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Tile the group

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1 Tiling the integers by arithmetic progressions

Try to partition the integers into a finite number of arithmetic progressions (such that every integer will appear in exactly one of these progressions).

An immediate example is the partition of $\mathbb{Z}$ into the even and odd integers, both are arithmetic progressions with modulus 2. More generally, for any natural number n, the integers can be decomposed to n arithmetic progressions, all of modulus n. That is

$$\mathbb{Z} = n\mathbb{Z} \cup (1 + n\mathbb{Z}) \cup \cdots \cup (n - 1 + n\mathbb{Z}), \tag{1}$$

where $k + n\mathbb{Z}$ denotes the subset of integers whose remainder after division by n is k, $0 \leq k \leq n - 1$.

Die einfachste Partition der ganzen Zahlen $\mathbb{Z}$ in arithmetische Folgen ist die Aufteilung in gerade und ungerade Zahlen. Paul Erdős vermutet, dass jede nichttriviale Partition der ganzen Zahlen in endlich viele arithmetische Folgen mindestens zwei Folgen zur selben Schrittweite aufweist. Diese Vermutung wurde in den 1950er Jahren bestätigt. 1974 schlugen Marcel Herzog und Jochanan Schönheim vor, das Problem zu verallgemeinern und statt $\mathbb{Z}$ beliebige Gruppen und anstelle der arithmetischen Folgen Nebenklassen von endlichem Index zu betrachten. Sie stellten also die Frage: Gibt es in jeder Partition einer Gruppe durch endlich viele Nebenklassen von endlichem Index stets mindestens zwei Nebenklassen zum selben Index? In dieser Allgemeinheit ist die Frage noch heute offen, sogar bei endlichen Gruppen. Der Autor der vorliegenden Arbeit beweist nun die Herzog–Schönheim-Vermutung für alle Gruppen deren Ordnung eine bestimmte zahlentheoretische Eigenschaft aufweist, insbesondere für alle Gruppen bis zur Ordnung 240. Dabei spielen ägyptische Brüche eine wichtige Rolle. Auch die unsprüngliche Erdős-Vermutung erscheint in neuem Licht.

Question. *Are these partitions, considered as* **trivial tilings**, *the only examples?*

It turns out that non-trivial tilings abound. For example, the even integers are a disjoint union of two arithmetic progressions, namely, the integers which are divisible by 4 and those who have a remainder 2 after division by 4. These two progressions, together with the odd integers, yield the non-trivial tiling

$$\mathbb{Z} = 4\mathbb{Z} \cup (2 + 4\mathbb{Z}) \cup (1 + 2\mathbb{Z}).$$

More generally, given any trivial tiling (1), take one of the progressions, e.g., $k + n\mathbb{Z}$, and split it by partitioning it with modulus m

$$k + n\mathbb{Z} = \bigcup_{i=0}^{m-1} (k + in + mn\mathbb{Z}).$$

A new tiling of the integers is obtained

$$\mathbb{Z} = \bigcup_{0 \le l (\ne k) \le n-1} (l + n\mathbb{Z}) \cup \bigcup_{i=0}^{m-1} (k + in + mn\mathbb{Z}).$$

Moreover, if

$$\bigcup_{l=1}^{s} (a_l + d_l\mathbb{Z})$$

is a partition of the integers, then so is

$$\bigcup_{1 \le l (\ne k) \le s} (a_l + d_l\mathbb{Z}) \cup \bigcup_{i=0}^{m-1} (a_k + id_k + md_k\mathbb{Z}). \tag{2}$$

Starting with a trivial partition (1) and applying any sequence of splittings (2), one can obtain a variety of partitions of the integers. But even this procedure does not exhaust all the possible partitions of the integers into arithmetic progressions.

Example (Š. Porubský [8, §2]). *The following is a partition of the integers which is not obtained as a splitting of any coarser partition.*

$$\begin{aligned}\mathbb{Z} = 6\mathbb{Z} &\cup (1 + 10\mathbb{Z}) \cup (2 + 15\mathbb{Z}) \cup (3 + 30\mathbb{Z}) \cup (4 + 30\mathbb{Z}) \cup (5 + 30\mathbb{Z}) \\ &\cup (7 + 30\mathbb{Z}) \cup \cup (8 + 30\mathbb{Z}) \cup (9 + 30\mathbb{Z}) \cup (10 + 30\mathbb{Z}) \cup (13 + 30\mathbb{Z}) \\ &\cup (14 + 30\mathbb{Z}) \cup (15 + 30\mathbb{Z}) \cup \cup (16 + 30\mathbb{Z}) \cup (19 + 30\mathbb{Z}) \cup (20 + 30\mathbb{Z}) \\ &\cup (22 + 30\mathbb{Z}) \cup (23 + 30\mathbb{Z}) \cup (25 + 30\mathbb{Z}) \cup \cup (26 + 30\mathbb{Z}) \cup (27 + 30\mathbb{Z}) \\ &\cup (28 + 30\mathbb{Z}) \cup (29 + 30\mathbb{Z}).\end{aligned}$$

Note that in every step of the above procedure of splittings (2), the largest modulus appears more than once. P. Erdős conjectured that in any partition of the integers into (more than one) arithmetic progressions, the largest modulus appears at least twice.

Erdős' conjecture was proven in the 1950s independently by H. Davenport, L. Mirsky, D. Newman and R. Rado using generating functions and complex variables. Their beautiful argument (see, e.g., [1, 7]) would be regarded by Erdős himself as "from the book".

Partition by Arithmetic Progressions (PAP) Theorem. *Let*

$$\mathbb{Z} = \bigcup_{l=1}^{s} (a_l + d_l\mathbb{Z}) \tag{3}$$

be a disjoint union of the integers to arithmetic progressions of moduli $d_1 \le \cdots \le d_s$, *where* $s > 1$. *Then* $d_{s-1} = d_s$.

In fact, by [1, Theorem 5], [7, Theorem 2], d_s appears in the partition (3) at least p times, where p is the smallest prime dividing d_s. Another strengthening of PAP Theorem [1, Theorem 4] says that any modulus d_l in the partition (3) divides another modulus d_k in this partition for some $k \neq l$. In particular, all moduli which do not properly divide any other modulus in (3) appear at least twice.

Many more details about covers of the integers by arithmetic progressions can be found in [9, 10, 11].

2 The Herzog–Schönheim conjecture

Question. *Can one view PAP Theorem as a special instance of a general phenomenon?*

The integers $\mathbb{Z}$ are an example of a group, whereas an arithmetic progression $k + n\mathbb{Z}$ is nothing but a coset with respect to the subgroup $n\mathbb{Z} \le \mathbb{Z}$. The modulus n is the index of this coset in $\mathbb{Z}$. It is thus natural to pass from partitions of the integers into arithmetic progressions to partitions of general groups into cosets.

Definition. *Let* $\{H_l\}_{l=1}^{s}$ *be subgroups of a given group* G, *all of finite index. Suppose that*

$$G = \bigcup_{l=1}^{s} g_l H_l \tag{4}$$

is a decomposition of G *to a disjoint union of cosets* $g_l H_l$. *This partition has* **multiplicity** *if there exist* $1 \le j_1 \neq j_2 \le s$ *such that* H_{j_1} *and* H_{j_2} *are of equal indices in* G.

The following is a natural attempt to generalize PAP Theorem.

Conjecture (M. Herzog and J. Schönheim [3]). *A decomposition* (4) *of any group to a disjoint union of cosets of finite index with* $s > 1$ *has multiplicity.*

A group G is termed HS if it satisfies the conjecture. That is, if any coset partition (4) of G with $s > 1$ admits at least two cosets of the same index. For example, the group $\mathbb{Z}$ is HS by the PAP Theorem.

The HS conjecture can be reduced to finite groups [5]. In other words, if all finite groups are HS, then all groups are HS.

For example, to find multiplicities in the partition (3) it is enough to check all integers between 0 and $n - 1$, where $n :=\mathrm{lcm}\{d_1, \dots, d_s\}$. This is a partition of the finite group $\mathbb{Z}/n\mathbb{Z}$.

Question. *Can one prove that a finite group* G *is HS only by knowing its cardinality?*

For a natural number n, let $\Delta(n)$ be the set of proper divisors of n. For example,

$$\Delta(6) = \{1, 2, 3\}.$$

Recall that Lagrange's Theorem says that the order of any subgroup of a finite group divides the order of the group. Consequently, if a finite group G admits a coset partition without multiplicity, then there exists a subset $\Delta_0 \subset \Delta(|G|)$ such that

$$\sum_{d \in \Delta_0} d = |G|.$$

This condition says that if n cannot be expressed as a sum of a partial subset of its divisors, then any group of order n is HS.

An important family of natural numbers satisfying the above condition is the **deficient numbers**, i.e., those numbers which are larger than the sum of their proper divisors. Thus, if n is a deficient number, then all groups of order n are HS.

In particular, if p is a prime number and m any natural number, then

$$\sum_{d \in \Delta(p^m)} d = 1 + p + \cdots + p^{m-1} = \frac{p^m - 1}{p - 1} < p^m,$$

and hence prime powers are deficient. Thus, p-groups (groups of prime power order) are HS.

What about other natural numbers? The smallest number which is not a prime power is 6. In fact, it is equal to the sum of all its divisors (i.e., it is a **perfect number**)

$$1 + 2 + 3 = 6.$$

Can a group of order 6 admit a partition into cosets of cardinality 1, 2, and 3?

Note that two arithmetic progressions, one of modulus 2 and the other of modulus 3, must intersect. This follows from the well-known

Chinese Remainder Theorem (CRT). *Let m and n be coprime numbers. Then for every $a, b \in \mathbb{Z}$*

$$(a + m\mathbb{Z}) \cap (b + n\mathbb{Z}) \neq \emptyset.$$

Consequently, a coset partition of the integers cannot contain arithmetic progressions of coprime moduli.

Surprisingly, the above observation is valid for any group.

Group-Theoretical CRT [12, Remark 2.2]. *Let H_1 and H_2 be two subgroups of a group G such that their indices are mutually coprime. Then for every $a, b \in G$, $aH_1 \cap bH_2 \neq \emptyset$. In particular, if* (4) *is a coset partition of a group G, then the indices of H_i and H_j in G are not coprime for every $1 \leq i, j \leq s$.*

One can formulate a sufficient arithmetic condition for a finite group to be HS. Let (4) be a coset partition of a finite group G and let $n_l := [G : H_l]$ be the corresponding indices.

Then the tiling condition yields $|G| = \sum_{l=1}^{s} \frac{|G|}{n_l}$, or alternatively

$$1 = \sum_{l=1}^{s} \frac{1}{n_l}. \tag{5}$$

Suppose that (4) has no multiplicity. In this case (5) describes a representation of 1 as an **Egyptian fraction**, that is the sum of distinct unit fractions (i.e., whose numerators equal 1 and whose denominators are positive integers). Then the Chinese (Remainder Theorem) and the Egyptian (fractions) meet in the following

Corollary. *Let $n \in \mathbb{N}$. Suppose that* 1 *cannot be represented as an Egyptian fraction* (5) *such that*

(i) *all the denominators n_l divide n, and*

(ii) *for every $1 \le i, j \le s$, n_i and n_j are not coprime.*

Then every group of order n is HS.

In fact, by an induction argument, a minimal counterexample to the HS conjecture must admit a representation of 1 as an Egyptian fraction with the above conditions and, additionally,

(iii) $n_l > 2$ *for every* $1 \le l \le s$.

The third condition is there since if one of the cosets $g_j H_j$ in a coset partition (4) is of index 2, then $\bigcup_{l=1, l\neq j}^{s} g_l H_l$ is a disjoint union of the complementary coset of $g_j H_j$, say $g'_j H_j$. Shifting all the cosets by $g_j'^{-1}$ yields a partition of H_j itself. If the partition (4) has no multiplicity, then so does this partition of the group H_j, which is of smaller order.

Example. *A computer simulation shows that the smallest number n admitting an Egyptian fraction* (5) *satisfying conditions* (i), (ii) *and* (iii) *is* 240*:*

$$1 = \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{16} + \frac{1}{20} + \frac{1}{24} + \frac{1}{30} + \frac{1}{40} + \frac{1}{48} + \frac{1}{60} + \frac{1}{80} + \frac{1}{120} + \frac{1}{240}.$$

Consequently, all groups of orders smaller than 240 *are HS.*

Recently, L. Margolis and O. Schnabel have elaborated the above corollary, and by that have significantly improved the order 240 *to beat. Their work in progress employs computer simulation which yields the order of* 1,440 *as a minimal candidate for a counterexample to the HS conjecture.*

The Herzog–Schönheim Conjecture is still open for more than four decades. There has been much progress in the investigation of this problem over the years; visit [13] for a list of classified publications on the conjecture as well as on related topics. To end this section, here is a citation of a result which provides further sufficient conditions for finite groups to be HS only in terms of their cardinality.

Theorem ([2])**.** *Let G be a group of order $p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where $p_1 < p_2 < \cdots < p_k$ are prime numbers. Suppose that either*

1. $k \leq 2$, *or*
2. $k = 3$ *and* $p_2 > 3$ (*i.e.*, $|G|$ *is not divisible by* 6), *or*
3. $\prod_{i=1}^{k}(1 + \frac{1}{p_i - 1}) \leq 2$, *in particular if* $p_1 \geq \frac{1}{\sqrt[k]{2}-1} + 1$.

Then G is HS.

3 A Group-theoretical proof of the PAP theorem

Despite its beauty, it is not clear how the original proof of PAP Theorem can be used to treat arithmetic progressions as special instances of cosets with respect to subgroups. This is a good opportunity to adjust the proof into a group-theoretical language. For this part, a basic knowledge in character theory is requisite.

Let $\mathbb{C}G$ be the group algebra of a finite group G over the complex numbers. Any coset partition (4) of G gives rise to an equation in this algebra

$$\sum_{g \in G} g = \sum_{l=1}^{s} g_l \Big(\sum_{h \in H_l} h \Big) \in \mathbb{C}G. \tag{6}$$

As mentioned above, to show that the group $\mathbb{Z}$ of integers is HS, it suffices to deal with its finite quotient $\mathbb{Z}/n\mathbb{Z}$, where n is the lcm of the moduli in the partition. This is a cyclic group generated by one element. Using a multiplicative way, one writes

$$C_n = \{1, x, \ldots, x^{n-1}\},$$

where x is a generator of this cyclic group of order n. The subgroups of C_n are in one-to-one correspondence with the divisors of n. That is, for every $d|n$, the subgroup

$$H_d := \langle x^d \rangle,$$

generated by x^d, is of order $\frac{n}{d}$ and index d in $C_n = \langle x \rangle (= H_1)$.

Given a partition (3) of the integers with $n = \mathrm{lcm}\{d_l\}_{l=1}^{s}$, one can adopt the group theoretical (multiplicative) writing (6) with $G = C_n$ to obtain

$$\sum_{g \in H_1 (=C_n)} g = \sum_{l=1}^{s} x^{a_l} \Big(\sum_{h \in H_{d_l}} h \Big). \tag{7}$$

Next, the **character group** of C_n, denoted by $\check{C_n}$ is the group of morphisms $\mathrm{Hom}(C_n, \mathbb{C}^*)$ to the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The group $\check{C_n}$ contains exactly n characters $\{\chi_0^j\}_{j=1}^{n}$, where

$$\chi_0^j : \begin{array}{ccc} C_n & \to & \mathbb{C}^* \\ x^a & \mapsto & \exp\left(2\pi i \cdot \frac{aj}{n}\right). \end{array}$$

The standard basis $\{x^l\}_{l=0}^{n-1}$ of $\mathbb{C}C_n$ can be replaced with the transformed basis $\{e_\chi\}_{\chi \in \check{C_n}}$, where (see [4, Theorem 2.12])

$$e_\chi := \frac{1}{n} \sum_{l=0}^{n-1} \chi(x^{-l}) x^l.$$

The advantage of this basis is the orthogonality property of the above idempotents (see [4, Theorem 2.13])

$$e_{\chi_1}e_{\chi_2} = \delta_{\chi_1,\chi_2}e_{\chi_1} \tag{8}$$

for every $\chi_1, \chi_2 \in \check{C}_n$, where $\delta_{\cdot,\cdot}$ denotes Kronecker's delta function. Furthermore, the following projection property

$$x^a e_\chi = \frac{1}{n}\sum_{l=0}^{n-1} \chi(x^{a-l})x^l = \chi(x^a)e_\chi, \quad a \in \mathbb{Z}, \chi \in \check{C}_n \tag{9}$$

provides a convenient way to multiply elements of the two bases.

Denoting for simplicity

$$e_{\frac{j}{d}} := e_{\chi_0^{\frac{nj}{d}}}$$

for any divisor d of n, one has

Lemma. *Let* $0 < d$ *be a divisor of* n. *Then*

$$\sum_{h\in H_d} h = \frac{n}{d}\sum_{j=0}^{d-1} e_{\frac{j}{d}}. \tag{10}$$

Proof. Develop the right-hand side to a double summation using the above notation

$$\begin{aligned}\frac{n}{d}\sum_{j=0}^{d-1} e_{\frac{j}{d}} &= \frac{n}{d}\sum_{j=0}^{d-1}\left[\frac{1}{n}\sum_{l=0}^{n-1}\chi_0^{\frac{nj}{d}}(x^{-l})x^l\right] \\ &= \frac{1}{d}\sum_{j=0}^{d-1}\sum_{l=0}^{n-1}\exp\left(2\pi i\cdot\frac{-lj}{d}\right)x^l = \sum_{l=0}^{n-1}\alpha_l x^l,\end{aligned} \tag{11}$$

where

$$\alpha_l := \frac{1}{d}\sum_{j=0}^{d-1}\exp\left(2\pi i\cdot\frac{-lj}{d}\right) = \begin{cases} 1 & \text{if } l = kd,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}. \tag{12}$$

The rightmost equality in (12) follows from a standard result about the vanishing of sums of roots of unity (see, e.g., [6]). Now, (11) and (12) yield

$$\frac{n}{d}\sum_{j=0}^{d-1} e_{\frac{j}{d}} = \sum_{k=0}^{\frac{n}{d}-1} x^{kd} = \sum_{h\in H_d} h,$$

proving the lemma. □

The next step is to plug (10) in (7) for every subgroup H_{d_l} in the partition (3) and get

$$ne_1 = \sum_{l=1}^{s} x^{a_l}\left(\frac{n}{d_l}\sum_{j=0}^{d_l-1} e_{\frac{j}{d_l}}\right).$$

Using (9), one obtains

$$ne_1 = \sum_{l=1}^{s} \frac{n}{d_l} \sum_{j=0}^{d_l-1} \chi_0^{\frac{jn}{d_l}}(x^{a_l}) e_{\frac{j}{d_l}}. \tag{13}$$

Now, let d_m be maximal in (3), i.e., it does not properly divide any other modulus d_j participating in this disjoint union. Note that in this case $e_{\frac{1}{d_m}}$ does not appear in (13) as $e_{\frac{j}{d_l}}$ for any $j > 1$. It remains to show that the index d_m appears at least twice in (3). Otherwise, the idempotent $e_{\frac{1}{d_m}}$ would appear only once in (13), for $l = m$ and $j = 1$ in the double summation. Multiply both sides of (13) by $e_{\frac{1}{d_m}}$. Then the orthogonality property (8) yields 0 on the left-hand side, and

$$\frac{n}{d_m}\chi_0^{\frac{n}{d_m}}(x^{a_m})e_{\frac{1}{d_m}} = \frac{n}{d_m}\exp\left(2\pi i \cdot \frac{a_m}{d_m}\right)e_{\frac{1}{d_m}} \neq 0$$

on the right-hand side. This is a contradiction. □

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