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On the unsolvability of certain equations of Erdős–Moser type

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1 Introduction

For positive integers k and m, let $S_k(m) := \sum_{j=1}^{m-1} j^k$ be the sum of kth powers of the first $m-1$ positive integers. In 2011, Kellner [7] conjectured that for $m > 3$, the ratio $S_k(m+1)/S_k(m)$ cannot be an integer. Since $S_k(m+1) = S_k(m)+m^k$, one can reformulate Kellner's conjecture as follows: for any positive integer a, the equation $aS_k(m) = m^k$ has no solutions (m, k) with $m > 3$. The special case $a = 1$ of this conjecture is called the Erdős–Moser conjecture. It was proposed around 1950 by Paul Erdős in a letter to Leo Moser. Moser [13] proved that if (m, k) is a solution of $S_k(m) = m^k$ with $m > 3$, then $m > 10^{10^6}$. As another supporting fact of the Erdős–Moser conjecture, we mention the best-known lower bound $m > 2.7139 \cdot 10^{1\,667\,658\,416}$, which is due to Gallot, Moree and Zudilin [3].

Potenzsummen haben die Mathematikerinnen und Mathematiker über Jahrhunderte beschäftigt. So studierten unter anderem Johann Faulhaber und Jakob Bernoulli die Summen der Form $S_k(m) = 1^k + 2^k + \cdots + (m-1)^k$. Auch heute noch tauchen immer wieder neue interessante Fragen in diesem Gebiet auf: So stellte 2011 Bernd Kellner die Vermutung auf, dass für $m > 3$ das Verhältnis $S_k(m+1)/S_k(m)$ nie eine ganze Zahl ist, oder äquivalent ausgedrückt, dass für jede ganze Zahl a die Gleichung $aS_k(m) = m^k$ keine Lösung $k, m \in \mathbb{Z}^+$ mit $m > 3$ besitzt. Obwohl Kellners Frage in dieser Allgemeinheit nach wie vor offen ist, konnten Pieter Moree und die Autorin der vorliegenden Arbeit die Vermutung für bestimmte Fälle verifizieren. Im aktuellen Artikel betrachtet die Autorin ein verwandtes Problem für die Summen $T_k(m) = 1^k + 3^k + \cdots + (2m-1)^k$. Sie zeigt, dass die beiden Probleme eine enge Verbindung aufweisen, worauf sich gleich zwei neue Vermutungen aufdrängen.

While the Erdős–Moser conjecture remains open, the unsolvability of the equation $aS_k(m) = m^k$ (which we call the Kellner–Erdős–Moser equation) was recently [1] established for many integers $a > 1$. In particular, it was shown [1, Theorem 1.5] that if a is even or a has a regular prime divisor or $2 \le a \le 1500$, then the equation $aS_k(m) = m^k$ has no solutions (m, k) with $m > 3$.

For positive integers k and m, let $T_k(m) := \sum_{j=1}^{m}(2j-1)^k$ be the sum of kth powers of the first m odd positive integers. A natural question about $T_k(m)$ is whether there are positive integers k and $m > 1$ such that $T_k(m+1)/T_k(m)$ is an integer. Since $T_k(m+1) = T_k(m) + (2m+1)^k$, the question can be reformulated as follows: Do there exist positive integers a, k and $m > 1$ such that $aT_k(m) = (2m+1)^k$?

In this paper, we investigate the solvability in positive integers k and m of the equation $aT_k(m) = (2m+1)^k$, where a is a fixed positive integer. In order to do this, we introduce a new type of *helpful pairs*, which provides an important tool for establishing the unsolvability of the equation $aT_k(m) = (2m+1)^k$. The key observation is that for both equations $aS_k(m) = m^k$ and $aT_k(m) = (2m+1)^k$ the same set of helpful pairs can be used. Combining this with some known results on the Kellner–Erdős–Moser equation [1] leads us to our main theorem.

Theorem 1. *If a is even or a has a regular prime divisor or $2 \le a \le 1500$, then the equation $aT_k(m) = (2m+1)^k$ has no solutions (m, k) with $m > 1$.*

Furthermore, using some known facts about the Erdős–Moser equation [5, 11, 12], we deduce our next theorem.

Theorem 2. *Let k and m be positive integers satisfying*

$$1^k + 3^k + \cdots + (2m-1)^k = (2m+1)^k.$$

Then:

(a) $2^8 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot \prod_{23 \le p < 1000} p$ *divides k.*

(b) *Every prime divisor of $2m+1$ is greater than* 10000.

Motivated by Theorems 1 and 2, we propose the following conjectures.

Conjecture 1. $\{T_k(m+1)/T_k(m) \mid k, m \in \mathbb{Z}^+, m > 1\} \cap \mathbb{Z} = \varnothing$.

Conjecture 2.

$$\left\{ \frac{S_k(m+1)}{S_k(m)} \,\middle|\, k, m \in \mathbb{Z}^+, m > 3 \right\} \cap \mathbb{Z} = \left\{ \frac{T_k(m+1)}{T_k(m)} \,\middle|\, k, m \in \mathbb{Z}^+, m > 1 \right\} \cap \mathbb{Z}.$$

This paper is organized as follows. In Section 2, we recall some properties of power sums $S_k(m)$ and prove their analogues for $T_k(m)$. In Section 3, we obtain analogues of Theorem 3.11 and Corollaries 3.12 and 3.13 of [1]. In Section 4, we define helpful pairs of the first and second kind and prove a crucial result (Lemma 8) that allows us to deduce Theorems 1 and 2 as immediate consequences of the results of the previous section and some known facts about the Kellner–Erdős–Moser equation. Some other properties of integer solutions of $aT_k(m) = (2m+1)^k$ are briefly discussed in Section 5, as well as some related problems.

2 Preliminary Lemmas

The following result is known as Carlitz–von Staudt's theorem [2, 15] (see also [9] for a simpler proof).

Lemma 1. *Let k and m be positive integers. Then*

$$S_k(m) \equiv \begin{cases} 0 \pmod{\frac{m(m-1)}{2}} & \text{if } k \text{ is odd,} \\ -\sum_{p\mid m,(p-1)\mid k} \frac{m}{p} \pmod{m} & \text{if } k \text{ is even.} \end{cases}$$

Corollary 1. *Let k, m and n be positive integers and $p > 2$ be a prime with $(p-1) \nmid k$. If $m \equiv n \pmod{p}$, then $S_k(m) \equiv S_k(n) \pmod{p}$.*

Proof. We may assume that $m > n$. Since $(p-1) \nmid k$, Lemma 1 yields $p \mid S_k(m-n)$. Hence $S_k(m) = S_k(m-n) + \sum_{j=0}^{n-1}(m-n+j)^k \equiv S_k(n) \pmod{p}$. □

Our next lemma is the analogue of Lemma 1 for $T_k(m)$.

Lemma 2. *Let k and m be positive integers. Then*

$$T_k(m) \equiv \begin{cases} 0 \pmod{m} & \text{if } k \text{ is odd,} \\ (2^{k-1}-1)\sum_{p\mid(2m+1),(p-1)\mid k} \frac{2m+1}{p} \pmod{2m+1} & \text{if } k \text{ is even.} \end{cases}$$

Proof. First we observe that $T_k(m) = S_k(2m+1) - 2^k S_k(m+1)$. Assume that k is odd. By Lemma 1, we have $m(2m+1) \mid S_k(2m+1)$ and $m(m+1) \mid 2^k S_k(m+1)$, and thus $m \mid T_k(m)$. Now assume that k is even. In this case

$$S_k(2m+1) = S_k(m+1) + \sum_{j=1}^{m}(2m+1-j)^k \equiv 2S_k(m+1) \pmod{2m+1}.$$

Hence $T_k(m) \equiv (1-2^{k-1})S_k(2m+1) \pmod{2m+1}$, and the result follows by Lemma 1. □

The *Bernoulli numbers* $B_0, B_1, B_2, \ldots$ are defined by the generating function

$$\frac{z}{e^z-1} = \sum_{j=0}^{\infty} B_j \frac{z^j}{j!}, \qquad |z| < 2\pi.$$

They are rational numbers satisfying the recurrence relation $\sum_{j=0}^{k}\binom{k+1}{j}B_j = 0$ $(k \geq 1)$. It is easy to see that $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, and $B_j = 0$ for all odd $j > 1$. For a positive integer k, the kth *Bernoulli polynomial* $B_k(x)$ is defined by

$$B_k(x) = \sum_{j=0}^{k}\binom{k}{j}B_j x^{k-j}.$$

The following lemma is a special case of Raabe's result [14].

Lemma 3. *For any positive integer* k,

$$B_k(x) = 2^{k-1}\left(B_k\left(\frac{x}{2}\right) + B_k\left(\frac{x+1}{2}\right)\right).$$

The next lemma relates power sums to Bernoulli numbers and Bernoulli polynomials (for a proof, see [4, Chapter 15]).

Lemma 4. *Let* k *and* m *be positive integers. Then:*

(a) $S_k(m) = (B_{k+1}(m) - B_{k+1})/(k+1)$.

(b) $S_k(m) = \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \frac{m^{j+1}}{j+1}$.

We next obtain the analogue of Lemma 4(b) for $T_k(m)$ with even k.

Lemma 5. *Let* k *and* m *be positive integers, where* k *is even. Then*

$$T_k(m) = 2^k \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \frac{(2m+1)^{j+1}}{2^{j+1}(j+1)}.$$

Proof. Since $k > 0$ is even, we have $B_{k+1} = 0$. Then, by Lemmas 3 and 4(a),

$$\begin{aligned} T_k(m) = S_k(2m) - 2^k S_k(m) &= \frac{B_{k+1}(2m) - 2^k B_{k+1}(m)}{k+1} \\ &= \frac{2^k}{k+1} B_{k+1}\left(m + \frac{1}{2}\right) = \frac{2^k}{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} B_j \cdot \left(m + \frac{1}{2}\right)^{k+1-j} \\ &= 2^k \sum_{j=0}^{k} \frac{k!}{j!(k-j+1)!} B_j \cdot \left(m + \frac{1}{2}\right)^{k-j+1} = 2^k \sum_{j=0}^{k} \binom{k}{j} B_{k-j} \frac{(2m+1)^{j+1}}{2^{j+1}(j+1)}. \end{aligned}$$

□

Write $B_j = U_j/V_j$, where U_j and V_j are integers, $V_j > 0$, $\gcd(U_j, V_j) = 1$. Kellner [6] used Lemma 4(b) to derive the following result (see [6, Proposition 8.5]).

Lemma 6. *Let* k *and* m *be positive integers, where* k *is even. Then:*

(a) $m^2 \mid S_k(m)$ *if and only if* $m \mid U_k$.

(b) $m^3 \mid S_k(m)$ *if and only if* $m^2 \mid U_k$.

The next lemma can be proved in exactly the same way as Lemma 6, except that Lemma 5 is invoked instead of Lemma 4(b).

Lemma 7. *Let* k *and* m *be positive integers, where* k *is even. Then:*

(a) $(2m+1)^2 \mid T_k(m)$ *if and only if* $(2m+1) \mid U_k$.

(b) $(2m+1)^3 \mid T_k(m)$ *if and only if* $(2m+1)^2 \mid U_k$.

An odd prime p is said to be an *irregular prime* if p divides some U_r with even $r \le p-3$. Otherwise, the prime p is said to be a *regular prime*. The pairs (r, p) with $p \mid U_r$ and even $r \le p-3$ are said to be *irregular pairs*.

3 The Equation $aT_k(m) = (2m+1)^k$

First assume that $m = 1$. Then $a = 3^k$. Next assume that $m > 1$ and k is odd. Appealing to Lemma 2, we see that $(2m+1)^k$ must be divisible by m, which is impossible. This shows that we may restrict our study to solutions (m, k) with $m > 1$ and even k.

Proceeding exactly as in Section 3 of [1] (with m replaced by $2m+1$) and making use of Lemmas 2, 5 and 7, we obtain the following results.

Theorem 3. *Suppose that $aT_k(m) = (2m+1)^k$ with $m > 1$ and even k. Let p be a prime dividing $2m+1$. Then:*

(a) *p is an irregular prime.*

(b) $k \not\equiv 0, 2, 4, 6, 8, 10, 14 \pmod{p-1}$.

(c) $\mathrm{ord}_p(B_k/k) \geq 2\,\mathrm{ord}_p(2m+1) \geq 2$.

(d) $k \equiv r \pmod{p-1}$ *for some irregular pair (r, p).*

Corollary 2. *If a has a regular prime divisor, then the equation $aT_k(m) = (2m+1)^k$ has no solutions with $m > 1$.*

Corollary 3. *Let p_1 and p_2 be distinct irregular prime divisors of a. Assume that for every pair (r_1, p_1), (r_2, p_2) of irregular pairs, $\gcd(p_1 - 1, p_2 - 1) \nmid (r_1 - r_2)$. Then the equation $aT_k(m) = (2m+1)^k$ has no solutions.*

Remark. We observe that $aT_k(m)$ with $k > 1$ and $m > 1$ can be a perfect kth power even when k is odd or a has a regular prime divisor, for example, $315T_2(3) = 105^2$ and $12005T_3(5) = 245^3$. More examples can be constructed from formulas expressing $T_k(m)$ for small values of k.

4 Helpful Pairs

For a positive integer a let us call a pair $(t, q)_a$ with $q > 3$ a prime and $2 \leq t \leq q-3$ even to be a *potentially helpful pair* if $q \nmid a$ and, in case of irregular q, (t, q) is not an irregular pair.

Let $(t, q)_a$ be a potentially helpful pair. We say that $(t, q)_a$ is a *helpful pair of the first kind* if $aS_t(x) \equiv x^t \pmod{q}$ implies $x \equiv 0 \pmod{q}$. In view of Corollary 1 this definition is equivalent to the definition of a helpful pair given in Section 4 of [1]. We say that $(t, q)_a$ is a *helpful pair of the second kind* if $aT_t(x) \equiv (2x+1)^t \pmod{q}$ implies $2x + 1 \equiv 0 \pmod{q}$. The following lemma plays a crucial role in our argument.

Lemma 8. *Let $(t, q)_a$ be a potentially helpful pair. Then $(t, q)_a$ is a helpful pair of the first kind if and only if it is a helpful pair of the second kind.*

Proof. Since $(q-1) \nmid t$, we have, by Lemma 1, $q \mid S_t(q)$. Further, as t is even,

$$S_t(q) = S_t\left(\frac{q+1}{2}\right) + \sum_{j=1}^{(q-1)/2} (q-j)^t \equiv 2S_t\left(\frac{q+1}{2}\right) \pmod{q},$$

and so $q \mid S_t((q+1)/2)$. For a positive integer x, we have

$$S_t\Big(x+\frac{q+1}{2}\Big)=S_t\Big(\frac{q+1}{2}\Big)+\sum_{j=0}^{x-1}\Big(\frac{q+1}{2}+j\Big)^t.$$

Hence

$$2^tS_t\Big(x+\frac{q+1}{2}\Big)=2^tS_t\Big(\frac{q+1}{2}\Big)+\sum_{j=0}^{x-1}(q+1+2j)^t\equiv T_t(x)\pmod{q}.$$

This implies

$$2^t\left(aS_t\Big(x+\frac{q+1}{2}\Big)-\Big(x+\frac{q+1}{2}\Big)^t\right)\equiv aT_t(x)-(2x+1)^t\pmod{q},\qquad (1)$$

and

$$2^t\left(aS_t(x)-x^t\right)\equiv aT_t\Big(x+\frac{q-1}{2}\Big)-\Big(2\Big(x+\frac{q-1}{2}\Big)+1\Big)^t\pmod{q}.\qquad (2)$$

Assume that $(t,q)_a$ is a helpful pair of the first kind and for some positive integer x the congruence $aT_k(x)\equiv(2x+1)^t\pmod{q}$ holds. Then, by (1),

$$aS_t\Big(x+\frac{q+1}{2}\Big)\equiv\Big(x+\frac{q+1}{2}\Big)^t\pmod{q}.$$

Hence $x+\frac{q+1}{2}\equiv 0\pmod{q}$, and so $2x+1\equiv 0\pmod{q}$. Therefore $(t,q)_a$ is a helpful pair of the second kind.

In a similar manner, making use of (2), we conclude that if $(t,q)_a$ is a helpful pair of the second kind, then it is a helpful pair of the first kind. □

From now on, we will call both helpful pairs of the first kind and helpful pairs of the second kind simply helpful pairs. Next we prove an analogue of [1, Lemma 4.4] for the equation $aT_k(m)=(2m+1)^k$.

Lemma 9. *Let $q>3$ be a prime and $2\le t\le q-3$. If $(t,q)_a$ is a helpful pair and $aT_k(m)=(2m+1)^k$, then $k\not\equiv t\pmod{q-1}$.*

Proof. Assume that $k\equiv t\pmod{q-1}$. Then we find that $aT_t(m)\equiv aT_k(m)=(2m+1)^k\equiv(2m+1)^t\pmod{q}$. Hence $2m+1\equiv 0\pmod{q}$. Appealing to parts (a) and (d) of Theorem 3, we conclude that q is irregular and (t,q) is an irregular pair, which contradicts the definition of a potentially helpful pair. □

Lemma 9 shows that, like in the case of the Kellner–Erdős–Moser equation, one can try to use helpful pairs to prove that for a given a and given even positive integers c and d the equation $aT_k(m)=(2m+1)^k$ has no solutions with $k\equiv c\pmod{d}$. Namely, if there is a prime $q > 3$ such that $(q-1) \mid d$ and $(t, q)_a$ is a helpful pair, where t is the least nonnegative integer congruent to c modulo $q-1$, then, by Lemma 9, the equation $aT_k(m) = (2m+1)^k$ has no solutions with $k \equiv c \pmod{d}$. Otherwise, we multiply d by an integer $\ell \geq 2$ and consider ℓ congruences $k \equiv c + jd \pmod{\ell d}$, $0 \leq j < \ell$. For each of these congruences we proceed with the same argument as above. Moreover, for both $aS_k(m) = m^k$ and $aT_k(m) = (2m+1)^k$ one can use the *same* set of helpful pairs to rule out the possibility that $k \equiv c \pmod{d}$. This implies that if for a given a it has already been established by means of helpful pairs that the equation $aS_k(m) = m^k$ has no solutions with $k \equiv c \pmod{d}$, then one can conclude that the equation $aT_k(m) = (2m+1)^k$ has no solutions with $k \equiv c \pmod{d}$, and vice versa. In view of this, we have the following analogue of Proposition 7.1 of [1].

Proposition 1. *Suppose that $aT_k(m) = (2m+1)^k$ with $m > 1$. Then:*

(a) *If $a \equiv 1$ or 2 or $3 \pmod{5}$, then $4 \mid k$.*

(b) *If $a \equiv 1$ or 3 or $5 \pmod{7}$, then $6 \mid k$.*

(c) *If $a \equiv 6$ or $7 \pmod{11}$, then $10 \mid k$.*

(d) *If $a \equiv 2$ or 8 or $11 \pmod{13}$, then $12 \mid k$.*

(e) *If $a \equiv 1$ or $6 \pmod{13}$, then $6 \mid k$.*

(f) *If $a \equiv 1$ or $5 \pmod{11}$ and $a \equiv 15 \pmod{31}$, then $10 \mid k$.*

Furthermore, Proposition 4.5 of [1] can be extended as follows.

Proposition 2. *Let p be an irregular prime dividing a. Assume that for every irregular pair (r, p) there exists a positive integer ℓ_r such that for every $j = 0, 1, \ldots, \ell_r - 1$ there is a helpful pair $(t_j, q_j)_a$ with $(q_j - 1) \mid \ell_r(p-1)$ and $t_j \equiv r + j(p-1) \pmod{q_j - 1}$. Then both $aS_k(m) = m^k$ and $aT_k(m) = (2m+1)^k$ have no solutions.*

The procedure described in Proposition 2 has recently been used in [1] to show that if $2 \leq a \leq 1500$ and the prime divisors of a are all irregular, then the equation $aS_k(m) = m^k$ has no solutions (the corresponding helpful pairs are listed in Table 3 of [1]). As an immediate consequence of this, we obtain the following.

Proposition 3. *If $2 \leq a \leq 1500$ and all prime divisors of a are irregular, then the equation $aT_k(m) = (2m+1)^k$ has no solutions.*

Combining Proposition 3 with Corollary 2 and an obvious fact that for even a the equation $aT_k(m) = (2m+1)^k$ has no solutions, we deduce Theorem 1.

If an irregular prime p divides a, then we have a reasonably good chance to establish the unsolvability of the equations $aS_k(m) = m^k$ and $aT_k(m) = (2m+1)^k$ by applying Proposition 2. Unfortunately, it is not the case when $a = 1$. However, helpful pairs can still be used to develop some divisibility properties of positive integers k and m satisfying either $S_k(m) = m^k$ or $T_k(m) = (2m+1)^k$. For the original Erdős–Moser equation $S_k(m) =$

m^k, Moree, te Riele and Urbanowicz [12] have proved by means of helpful pairs that $2^8 \cdot 3^5 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot \prod_{23 \le p < 200} p$ divides k and every prime divisor of m is greater than 10000, provided that $k > 1$ (see also [11] for more details). Kellner [5] has shown that also all primes $200 < p < 1000$ divide k. Combining these results with the above remarks, we deduce Theorem 2.

Remark. Although the theoretical results of [11] and [12] are stated and proved in terms of good pairs (of which the helpful pairs are a special case), the numerical results mentioned above have been obtained using only helpful pairs.

5 Concluding Remarks

It is not difficult to find some other similarities between the Kellner–Erdős–Moser equation $aS_k(m) = m^k$ and the equation $aT_k(m) = (2m+1)^k$. For example, using Lemma 10.2 of [1] and the relation $T_k(m) = S_k(2m) - 2^k S_k(m)$ one can readily deduce the following result (cf. [1, Theorem 10.1]).

Theorem 4. *Suppose that $aT_k(m) = (2m+1)^k$ with $m > 1$. Then*

$$\operatorname{ord}_2(am-1) \begin{cases} = 2 + \operatorname{ord}_2 k & \text{if } a \equiv 1 \pmod 4, \\ \ge 3 + \operatorname{ord}_2 k & \text{if } a \equiv 3 \pmod 4. \end{cases}$$

Furthermore, one can easily show that if (m_1, k_1) and (m_2, k_2) are two distinct solutions of $aT_k(m) = (2m+1)^k$, then $m_1 \ne m_2$ and $k_1 \ne k_2$ (cf. [1, Proposition 1.7]).

By employing the same type of argument as in [1] and in Sections 3 and 4 of this paper, one can derive some divisibility properties of integers k and m satisfying either $S_k(m) = am^k$ or $T_k(m) = a(2m+1)^k$, where a is a fixed positive integer (see [8] and [10] for some results on $S_k(m) = am^k$). In particular, we are able to prove analogues of Theorem 3 and Proposition 1, however, so far we cannot make any conclusions about the unsolvability of these equations.

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