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**Short note**

**Composite values of irreducible polynomials**

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Franz Lemmermeyer

In his letter to Euler from September 1743 in [3, Letter 73], Goldbach remarked that it

*is very easy to prove that no algebraic formula such as  $a+bx+cx^2+dx^3+\dots$ , where  $x$  is the index of the terms, can yield none but prime numbers, whatever integers the coefficients  $a, b, c, \dots$  may be; but all the same there are formulae which comprise a greater number of primes than many others; the series  $x^2 + 19x - 19$  is of this kind, as in its 47 initial terms it comprises only 4 non-prime numbers.*

In this note we will give a very short proof of Goldbach's claim based on a simple identity, which shows not only the existence of infinitely many composite values of a given polynomial, but presents identities from which the claim follows directly. As an example, applying our result to Goldbach's polynomial  $f(x) = x^2 + 19x - 19$  provides us with the identity

$$f(x^2 + 20x - 19) = f(x) \cdot g(x) \quad \text{with} \quad g(x) = x^2 + 21x + 1 = f(x + 1),$$

which implies that  $f$  attains infinitely many composite values. In particular,  $f(25) = f(f(2) + 2) = f(2) \cdot g(2) = 23 \cdot 47$ .

Observe that Goldbach's claim is trivial if  $f$  is reducible or if its content (the greatest common divisor of its coefficients) is not a unit. Our main result is the following

**Theorem 1.** *Let  $f \in \mathbb{Z}[x]$  be an irreducible polynomial with integral and coprime coefficients. Then for an arbitrarily chosen polynomial  $q(x) \in \mathbb{Z}[x]$  there exists a polynomial  $g \in \mathbb{Z}[x]$  such that*

$$f(q(x)f(x) + x) = f(x)g(x). \tag{1}$$

We have to show that for every choice of  $q(x) \in \mathbb{Z}[x]$ , the polynomial

$$h(x) = f(q(x)f(x) + x)$$

is divisible by  $f(x)$ . Since  $f$  is irreducible, a polynomial  $h$  is divisible by  $f$  in the ring  $\mathbb{Q}[x]$  if and only if  $h(\alpha) = 0$  for all the complex roots  $\alpha$  of  $f$ . But if  $f(\alpha) = 0$ , then

$$h(\alpha) = f(q(\alpha)f(\alpha) + \alpha) = f(\alpha) = 0,$$

and we are done. Gauss's Lemma for polynomials (see [4] and [2] for the history of this result) now tells us that if  $h = fg$  for polynomials  $f, h \in \mathbb{Z}[x]$  for a primitive polynomial  $f$ , then the coefficients of  $g$  must be integral. This completes the proof.

This implies in particular that polynomials  $f$  with degree  $\geq 1$  represent infinitely many composite numbers of the form  $f(f(x) + x)$ . In fact, assume that  $f(x) = a_n x^n + \dots + a_0$  with  $a_n \geq 1$ . Then there is a constant  $C > 0$  such that  $f(x) > 1$  and  $f'(x) > 0$  for all  $x > C$ . But then  $f(x) + x > x$ , hence  $f(f(x) + x) > f(x)$  and thus also  $g(x) > 1$ .

**A second proof.** Theorem 1 may also be proved by setting  $h = q(x)f(x)$  in the Taylor identity

$$f(x+h) = f(x) + f'(x) \cdot h + \frac{f''(x)}{2!} h^2 + \dots + \frac{f^{(n+1)}(x)}{(n+1)!} h^{n+1}.$$

This implies

$$f(x+f(x)) = f(x) \left[ 1 + f'(x)q(x) + \frac{f''(x)}{2!} f(x)q(x)^2 + \dots + \frac{f^{(n+1)}(x)}{(n+1)!} f(x)^n q(x)^{n+1} \right].$$

Observe that the polynomials  $\frac{1}{k!} f^{(k)}(x)$  have integral coefficients since the product of  $k$  consecutive integers is divisible by  $k!$ .

**Applications to Goldbach's polynomial.** Out of the four composite values of  $f(n)$  for  $0 \leq n \leq 47$ , where  $f(x) = x^2 + 19x - 19$  is Goldbach's polynomial, the numbers  $f(19)$  and  $f(38)$  (and more generally  $f(19k)$  for integers  $k \geq 1$ ) are composite for trivial reasons: they are clearly divisible by 19. The other two composite values are  $f(25) = f(2 + f(2))$  and  $f(36) = f(-f(-1) - 1)$ . The next few composite values also follow from our theorem.

**André Gérardin.** I was led to the problem addressed here by a remark by André Gérardin (1879–1953) in [1], in which he claimed that the numbers of the form  $2a^2 - 1$  are composite for  $a = 9, 89, 881$  etc. I quickly found that these numbers are solutions of the diophantine equation  $3a^2 - 2b^2 = 1$ , and that Gérardin's claim follows from the observation that

$$9(2x^2 - 1) = 3(4y^2 - 1) = 3(2y - 1)(2y + 1).$$

This is reminiscent of the well-known fact that there are infinitely many composite integers of the form  $f(x) = 4x^2 + 1$  since

$$f(a^2) = 4a^4 + 1 = (2a^2 + 1)^2 - 4a^2 = (2a^2 + 2a + 1)(2a^2 - 2a + 1).$$

This begs the question whether there is a result that encompasses all these examples.

## References

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