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Autor: Fortman, Margaret / Kupiec, Kevin / Rawlings, Marina
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Elemente der Mathematik

Walking on rational numbers and a self-referential formula

Margaret Fortman, Kevin Kupiec,
Marina Rawlings and Enrique Treviño

Margaret Fortman is a student at Lake Forest College studying mathematics and physics. She completed three summers of undergraduate research in mathematics, applied mathematics, and materials science at Lake Forest College, San Diego State University, and Northwestern University, respectively.

Kevin Kupiec received a Bachelor of Arts Degree in Mathematics from Lake Forest College with specializations in statistics and computer science. He has contributed to research programs at Harvard University, as well. Kevin is pursuing a career in software development. He is from Chicago, Il.

Marina Rawlings graduated from Lake Forest College in 2017 with a B.A. in Mathematics and Economics. She is currently in a rotational program for developing leadership with a top U.S. insurance corporation. She also has various artistic pursuits in her spare time.

Enrique Treviño is an Assistant Professor at Lake Forest College. He got his PhD from Dartmouth College in 2011 working in number theory with Carl Pomerance. When not doing mathematics, he enjoys playing fútbol with his students and running around the house with his 3 year old daughter Katya.

Eine in Basis vier notierte reelle Zahl kann als zweidimensionaler Gitterweg interpretiert werden: Man startet im Ursprung und betrachtet die Ziffern 0, 1, 2 und 3 je als Schritt nach rechts, oben, links und unten. Umgekehrt kann man jeden Gitterweg, der nicht mit einer Bewegung nach rechts beginnt, in entsprechender Weise als reelle Zahl auffassen. Insbesondere lassen sich beliebige Schriftzeichen, die man sich als Gitterpfade gezeichnet denkt, in reelle Zahlen verwandeln. Die Autoren der vorliegenden Arbeit geben eine Formel an, die als Input einen Satz aus gegebenen Schriftzeichen hat, und als Output eine rationale Zahl liefert, die genau diesen Satz ins Gitter schreibt. Zudem wird eine Variante von Tuppers Formel diskutiert, die mit Pixeln statt mit Pfaden arbeitet: Tuppers Formel kann in Abhängigkeit einer natürlichen Zahl jede Graphik in einem 106×17-Display darstellen, inklusive die Formel selber.

1 Walking on numbers

In [1], Aragón Artacho, et al. describe the process of a walk on the plane using the digits of a number base 4.[1] Consider a number x written in base 4, $x = d_n d_{n-1} \ldots d_0.d_{-1}d_{-2}\ldots$. We start at the origin in the Cartesian plane. If $d_n = 0$ we move a unit to the right, if it is 1, we move a unit upwards, if it is 2, we move a unit to the left, and if it is 3, we move downwards. We continue this process with $d_{n-1}, d_{n-2}, \ldots, d_1, d_0, d_{-1}, \ldots$, and this process creates a "walk." For example, the number 419636198 is rewritten in base 4 as 121000302033212_4. The walk would look like Figure 1.

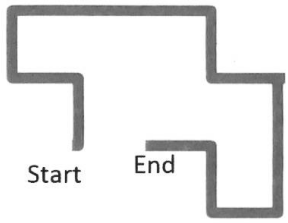

Figure 1 Walk for the number 419636198, which is 121000302033212_4.

In [1], they show several other walks, including walks on π and e using 100 billion digits. The one that inspired this paper is the following: Consider the rational number

$$
\begin{aligned}
Q = {} & 1049012271677499437486619280565448601617567358491560876166848380843144 3 \\
& 5844725287555162924702775955557045371567931305878324772977202177081818 7 \\
& 9659063736576748798142280132859202786101925814095713574870471229026746 5 \\
& 1513128059541953997504202061380373822338959713391954 \\
& / \\
& 1612226962694290912940490066273549214229880755725468512353395718465191 3 \\
& 5301734881431401750453996944547935301206438332726709700793305262920303 5 \\
& 0920973600450955456136596649325078391464772840162385651374295294530896 1 \\
& 2268152748875615658076162410788075184599421938774835.
\end{aligned}
$$

The walk on this number is Figure 2.[2]

Figure 2 Walk on Q.

In Section 2 we describe an algorithm that, given any sentence σ, it can find a rational number whose walk creates an image that spells out σ. We do something similar in Section 4, but instead of using walks as our starting point we use Tupper's self-referential formula, which is described in Section 3.

[1] They describe the process in any base, but for the purposes of this paper, we'll focus on base 4.

[2] This figure comes from [1]. As the authors explain, the color in this walk indicates the path followed by the walk. It is shifted up the spectrum (red-orange-yellow-green-cyan-blue-purple-red).

2 Writing using walks

For each letter of the alphabet α, we wish to find a rational r whose walk produces an image of α, just as the previous rational has produced the Q in Figure 2. How do you find the rational for a particular letter? First, find a string of digits that "spell out" your letter in such a way that you finish where you started. You can do this by hand. For example, for the letter "D", you can draw a stylish D as in Figure 3, then translate every movement as digits. In this case we obtain 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 0, 0, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 2, 2. Now, grab the first digit and multiply it by $\frac{1}{4}$, then add to it the second digit multiplied by $\frac{1}{4^2}$, and so on. In our example we'd have

$$\delta = 2\left(\frac{1}{4}\right) + 1\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^4}\right) + \cdots = \frac{6.38477\ldots \times 10^{74}}{2.32588\ldots \times 10^{74}}.^{3} \tag{1}$$

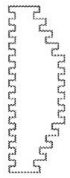

Figure 3 Walk on the rational δ creating the letter "D".

If we do a walk of δ with 124 steps, then we'll get Figure 3. However, this rational has only 124 significant digits base 4, i.e., every digit afterwards is 0. Therefore, if you consider the "infinite walk", then it will not spell "D" anymore, it will start heading eastwards, creating an underlined "D". To fix this, we create copies of the digits that form one loop of "D" and glue them every 124 steps, which is equivalent to multiplying δ by powers of 4^{-124} and adding them up. We get the geometric series:

$$\mathcal{D} = \delta + 4^{-124}\delta + \left(4^{-124}\right)^2\delta + \cdots = \frac{1}{1-4^{-124}}\delta = \frac{4^{124}}{4^{124}-1}\delta. \tag{2}$$

The walk for $\mathcal{D}$ for any number of steps ≥ 124 will create Figure 3.

Now, suppose that for every letter α (a variable representing an uppercase letter from the English alphabet), you find a rational r_α and an integer n_α, such that the walk of r_α with n_α steps spells the letter α in such a way that the last step ends at the origin. We will also define $r_{\text{blank}} = n_{\text{blank}} = 0$, i.e., representing a blank space. Finally, suppose the base of each letter is at most w, i.e., the length of a blank space is w.

Theorem 1. *Suppose we are given a sentence σ which we'll write as $\sigma = \alpha_1\alpha_2\alpha_3\cdots\alpha_k$, where α_i is a letter or a space. Let*

$$n = \sum_{i=1}^{k} n_{\alpha_i} + 2(k-1)w, \tag{3}$$

[3]Because of the lentgh of many numbers in this paper, we will restrict most of them to their first six digits. In the case of fractions we also keep the power of 10 to show how many digits a number has. The complete numbers can be found in the Appendix.

and

$$r = \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_j}+w)\right)}} + \frac{2 \cdot 4^{w}\left(1 - \frac{1}{4^{(k-1)w}}\right)}{3 \cdot 4^{\left(\sum_{j=1}^{k}(n_{\alpha_j}+w)\right)}}. \tag{4}$$

Then a walk on r of length n spells out σ. Furthermore, a walk on $\frac{4^n}{4^n-1}r$ of any length $m \geq n$ spells out σ.

Proof. The idea is to focus on the digits first. We have n_{α_1} digits to represent α_1, then we include w zeroes to give space for the next letter. We follow this with n_{α_2} digits of the second letter, followed by w zeroes, and so on. When we "write" the last letter, we have used $n_{\alpha_1} + n_{\alpha_2} + \cdots + n_{\alpha_k} + (k-1)w$ digits. But the walk is $(k-1)w$ steps to the right. To get back to the origin, we need to take $(k-1)w$ steps to the left, i.e., we need $(k-1)w$ 2's in the digit expansion. Therefore we've used n digits where n is the same as in (3).

Now we want to find the rational that has this digit expansion. To account for the letter α_i in the desired position, we need to multiply it by 4^{-x} where x is the number of digits used so far. For α_1, we've used 0, for α_2 we've used $n_{\alpha_1} + w$, for α_3 we've used $(n_{\alpha_1} + w) + (n_{\alpha_2} + w)$, and in general for α_i, we've used $\sum_{j=1}^{i-1}(n_{\alpha_j} + w)$. Finally, we have to take into account the final $(k-1)w$ 2's. To do this, we can think of $r_{\alpha_{k+1}} = 0.22\cdots 2_4$ and place it after all the digits so far, which have been $\sum_{i=1}^{k-1}(n_{\alpha_i} + w) + n_{\alpha_k}$. Therefore

$$\begin{aligned} r &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{r_{\alpha_{k+1}}}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)-w\right)}} \\ &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{4^{w}}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)\right)}}\left(\frac{2}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \cdots + \frac{2}{4^{(k-1)w}}\right). \end{aligned}$$

By completing the geometric series we can verify this matches (4). By construction, the walk for r with n steps spells out σ. Furthermore, by the same process as that in (2), we find the rational whose infinite walk spells out σ. □

Theorem 1 suggests how to build a program to find a rational number for any sentence. As an example, a certain rational

$$r_\sigma = \frac{3.47783\ldots \times 10^{3195}}{5.42542\ldots \times 10^{3195}}, \tag{5}$$

creates Figure 4.

Figure 4 The walk for r_σ as in (5) after 10000 steps.

3 Tupper's self-referential formula

In [4], Tupper introduced the formula[4]

$$\frac{1}{2} < \left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \mathrm{mod}(\lfloor y \rfloor, 17)}, 2\right)\right\rfloor. \tag{6}$$

This formula has the amazing property that if you graph the equation[5] for $0 \leq x < 106$ and $k \leq y < k + 17$ for k as in (7) [6] you get Figure 5.

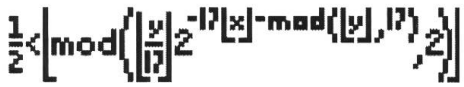

Figure 5 Graph of (6) in the range $0 \leq x < 106$ and $k \leq y < k + 17$.

$$\begin{aligned} k = {} & 4858450636189713423582095962494202044581400587983244549483093085061934704708809 \\ & 9284506447698655243648499972470249151191104116057391774078569197543265718554420 \\ & 5721044573588368182982375413963433822519945219165128434833290513119319995350241 \\ & 3758765239264874613394906870130562295813219481113685339535565290850023875092856 \\ & 8926945559742815463865107300491067230589335860525440966643512653493636439571255 \\ & 6569593681518433485760526694016125126695142155053955451915378545752575659074054 \\ & 0157929001765967965480064427829131488548259914721248506352686630476300. \end{aligned} \tag{7}$$

It turns out that the formula doesn't only graph itself, by considering different values of k, we can graph anything that can be represented by pixels in a 106×17 table. For example, a certain value

$$k_0 = 1.4452\ldots \times 10^{536} \tag{8}$$

gives the interval in which the graph looks like Figure 6.

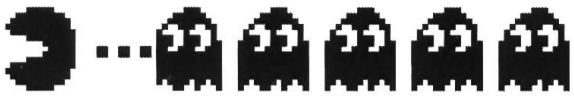

Figure 6 Graph of (6) in the range $0 \leq x < 106$ and $k_0 \leq y < k_0 + 17$.

The main reason why we can build anything in a 106×17 grid is the following lemma:

Lemma 1. *Let $k = 17k'$ for a nonnegative integer $k' < 2^{106 \times 17}$. Suppose we write k' in binary as follows:*

$$k' = \sum_{m=0}^{105} \sum_{n=0}^{16} a_{17m+n} 2^{17m+n}. \tag{9}$$

[4]The formula was given as an example of a formula that graphing software had difficulties with, but Tupper's graphing software can handle.

[5]By this we mean that the point (x, y) is painted black if it satisfies the inequality and not painted if it doesn't.

[6]In [2] and many other places, the value of k is given differently because of the convention in computer science that positive y go downwards.

Then

$$\left\lfloor \mathrm{mod}\left(\left\lfloor \frac{y}{17}\right\rfloor 2^{-17\lfloor x\rfloor - \mathrm{mod}\,(\lfloor y\rfloor,17)}, 2\right)\right\rfloor = a_b, \tag{10}$$

for $b = 17\lfloor x\rfloor + \mathrm{mod}(\lfloor y\rfloor, 17)$.

Therefore, the point (x, y) *is painted whenever* $a_b = 1$ *and not painted when* $a_b = 0$*, i.e., it depends only on the binary expansion of* k'.

Proof. Let $\lfloor x\rfloor = i$ and $\lfloor y\rfloor = j = 17k' + j'$ for some $0 \le j' \le 16$. Then $\left\lfloor \frac{y}{17}\right\rfloor = k'$, $j' = \mathrm{mod}(\lfloor y\rfloor, 17)$, and $b = 17i + j'$. Now

$$\left\lfloor \frac{y}{17}\right\rfloor 2^{-17\lfloor x\rfloor - \mathrm{mod}\,(\lfloor y\rfloor,17)} = k'2^{-17i-j'} = \sum_{m=0}^{105}\sum_{j=0}^{16} a_{17m+n}2^{17m+n-17i-j'}.$$

When we consider mod 2, we can eliminate any term where the exponent of 2 is at least 1, i.e., we're left with exponents satisfying $17m + n - 17i - j' \le 0$. When we take the floor, we exclude any of the small exponents because $1/2 + 1/4 + \cdots + 1/2^c < 1$ for any finite c. Therefore the only exponent of 2 we allow is 0. Hence $17m + n - 17i - j' = 0$. This implies $n \equiv j' \bmod 17$, but both n and j' are between 0 and 16, so $n = j'$, and then $m = i$, which is what we wanted to prove. □

4 Writing using Tupper's self-referential formula

From Lemma 1 we can extrapolate an algorithm to find a k to build any picture in a 106×17 grid. Indeed, write a 1 on any unit square that is painted black and a 0 otherwise. Now starting at the square with bottom-left corner $(0, 0)$, read the digits from bottom to top on each column. This binary number (read from right to left) will be k' and so $k = 17k'$.

Problem: For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \le x < 106$ and $k \le y < k + 17$.

As in the walk example, the key is figuring out how to do a letter first. Let's demonstrate how to do the letter a. Consider Figure 7. We read the number as 11101 10101 11111. To transform it into a number that fits in the 106×17 grid, we need to fill in the necessary 0's, which is equivalent to multiplying numbers in the ℓ-column by $2^{17(\ell-1)}$. Therefore, we associate the letter "a" with the number

$$17\left((1+2+4+16) + (1+4+16)2^{17} + (1+2+4+8+16)2^{34}\right). \tag{11}$$

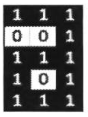

Figure 7 Breaking down the letter "a" in binary.

We can now move a letter around the grid by multiplying it by 2^{17m+n} to place it where the bottom-left corner of the letter is (m, n). If we create all letters with a height of at most 5 squares and width of at most 5 squares (the letters "m" and "w" need 5 squares, and the

rest need 3), we can then fit up to three rows of letters to spell a short sentence. Given a letter α, let $f(\alpha)$ be the number we associate with α with bottom-left corner on $(0, 0)$. We'll let $f_{\text{blank}} = 0$ and for letters with width 3, we'll multiply their numbers by 2^{17},[7] to create a buffer between letters.

Theorem 2. *Given a sentence $\sigma = \alpha_1\alpha_2\cdots\alpha_k$, where α_i represents a single letter or a blank space and $k \leq 63$, we use the following formula to figure out the value of k for the range where the plot of Tupper's formula is σ:*

$$\sum_{i=1}^{\min(21,k)} 2^{85(i-1)+12} f(\alpha_i) + \sum_{i=22}^{\min(42,k)} 2^{85(i-22)+6} f(\alpha_i) + \sum_{i=43}^{k} 2^{85(i-43)} f(\alpha_i). \tag{12}$$

Proof. Each letter fits in a block of width 5 and height 5. To move from one letter to the next (to the right), we need to multiply by $2^{17\times 5} = 2^{85}$. This is where the 85's in the exponents come from. The reason we add 12 and 6 (depending on how many letters we have) is because the first row consists of numbers in the top "strip" ($k+12 \leq y < y+17$), so we have to multiply by 2^{12} to move upwards. The numbers in the middle strip ($k+6 \leq y \leq y+11$) need a shift of 2^6, and the bottom row needs no translation. The formula follows. □

As an example of finding a k for a particular phrase, Figure 8 is the plot of Tupper's formula for $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$ for a certain

$$k_1 = 6.20234\ldots \times 10^{461}. \tag{13}$$

Figure 8 Graph of (6) in the range $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$.

Appendix: Full decimal digit expansion of constants in the paper.

The value of δ in (1) is

$$\begin{aligned}\delta =&6384779382043951036217348661253680515005885357484535471589654514956414794662\\&721006368542597248986985323127416704519810815261318970154183\\&/\\&2325883917745942049757836185241614509931652354199417792900768637378045721962\\&8733546438113622840434097944400691400517693873107252115668992.\end{aligned} \tag{14}$$

The value of k_0 in (8) is

$$\begin{aligned}k_0 =&1445202489708975828479425373371945674812777822151507024797188139685490\\&8873568298734888825132090576643817888323197692344001666776474924212512\end{aligned}$$

[7]The only letters not multiplied by 2^{17} are "m" and "n".

$$
\begin{aligned}
&8995265907053708020473915320841631792025549005418004768657201699730466\\
&3833949016013743197155209961811452497819450190683595005106578043256408\\
&0119786755686314228025969420625409608166564241736740394638417077453742\\
&7319606443899923010379398938675025786929455234476319291860957618345432\\
&2480049217280333494198162067498544720381939397385138489604767597826733\\
&1343769705199458068186981933044633677404726886 4,
\end{aligned}
\tag{15}
$$

The value of k_1 in (13) is

$$
\begin{aligned}
k_1 = &6202342045523518696372190728132145377913289497819263812843155643364944\\
&0461939615515996102292719598762206682015817244444562918664906697777197\\
&5007949955351598702405129648571930754026169504789347614953307622064658\\
&7662203381308047340029024837030531000814297140117523848644113896733785\\
&5616409282734138890208876466646383764272086299397454808405688789312744\\
&7832949435883695715278636348898143061593729742606126050532003884145813\\
&5744808540007473975236137965922728708669 44.
\end{aligned}
\tag{16}
$$

The value of r_σ in (5) is

$$
\begin{aligned}
r_\sigma = &\\
&3477838176632809766274027652998361070109305198798526822993346685629729543816284212324168906789690775547148059136594\\
&5650708415322762044997606235997855677049782095397960796362753436599841518280673260542696144332265810919906062866086\\
&7300310028185786493083481099698081973220156381116333517929438873090176653898133176194667067667046684812642590682921\\
&7676064926655968063069943083237148802645189802820278834066659917243976811957017195714300536678576755193478646225748\\
&0029718138397905599265918142472752199419439333830496840583774337111547232356603947260968981693338103053145854635253\\
&6556487569357115640975974485570693172272764417938367108726707881765559137578753568579282574517818911608998412741042\\
&0227212620869607503190020705525648927347045787545010799076183740241404923921318243481736923524124663788869995658064\\
&9792376238237930407962450320389159643655424259838617670381363425821742408854318599877016505639048938102659648790392\\
&9595397634307377756861919915276546513631911373910771937588542354894963935837136025758958018813262204515835086096680\\
&5708164505804035890264351831753336265274683965178474945976961829762028394224797947352146364669544961757542892363017 2\\
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Margaret Fortman
Lake Forest College student
e-mail:
`fortmanma@mx.lakeforest.edu`

Marina Rawlings
Lake Forest College 17'
e-mail:
`rawlingsmn@mx.lakeforest.edu`

Kevin Kupiec
Lake Forest College 17'
e-mail:
`kupieck@mx.lakeforest.edu`

Enrique Treviño
Department of Mathematics and Computer Science
at Lake Forest College
e-mail:
`trevino@lakeforest.edu`