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## A short combinatorial proof of derangement identity

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### 1 Introduction

Having a permutation  $\sigma \in S_n$ ,  $\sigma : [n] \rightarrow [n]$  where  $[n] := \{1, 2, \dots, n\}$ , it is said that  $k \in [n]$  is a *fixed point* if it is mapped to itself,  $\sigma(k) = k$ . Permutations without fixed points are of particular interest and are usually called *derangements*. We let  $D_n$  denote the number of derangements of the set  $[n]$ ,  $D_n = |S_n^{(0)}|$ ,

$$S_n^{(0)} := \{\sigma \in S_n : \sigma(k) \neq k, k = 1, \dots, n\}.$$

Derangements are usually introduced in the context of the inclusion-exclusion principle

Die Subfakultät  $!n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = \left\lfloor \frac{n!+1}{e} \right\rfloor$  gibt die Anzahl der Derangements, d.h. der fixpunktfreien Permutationen von  $n$  unterscheidbaren Objekten an. Die Anzahl der Permutationen dieser Objekte mit genau  $r$  Fixpunkten wird als Rencontres-Zahl bezeichnet. Es gibt eine Vielzahl interessanter kombinatorischer Identitäten, bei denen die Subfakultät und Rencontres-Zahlen eine Rolle spielen. Am bekanntesten ist vermutlich das Inklusions-Exklusionsprinzip. In der vorliegenden Arbeit betrachten die Autoren gewichtete Summen von Subfakultäten. Diese lassen sich zwar auch algebraisch beweisen, hier werden jedoch elegante kombinatorische Abzählargumente für die Herleitung benützt.

[3, 5, 10], since this principle is used to provide an interpretation of  $D_n$  as a *subfactorial*,

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}. \quad (1)$$

The numbers  $D_0, D_1, D_2, \dots, D_n, \dots$  form the recursive sequence  $(D_n)_{n \geq 0}$  defined by the recurrence formula

$$D_n = (n-1)(D_{n-1} + D_{n-2}) \quad (2)$$

and initial terms  $D_0 = 1, D_1 = 0$  [7]. There is a counting argument to prove this. Let the number  $k$  be mapped by  $\sigma$  to the number  $j, j = 1, \dots, k-1, k+1, \dots, n$ . Note that there are  $(n-1)$  such permutations  $\sigma$ . Now, we separate the set of permutations  $\sigma$  into two disjoint sets  $\mathcal{A}$  and  $\mathcal{B}$ , such that

$$\begin{aligned} \mathcal{A} &:= \{\sigma \in S_n^{(0)} : \sigma(j) \neq k, \sigma(k) = j\} \\ \mathcal{B} &:= \{\sigma \in S_n^{(0)} : \sigma(j) = k, \sigma(k) = j\}. \end{aligned}$$

This means that

$$D_n = (n-1)(|\mathcal{A}| + |\mathcal{B}|).$$

The set  $\mathcal{A}$  counts  $D_{n-1}$  elements while the set  $\mathcal{B}$  counts  $D_{n-2}$  elements. The fact that the number  $k$  in this reasoning is chosen without loss of generality, completes the proof of (2).

There is another recurrence for the sequence  $(D_n)_{n \geq 0}$ ,

$$D_n = nD_{n-1} + (-1)^n. \quad (3)$$

Namely, set  $\delta_n = D_n - nD_{n-1}$  for every  $n \geq 1$ . Then  $\delta_1 = -1$ , and formula (2) implies that one has for every  $n \geq 2$

$$\delta_n = D_n - nD_{n-1} = (n-1)D_{n-2} - D_{n-1} = -\delta_{n-1},$$

hence one gets immediately  $\delta_n = (-1)^n$ , which proves (3).

When we iteratively apply recurrence (3) to the derangement number on the r.h.s. of this relation we get

$$nD_{n-1} + (-1)^n = n[(n-1)D_{n-2} + (-1)^{n-1}] + (-1)^n$$

which finally results with

$$n(n-1)(n-2) \cdots 3(-1)^2 + n(n-1)(n-2) \cdots 4(-1)^3 + \cdots + (-1)^n. \quad (4)$$

on the r.h.s. of (3), which completes the proof of (1).

A few identities for the sequence  $(D_n)_{n \geq 0}$  are known [4, 6, 8]. In [4] Deutsche and Elizalde give a nice identity

$$D_n = \sum_{k=2}^n (k-1) \binom{n}{k} D_{n-k}. \quad (5)$$

Recently, Bhatnagar presents families of identities for some sequences including the shifted derangement numbers [1], deriving it using an Euler identity [2]. In what follows we demonstrate a combinatorial proof for that derangement identity, with weighted sum.

## 2 A pair of weighted sums for derangements

We define the *rencontres number*  $D_n(r)$  as the number of permutations  $\sigma \in S_n$  having exactly  $r$  fixed points. Thus,  $D_n(0) = D_n$ . For a given  $r \in \mathbb{N}$ , we define the sequence  $D_0(r), D_1(r), \dots, D_n(r), \dots$ , denoted by  $(D_n(r))_{n \geq r}$ .

Applying an analogue counting argument that we used when proving relation (2), one can represent rencontres numbers by the derangement numbers,

$$D_n(r) = \binom{n}{r} D_{n-r}. \quad (6)$$

On the other hand, relation (6) follows immediately from the fact that fixed points here are  $r$ -combinations over the set of  $n$  elements.

A few other notable properties of the rencontres numbers are also known. It follows from (3) that  $D_n - D_n(1) = (-1)^n$  for every  $n \geq 1$ . According to the definition of rencontres numbers, the sum of the  $n$ th row in the array of numbers  $(D_n(r))_{n \geq r}$  is equal to  $n!$ ,

$$n! = \sum_{k=0}^n D_n(k). \quad (7)$$

Moreover, identity (6) shows that  $D_n$  can be interpreted as a weighted sum of rencontres numbers in the  $n$ th row of the array, by means of relation (5),

$$D_n = \sum_{k=2}^n (k-1) D_n(k). \quad (8)$$

The number  $D_n/(n-1)$  is also a weighted sum of previous consecutive derangement numbers. For example,  $24 + 12D_2 + 4D_3 + D_4 = \frac{D_6}{5}$ . In general we have

$$n! + \sum_{k=2}^n \frac{n!}{k!} D_k = \frac{D_{n+2}}{n+1}, \quad (9)$$

as follows from Theorem 1.

**Theorem 1.** For  $n \in \mathbb{N}$  and the sequence of derangement numbers  $(D_n)_{n \geq 0}$  we have

$$1 + \sum_{k=1}^n \frac{D_k}{k!} = \frac{D_{n+2}}{(n+1)!}. \quad (10)$$

*Proof.* Within a derangement  $\sigma$ , the number  $k$ ,  $k = 1, \dots, n$  can be mapped to any  $j$ ,  $j = 1, \dots, k-1, k+1, \dots, n$ . We let  $\mathcal{A}_n$  denote the set of derangements with  $\sigma(k) = j$ , where  $j \neq k$ ,

$$\mathcal{A}_n := \{\sigma \in S_n^{(0)} : \sigma(k) = j\}.$$

Obviously, cardinality of the set  $\mathcal{A}_n$  is independent of  $j$ ,  $j \neq k$ . More precisely,

$$|\mathcal{A}_n| = \frac{D_n}{n-1}.$$

Furthermore, we separate the set  $\mathcal{A}_n$  into two disjoint sets of derangements, sets  $\mathcal{B}_n$  and  $\mathcal{C}_n$ ,

$$\begin{aligned}\mathcal{B}_n &:= \{\sigma \in \mathcal{A}_n : \sigma(j) = k\} \\ \mathcal{C}_n &:= \{\sigma \in \mathcal{A}_n : \sigma(j) \neq k\}.\end{aligned}$$

Obviously, the set  $\mathcal{B}_n$  counts  $D_{n-2}$  elements. For derangements in  $\mathcal{C}_n$  there are now  $(n-2)$  equivalent ways to map  $j$  (excluding  $j$  and  $k$ ), as Figure 1 illustrates. Thus, we have

$$|\mathcal{C}_n| = (n-2)|\mathcal{A}_{n-1}|,$$

which gives the recurrence relation

$$|\mathcal{A}_n| = D_{n-2} + (n-2)|\mathcal{A}_{n-1}|. \quad (11)$$

After repeating usage of (11) we get identity (9) which completes the proof.  $\square$

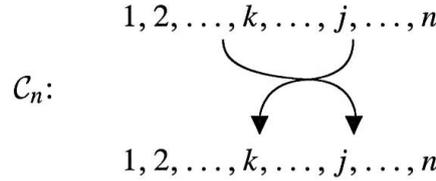


Figure 1 In case of derangements in the set  $\mathcal{C}_n$  there are  $(n-2)$  equivalent ways to map  $j$ .

In order to prove Theorem 1 algebraically, we apply recurrence (2) to get

$$\begin{aligned}\frac{D_{n+2}}{(n+1)!} &= \frac{(n+1)(D_{n+1} + D_n)}{(n+1)!} = \frac{D_{n+1}}{n!} + \frac{D_n}{n!} \\ &= \frac{n(D_n + D_{n-1})}{n!} + \frac{D_n}{n!} = \frac{D_n}{(n-1)!} + \frac{D_{n-1}}{(n-1)!} + \frac{D_n}{n!} \\ &= 1 + \frac{D_1}{1!} + \dots + \frac{D_n}{n!} = 1 + \sum_{k=1}^n \frac{D_k}{k!}.\end{aligned}$$

**Theorem 2.** For  $n \in \mathbb{N}$  and the sequence of derangement numbers  $(D_n)_{n \geq 0}$  we have

$$1 + \sum_{k=1}^n \frac{(-1)^k D_{k+3}}{k+2} = (-1)^n D_{n+2}. \quad (12)$$

*Proof.* By applying recurrence (2) we have

$$\begin{aligned}\sum_{k=0}^n \frac{(-1)^k D_{k+3}}{k+2} &= \frac{2(D_2 + D_1)}{2} - \frac{3(D_3 + D_2)}{3} + \dots + (-1)^n \frac{(n+2)(D_{n+2} + D_{n+1})}{n+2} \\ &= (D_2 + D_1) - (D_3 + D_2) + \dots + (-1)^n (D_{n+2} + D_{n+1}) \\ &= (-1)^n D_{n+2}\end{aligned}$$

which completes the proof.  $\square$

Once having Theorem 1, substitution of (6) in identity (10) gives the generalization (13).

$$1 + \sum_{k=1}^n \frac{D_{k+r}(r)}{k! \binom{k+r}{r}} = \frac{D_{n+r+2}(r)}{(n+1)! \binom{n+r+2}{r}}. \quad (13)$$

The identity (14) follows by substitution of (6) in (12),

$$1 + \sum_{k=1}^n \frac{(-1)^k D_{k+r+3}(r)}{(k+2) \binom{k+r+3}{r}} = \frac{(-1)^n D_{n+r+2}(r)}{\binom{n+r+2}{r}}. \quad (14)$$

Note that the terms in identity (14) are always integers, which can be seen as a consequence of recurrence relation (2).

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