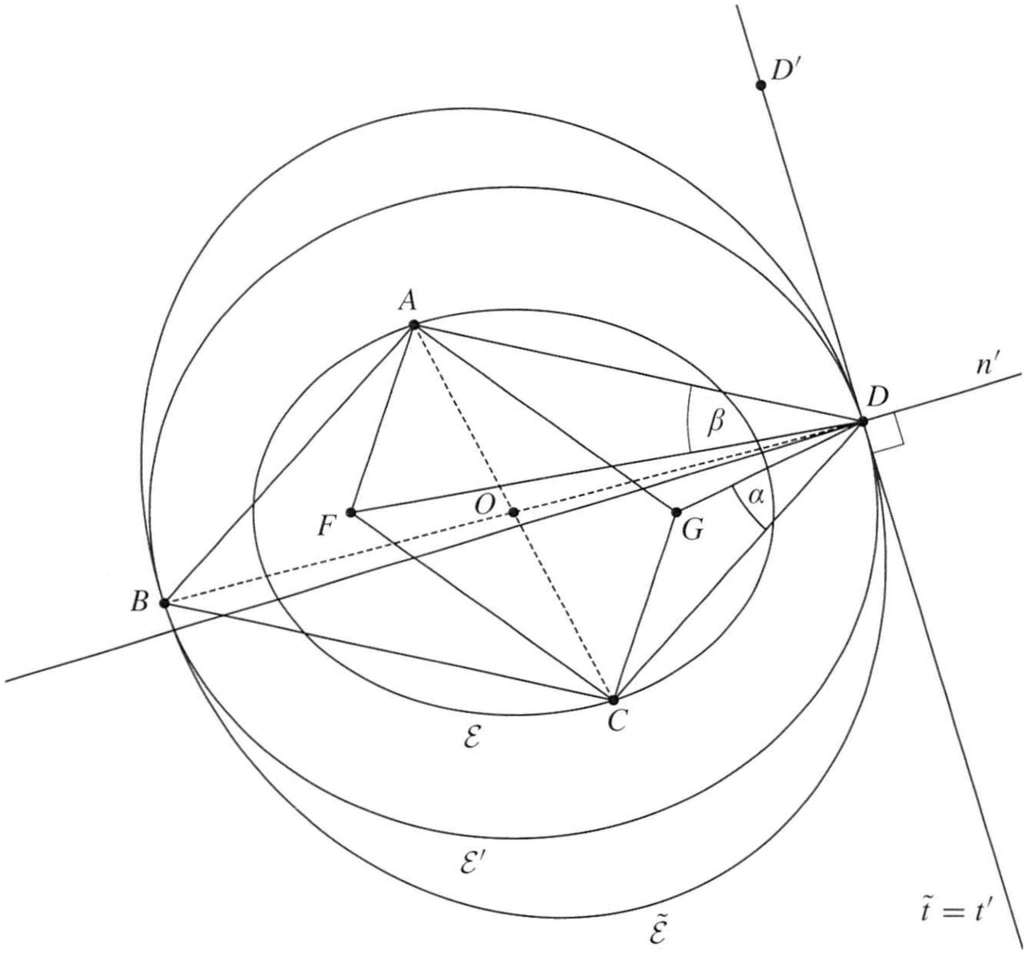

Figure 7 The parallelogram $ABCD$ of the maximal perimeter $p_{AC} = 2(AD + DC)$ and $\alpha = \beta$.

Now consider the tangent line $\tilde{t}$ of $\tilde{\mathcal{E}}$ at $D \in \mathcal{E}' \cap \tilde{\mathcal{E}} : \tilde{t} \cap \tilde{\mathcal{E}} = \{D\}$. We claim that $\tilde{t}$ is also tangent to the ellipse $\mathcal{E}'$. To prove this claim suppose the contrary. Then there exists $D' \in \tilde{t}$, $D' \neq D$ such that $D' \in \mathcal{E}'$; see Figure 7. From the focal property of $\tilde{\mathcal{E}}$ (Proposition 1) we get $AD' + D'C > AD + DC$. But our special choice of $D \in \mathcal{E}'$ implies $AD + DC = \frac{1}{2} p_{AC} \geq AD' + D'C$. We get a contradiction, and this proves the claim. From now on we use the notation t' for the line $\tilde{t}$ as well: $t' = \tilde{t}$. It can be observed that the two diameters KL and MN of $\mathcal{E}'$ have merged to form the single diameter BD.

Let us notice that we have just shown that the points B and D are two double common points of the two different ellipses $\mathcal{E}'$ and $\tilde{\mathcal{E}}$. So it is possible to resort to elementary Cartesian algebraic geometry to see that a diameter BD of $\mathcal{E}'$ with the property that $p(AC, BD) = p_{AC}$ is unique; see [3, Section 16.4]. But we stress that the uniqueness of the diameter BD will also follow from the important geometric step in our proof to be established below (see (14)) that the tangent t' of $\mathcal{E}'$ at D is perpendicular to the tangent t of $\mathcal{E}$ at A; cf. the *Geometric Criterion* in the introduction.

From the optical property (Proposition 2) it follows that the radii AD and CD of $\tilde{\mathcal{E}}$ make equal angles with the tangent $\tilde{t} = t'$ at D and also the radii FD and DG of $\mathcal{E}'$ do the same.

So we conclude that

$$\alpha := \angle GDC = \beta := \angle FDA;$$

see Figure 7.

Our next claim is that a configuration depicted in Figure 7 of the convex angles FDG and ADC, both symmetric with respect to the common normal n' of $\mathcal{E}'$ and $\tilde{\mathcal{E}}$ at D, is generic, i.e., the angle FDG is included properly in the angle ADC, so $\alpha = \beta > 0$.

First, we have the sharp inequality $\tilde{a} = \frac{1}{4}p_{AC} > a'$, because if RS is the greatest diameter of $\mathcal{E}'$ and the parallelogram $ARCS$ does not degenerate to a segment, then $\tilde{a} \geq \frac{1}{4}p\,(AC, RS) = \frac{1}{2}(AR + AS) > \frac{1}{2}RS = a'$, and if $ARCS$ degenerates to a segment, then AC is the greatest diameter AC of $\mathcal{E}$ and we choose the smallest diameter XZ of $\mathcal{E}'$ (perpendicular to AC) instead of RS to get by Pythagoras' theorem [6, I. 47] that $\tilde{a} = \frac{1}{4}p_{AC} \geq \frac{1}{4}p(AC, XZ) = \sqrt{a^2 + b'^2} = \sqrt{b^2 + b'^2 + c^2} > \sqrt{b'^2 + c^2} = a'$, as $b > 0$.

Next project orthogonally the points O, F, A on the line $t' = \tilde{t}$ to the points $\dot{O}$, $\dot{F}$, $\dot{A}$, respectively; see Figure 8. Apply Proposition 4 to the ellipses $\mathcal{E}'$ and $\tilde{\mathcal{E}}$ to get $O\dot{F} = a'$, $O\dot{A} = \tilde{a} > a'$.

In Figure 8 denote

$$\varphi := \angle O\dot{A}\dot{O}, \ \psi := \angle O\dot{F}\dot{O}.$$

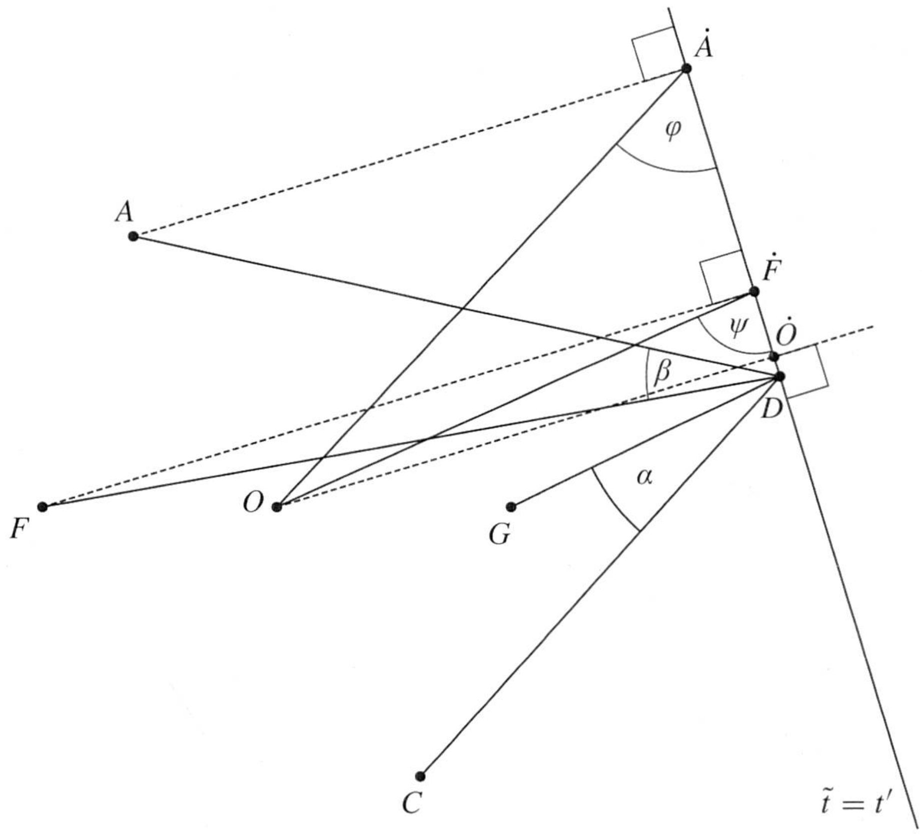

Figure 8 Generic configuration of the angles FDG and ADC, $\alpha = \beta = \psi - \varphi > 0$. Details from Figure 7.

We see that

$$\sin\varphi = \frac{O\dot{O}}{O\dot{A}} = \frac{O\dot{O}}{\tilde{a}}, \quad \sin\psi = \frac{O\dot{O}}{O\dot{F}} = \frac{O\dot{O}}{a'} \quad \left(0 < \varphi,\ \psi \le \frac{\pi}{2}\right).$$

Because $\tilde{a} > a'$, it follows that

$$0 < \varphi < \psi \le \frac{\pi}{2}.$$

Now observe that in Figure 8 we have

$$O\dot{A} \parallel CD, \ O\dot{F} \parallel GD;$$

cf. Remark 1 in Section 2. Therefore

$$\alpha = \beta = \psi - \varphi > 0.$$

This ends the verification of our claim that a configuration of the convex angles FDG and ADC in Figure 7 is generic.

Now we are prepared to apply the Coxeter–Greitzer Lemma (Proposition 6) to the parallelogram $AGCF$ and the point D outside it; see Figure 7. We get

$$\gamma := \angle GAD = \angle GCD > 0;$$

see Figure 9.

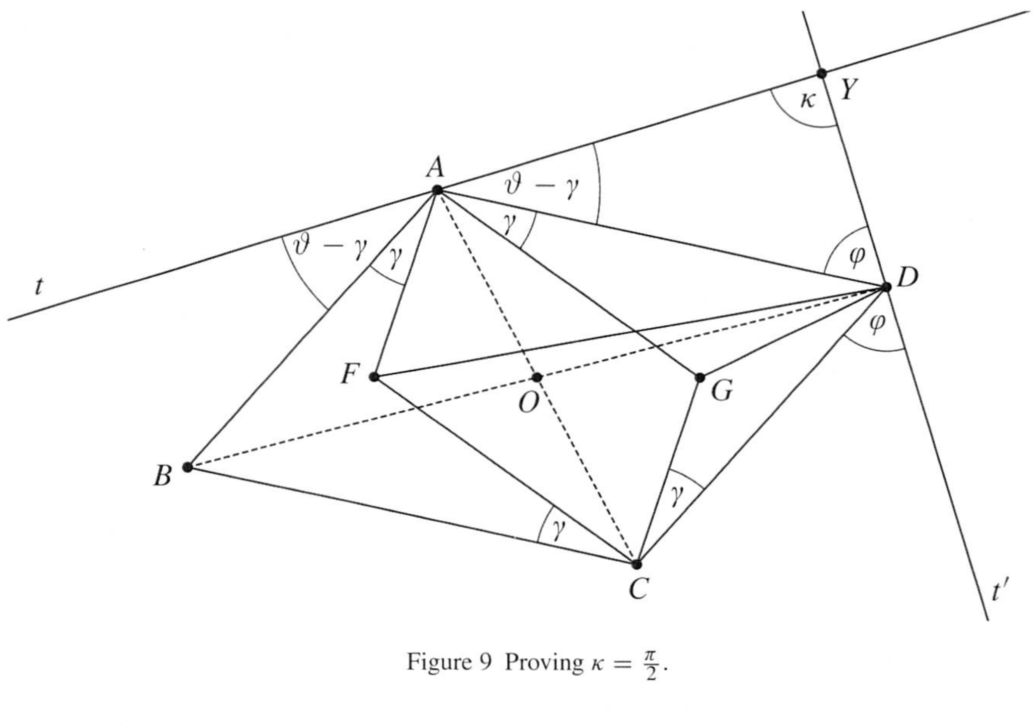

Figure 9 Proving $\kappa = \frac{\pi}{2}$.

Both the parallelograms $AFCG$ and $ABCD$ are symmetric with respect to O. So the triangles BAF and DCG are also symmetric with respect to O. Therefore

$$\angle BAF = \angle GAD = \gamma > 0.$$

Also, for symmetry reasons, the configuration of the radii FA and GA of $\mathcal{E}$ and of the segments BA and AD in Figure 9 is generic, i.e., the convex angle FAG lies inside the convex angle BAD properly.

Let t be a tangent of $\mathcal{E}$ at A. By the optical property of $\mathcal{E}$ (Proposition 2) the radii FA and GA make equal angles $= \vartheta$ with t. It follows that the segments BA and AD make equal angles $= \vartheta - \gamma$ with t. In fact $0 < \vartheta - \gamma < \frac{\pi}{2}$, because in the parallelogram $ABCD$ we have $\angle BAD = \pi - \angle ADC = 2\varphi$, so $2(\vartheta - \gamma) = \pi - \angle BAD = \pi - 2\varphi$ and

$$0 < \vartheta - \gamma = \frac{\pi}{2} - \varphi < \frac{\pi}{2}, \tag{13}$$

for we proved before that $0 < \varphi < \psi \leq \frac{\pi}{2}$. The relation (13) means that the tangents t and t' intersect at the right angle, so the *Geometric Criterion* from the introduction for the maximality of the perimeter of $ABCD$ holds; see Figures 9 and 1. In these figures the intersection point of t and t' is denoted by Y. Let us formulate the perpendicularity relation (13) as

$$\kappa := \angle(t, t') = \angle AYD = \frac{\pi}{2}. \tag{14}$$

As the diameter AC of $\mathcal{E}$ determines a direction of the tangent t of $\mathcal{E}$ ([1, II.49]) and for any ellipse there exists only one pair of tangents of the ellipse having a given direction ([1, II.50]), we conclude from (14) that a diameter BD of $\mathcal{E}'$ with the property that $p(AC, BD) = p_{AC}$ is unique. We mentioned in the beginning of proof that the uniqueness of BD (just proved geometrically) is an algebraic consequence of the coincidence of the tangents t' of $\mathcal{E}'$ and $\tilde{t}$ of $\tilde{\mathcal{E}}$ at D as well.

Due to (14) the assumptions of the Monge circle theorem for a pair of confocal ellipses (Proposition 7) are satisfied. So we get

$$OY = \sqrt{b^2 + b'^2 + c^2}.$$

Recall that the ellipse $\tilde{\mathcal{E}}$ with foci A and C touches the straight line $t' = \tilde{t}$ at D; see Figures 7 and 9. Proposition 4 applied to $\tilde{\mathcal{E}}$ and $M := D$, $\dot{F} := Y$ gives

$$OY = \tilde{a}.$$

It follows that

$$p_{AC} = 2(AD + DC) = 4\tilde{a} = 4\ OY = 4\sqrt{b^2 + b'^2 + c^2}.$$

The theorem is proved.

Acknowledgement

We thank the referee for a number of proposals aimed at improving the presentation of the note. We incorporated many of them into the present version. We thank very much Gerhard Wanner for his permission to include his neat proof of the generalized Monge circle theorem (Proposition 7) in the present note; see Section 4.

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