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Walking on rational numbers and a self-referential formula

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Eine in Basis vier notierte reelle Zahl kann als zweidimensionaler Gitterweg interpretiert werden: Man startet im Ursprung und betrachtet die Ziffern 0, 1, 2 und 3 je als Schritt nach rechts, oben, links und unten. Umgekehrt kann man jeden Gitterweg, der nicht mit einer Bewegung nach rechts beginnt, in entsprechender Weise als reelle Zahl auffassen. Insbesondere lassen sich beliebige Schriftzeichen, die man sich als Gitterpfade gezeichnet denkt, in reelle Zahlen verwandeln. Die Autoren der vorliegenden Arbeit geben eine Formel an, die als Input einen Satz aus gegebenen Schriftzeichen hat, und als Output eine rationale Zahl liefert, die genau diesen Satz ins Gitter schreibt. Zudem wird eine Variante von Tuppers Formel diskutiert, die mit Pixeln statt mit Pfaden arbeitet: Tuppers Formel kann in Abhängigkeit einer natürlichen Zahl jede Graphik in einem 106×17-Display darstellen, inklusive die Formel selber.

1 Walking on numbers

In [1], Aragón Artacho, et al. describe the process of a walk on the plane using the digits of a number base 4.[1] Consider a number x written in base 4, $x = d_n d_{n-1} \dots d_0.d_{-1}d_{-2}\dots$. We start at the origin in the Cartesian plane. If $d_n = 0$ we move a unit to the right, if it is 1, we move a unit upwards, if it is 2, we move a unit to the left, and if it is 3, we move downwards. We continue this process with $d_{n-1}, d_{n-2}, \dots, d_1, d_0, d_{-1}, \dots,$ and this process creates a "walk." For example, the number 419636198 is rewritten in base 4 as 121000302033212_4. The walk would look like Figure 1.

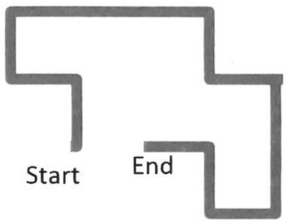

Figure 1 Walk for the number 419636198, which is 121000302033212_4.

In [1], they show several other walks, including walks on π and e using 100 billion digits. The one that inspired this paper is the following: Consider the rational number

$$
\begin{aligned}
Q =\,&1049012271677499437486619280565448601617567358491560876166848380843 1443\\
&5844725287555162924702775955557045371567931305878324772977202177081 8187\\
&9659063736576748798142280132859202786101925814095713574870471229026 7465\\
&1513128059541953997504202061380373822338959713391954\\
&/\\
&1612226962694290912940490066273549214229880755725468512353395718465 1913\\
&5301734881431401750453996944547935301206438332726709700793305262920 3035\\
&0920973600450955456136596649325078391464772840162385651374295294530 8961\\
&226815274887561565807616241078807518459942193877 4835.
\end{aligned}
$$

The walk on this number is Figure 2.[2]

Figure 2 Walk on Q.

In Section 2 we describe an algorithm that, given any sentence σ, it can find a rational number whose walk creates an image that spells out σ. We do something similar in Section 4, but instead of using walks as our starting point we use Tupper's self-referential formula, which is described in Section 3.

[1] They describe the process in any base, but for the purposes of this paper, we'll focus on base 4.

[2] This figure comes from [1]. As the authors explain, the color in this walk indicates the path followed by the walk. It is shifted up the spectrum (red-orange-yellow-green-cyan-blue-purple-red).

2 Writing using walks

For each letter of the alphabet α, we wish to find a rational r whose walk produces an image of α, just as the previous rational has produced the Q in Figure 2. How do you find the rational for a particular letter? First, find a string of digits that "spell out" your letter in such a way that you finish where you started. You can do this by hand. For example, for the letter "D", you can draw a stylish D as in Figure 3, then translate every movement as digits. In this case we obtain 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 0, 0, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 3, 0, 3, 2, 2, 2, 2. Now, grab the first digit and multiply it by $\frac{1}{4}$, then add to it the second digit multiplied by $\frac{1}{4^2}$, and so on. In our example we'd have

$$\delta = 2\left(\frac{1}{4}\right) + 1\left(\frac{1}{4^2}\right) + 1\left(\frac{1}{4^4}\right) + \cdots = \frac{6.38477\ldots \times 10^{74}}{2.32588\ldots \times 10^{74}}.^{3} \tag{1}$$

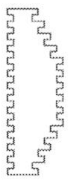

Figure 3 Walk on the rational δ creating the letter "D".

If we do a walk of δ with 124 steps, then we'll get Figure 3. However, this rational has only 124 significant digits base 4, i.e., every digit afterwards is 0. Therefore, if you consider the "infinite walk", then it will not spell "D" anymore, it will start heading eastwards, creating an underlined "D". To fix this, we create copies of the digits that form one loop of "D" and glue them every 124 steps, which is equivalent to multiplying δ by powers of 4^{-124} and adding them up. We get the geometric series:

$$\mathcal{D} = \delta + 4^{-124}\delta + \left(4^{-124}\right)^2 \delta + \cdots = \frac{1}{1 - 4^{-124}}\delta = \frac{4^{124}}{4^{124} - 1}\delta. \tag{2}$$

The walk for $\mathcal{D}$ for any number of steps ≥ 124 will create Figure 3.

Now, suppose that for every letter α (a variable representing an uppercase letter from the English alphabet), you find a rational r_α and an integer n_α, such that the walk of r_α with n_α steps spells the letter α in such a way that the last step ends at the origin. We will also define $r_{\text{blank}} = n_{\text{blank}} = 0$, i.e., representing a blank space. Finally, suppose the base of each letter is at most w, i.e., the length of a blank space is w.

Theorem 1. *Suppose we are given a sentence σ which we'll write as $\sigma = \alpha_1\alpha_2\alpha_3\cdots\alpha_k$, where α_i is a letter or a space. Let*

$$n = \sum_{i=1}^{k} n_{\alpha_i} + 2(k-1)w, \tag{3}$$

[3] Because of the lentgh of many numbers in this paper, we will restrict most of them to their first six digits. In the case of fractions we also keep the power of 10 to show how many digits a number has. The complete numbers can be found in the Appendix.

and

$$r = \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_j}+w)\right)}} + \frac{2 \cdot 4^{w}\left(1 - \frac{1}{4^{(k-1)w}}\right)}{3 \cdot 4^{\left(\sum_{j=1}^{k}(n_{\alpha_j}+w)\right)}}. \tag{4}$$

Then a walk on r of length n spells out σ. Furthermore, a walk on $\frac{4^n}{4^n-1}r$ of any length $m \geq n$ spells out σ.

Proof. The idea is to focus on the digits first. We have n_{α_1} digits to represent α_1, then we include w zeroes to give space for the next letter. We follow this with n_{α_2} digits of the second letter, followed by w zeroes, and so on. When we "write" the last letter, we have used $n_{\alpha_1} + n_{\alpha_2} + \cdots + n_{\alpha_k} + (k-1)w$ digits. But the walk is $(k-1)w$ steps to the right. To get back to the origin, we need to take $(k-1)w$ steps to the left, i.e., we need $(k-1)w$ 2's in the digit expansion. Therefore we've used n digits where n is the same as in (3).

Now we want to find the rational that has this digit expansion. To account for the letter α_i in the desired position, we need to multiply it by 4^{-x} where x is the number of digits used so far. For α_1, we've used 0, for α_2 we've used $n_{\alpha_1} + w$, for α_3 we've used $(n_{\alpha_1} + w) + (n_{\alpha_2} + w)$, and in general for α_i, we've used $\sum_{j=1}^{i-1}(n_{\alpha_j} + w)$. Finally, we have to take into account the final $(k-1)w$ 2's. To do this, we can think of $r_{\alpha_{k+1}} = 0.22\cdots2_4$ and place it after all the digits so far, which have been $\sum_{i=1}^{k-1}(n_{\alpha_i} + w) + n_{\alpha_k}$. Therefore

$$\begin{aligned} r &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{r_{\alpha_{k+1}}}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)-w\right)}} \\ &= \sum_{i=1}^{k} \frac{r_{\alpha_i}}{4^{\left(\sum_{j=1}^{i-1}(n_{\alpha_i}+w)\right)}} + \frac{4^{w}}{4^{\left(\sum_{i=1}^{k}(n_{\alpha_i}+w)\right)}}\left(\frac{2}{4} + \frac{2}{4^2} + \frac{2}{4^3} + \cdots + \frac{2}{4^{(k-1)w}}\right). \end{aligned}$$

By completing the geometric series we can verify this matches (4). By construction, the walk for r with n steps spells out σ. Furthermore, by the same process as that in (2), we find the rational whose infinite walk spells out σ. □

Theorem 1 suggests how to build a program to find a rational number for any sentence. As an example, a certain rational

$$r_\sigma = \frac{3.47783\ldots \times 10^{3195}}{5.42542\ldots \times 10^{3195}}, \tag{5}$$

creates Figure 4.

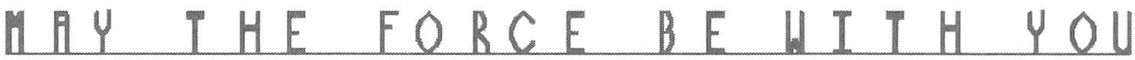

Figure 4 The walk for r_σ as in (5) after 10000 steps.

3 Tupper's self-referential formula

In [4], Tupper introduced the formula[4]

$$\frac{1}{2} < \left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x \rfloor - \operatorname{mod}(\lfloor y \rfloor, 17)}, 2 \right) \right\rfloor . \tag{6}$$

This formula has the amazing property that if you graph the equation[5] for $0 \leq x < 106$ and $k \leq y < k + 17$ for k as in (7) [6] you get Figure 5.

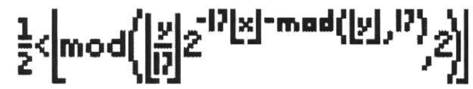

Figure 5 Graph of (6) in the range $0 \leq x < 106$ and $k \leq y < k + 17$.

$$\begin{aligned} k = {} & 4858450636189713423582095962494202044581400587983244549483093085061934704708809 \\ & 9284506447698655243648499972470249151191104116057391774078569197543265718554420 \\ & 5721044573588368182982375413963433822519945219165128434833290513119319995350241 \\ & 3758765239264874613394906870130562295813219481113685339535565290850023875092856 \\ & 8926945559742815463865107300491067230589335860525440966643512653493636439571255 \\ & 6569593681518433485760526694016125126695142155053955451915378545752575659074054 \\ & 0157929001765967965480064427829131488548259914721248506352686630476300. \end{aligned} \tag{7}$$

It turns out that the formula doesn't only graph itself, by considering different values of k, we can graph anything that can be represented by pixels in a 106×17 table. For example, a certain value

$$k_0 = 1.4452 \ldots \times 10^{536} \tag{8}$$

gives the interval in which the graph looks like Figure 6.

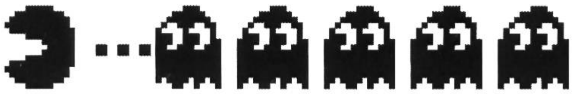

Figure 6 Graph of (6) in the range $0 \leq x < 106$ and $k_0 \leq y < k_0 + 17$.

The main reason why we can build anything in a 106×17 grid is the following lemma:

Lemma 1. *Let $k = 17k'$ for a nonnegative integer $k' < 2^{106 \times 17}$. Suppose we write k' in binary as follows:*

$$k' = \sum_{m=0}^{105} \sum_{n=0}^{16} a_{17m+n} 2^{17m+n}. \tag{9}$$

---

[4]The formula was given as an example of a formula that graphing software had difficulties with, but Tupper's graphing software can handle.

[5]By this we mean that the point (x, y) is painted black if it satisfies the inequality and not painted if it doesn't.

[6]In [2] and many other places, the value of k is given differently because of the convention in computer science that positive y go downwards.

Then

$$\left\lfloor \operatorname{mod}\left(\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x\rfloor - \operatorname{mod}(\lfloor y\rfloor, 17)}, 2\right)\right\rfloor = a_b, \tag{10}$$

for $b = 17\lfloor x\rfloor + \operatorname{mod}(\lfloor y\rfloor, 17)$.

Therefore, the point (x, y) *is painted whenever* $a_b = 1$ *and not painted when* $a_b = 0$*, i.e., it depends only on the binary expansion of* k'.

Proof. Let $\lfloor x\rfloor = i$ and $\lfloor y\rfloor = j = 17k' + j'$ for some $0 \le j' \le 16$. Then $\left\lfloor \frac{y}{17}\right\rfloor = k'$, $j' = \operatorname{mod}(\lfloor y\rfloor, 17)$, and $b = 17i + j'$. Now

$$\left\lfloor \frac{y}{17} \right\rfloor 2^{-17\lfloor x\rfloor - \operatorname{mod}(\lfloor y\rfloor, 17)} = k'2^{-17i-j'} = \sum_{m=0}^{105}\sum_{j=0}^{16} a_{17m+n}2^{17m+n-17i-j'}.$$

When we consider mod 2, we can eliminate any term where the exponent of 2 is at least 1, i.e., we're left with exponents satisfying $17m + n - 17i - j' \le 0$. When we take the floor, we exclude any of the small exponents because $1/2 + 1/4 + \cdots + 1/2^c < 1$ for any finite c. Therefore the only exponent of 2 we allow is 0. Hence $17m + n - 17i - j' = 0$. This implies $n \equiv j' \bmod 17$, but both n and j' are between 0 and 16, so $n = j'$, and then $m = i$, which is what we wanted to prove. □

4 Writing using Tupper's self-referential formula

From Lemma 1 we can extrapolate an algorithm to find a k to build any picture in a 106×17 grid. Indeed, write a 1 on any unit square that is painted black and a 0 otherwise. Now starting at the square with bottom-left corner $(0, 0)$, read the digits from bottom to top on each column. This binary number (read from right to left) will be k' and so $k = 17k'$.

Problem: For a given sentence, find the integer k such that the graph of Tupper's formula looks like that sentence for $0 \le x < 106$ and $k \le y < k + 17$.

As in the walk example, the key is figuring out how to do a letter first. Let's demonstrate how to do the letter a. Consider Figure 7. We read the number as 11101 10101 11111. To transform it into a number that fits in the 106×17 grid, we need to fill in the necessary 0's, which is equivalent to multiplying numbers in the ℓ-column by $2^{17(\ell-1)}$. Therefore, we associate the letter "a" with the number

$$17\left((1 + 2 + 4 + 16) + (1 + 4 + 16)2^{17} + (1 + 2 + 4 + 8 + 16)2^{34}\right). \tag{11}$$

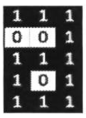

Figure 7 Breaking down the letter "a" in binary.

We can now move a letter around the grid by multiplying it by 2^{17m+n} to place it where the bottom-left corner of the letter is (m, n). If we create all letters with a height of at most 5 squares and width of at most 5 squares (the letters "m" and "w" need 5 squares, and the rest need 3), we can then fit up to three rows of letters to spell a short sentence. Given a letter α, let $f(\alpha)$ be the number we associate with α with bottom-left corner on $(0,0)$. We'll let $f_{\text{blank}} = 0$ and for letters with width 3, we'll multiply their numbers by 2^{17},[7] to create a buffer between letters.

Theorem 2. *Given a sentence $\sigma = \alpha_1\alpha_2\cdots\alpha_k$, where α_i represents a single letter or a blank space and $k \leq 63$, we use the following formula to figure out the value of k for the range where the plot of Tupper's formula is σ:*

$$\sum_{i=1}^{\min(21,k)} 2^{85(i-1)+12} f(\alpha_i) + \sum_{i=22}^{\min(42,k)} 2^{85(i-22)+6} f(\alpha_i) + \sum_{i=43}^{k} 2^{85(i-43)} f(\alpha_i). \tag{12}$$

Proof. Each letter fits in a block of width 5 and height 5. To move from one letter to the next (to the right), we need to multiply by $2^{17\times 5} = 2^{85}$. This is where the 85's in the exponents come from. The reason we add 12 and 6 (depending on how many letters we have) is because the first row consists of numbers in the top "strip" ($k+12 \leq y < y+17$), so we have to multiply by 2^{12} to move upwards. The numbers in the middle strip ($k+6 \leq y \leq y+11$) need a shift of 2^6, and the bottom row needs no translation. The formula follows. □

As an example of finding a k for a particular phrase, Figure 8 is the plot of Tupper's formula for $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$ for a certain

$$k_1 = 6.20234\ldots \times 10^{461}. \tag{13}$$

Figure 8 Graph of (6) in the range $0 \leq x < 106$ and $k_1 \leq y < k_1 + 17$.

Appendix: Full decimal digit expansion of constants in the paper.

The value of δ in (1) is

$$\delta = 6384779382043951036217348661253680515005885357484535471589654514956414794662\\
7210063685425972489869853231274167045198108152613189701541 83\\
/\\
2325883917745942049757836185241614509931652354199417792900768637378045721962\\
873354643811362284043409794440069140051769387310725211 5668992. \tag{14}$$

The value of k_0 in (8) is

$$k_0 = 1445202489708975828479425373371945674812777822151507024797188139685490\\
8873568298734888825132090576643817888323197692344001666776474924212512$$

[7] The only letters not multiplied by 2^{17} are "m" and "n".

8995265907053708020473915320841631792025549005418004768657201699730466
3833949016013743197155209961811452497819450190683595005106578043256408
0119786755686314228025969420625409608166564241736740394638417077453742
7319606443899923010379398938675025786929455234476319291860957618345432
2480049217280333494198162067498544720381939397385138489604767597826733
13437697051994580681869819330446336774047268864, (15)

The value of k_1 in (13) is

k_1 =6202342045523518696372190728132145377913289497819263812843155643364944
0461939615515996102292719598762206682015817244444562918664906697777197
5007949955351598702405129648571930754026169504789347614953307622064658
7662203381308047340029024837030531000814297140117523848644113896733785
5616409282734138890208876466646383764272086299397454808405688789312744
7832949435883695715278636348898143061593729742606126050532003884145813
574480854000747397523613796592272870866944. (16)

The value of r_σ in (5) is

r_σ =
3477838176632809766274027652998361070109305198798526822993346685629729543816284212324168906789690775547148059136594
5650708415322762044997606235997855677049782095397960796362753436599841518280673260542696144332265810919906062866086
7300310028185786493083481099698081973220156381116333517929438873090176653898133176194667067667046684812642590682921
7676064926655968063069943083237148802645189802820278834066659917243976811957017195714300536678576755193478646225748
0029718138397905599265918142472752199419439333830496840583774337111547232356603947260968981693338103053145854635253
6556487569357115640975974485570693172272764417938367108726707881765559137578753568579282574517818911608998412741042
0227212620869607503190020705525648927347045787545010799076183740241404923921318243481736923524124663788869995658064
9792376238237930407962450320389159643655424259838617670381363425821742408854318599877016505639048938102659648790392
9595397634307377756861919915276546513631911373910771937588542354894963935837136025758958018813262204515835086096680
5708164505804035890264351831753362652746839651784749459769618297620283942247979473521463646695449617575428923630172
0483591538711675276566969776077077093216794342308880360116044571717992965087557491510480694730625476830869629084918
0330152435295444522317092256763450131932664980244104376549124516719713135304140238008592647538068871086358474426810
3228839275062729870736963264331850092569189868395203739083590687341840855847621655276649131461408882329507967011133
9673513989596387755006734087403631374900979733345825183520000390763148149610355789986385442991553413321157523278681
4570776631231717821839613245063461887765374214685496293414305359730563242820842921560948381487459668207389426718341
0208201125760638630177927567438294560420792809301856500683312377648424972104939732771020518033666376045372522152359
6365323093742860258648736509758715341472162499794997150383376599167041645495565423248191931830700401690739885894446
5925138828780782529332465419011534533168166732703212020724420330540765914808719296089874268534962295074250221700319
4354384010924295515836192696117735836407353432915120743106575043357828096478475426584678135447571530015321875975666
0288499555262323407672275144428184247080572013967367202580538705898188934186489884901842932385285873056974563406043
2972003424454777927419274342586200406369883986949725285622098211132577303257888674921256677207569164691929862742553
6434760361740959640882424911067801423858038854691012806455532141473569316256449253261282819184164210728465538046128
8689316433179357074775456515481274301511980843961573141008058309365269614398459837204683190833300361309048467141298
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