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Elemente der Mathematik

Density property of certain sets and their applications

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1 Introduction

It is well known that the set $\mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$ is a dense set in $\mathbb{R}$ if q is irrational. Here we provide a proof using the Engel expansion. Let $p, q \in \mathbb{R}^+ \setminus \{1\}$ be fixed. In [1], the authors proved that a set of the form $\{\pm p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number. Here we give a different proof. The authors in [1] also proved that, if $\frac{\ln p}{\ln q}$ is an irrational number and f is a continuous function on $\mathbb{R} \setminus \{0\}$, then $\int_x^{px} f(t)dt$ and $\int_x^{qx} f(t)dt$ are constant functions of x if and only if $f(t) = \frac{c}{t}$, where c is a real number. We extend this result to the class of integrable functions. In this paper we also obtain an equivalent characterization of irrational numbers. Using this characterization we

Friedrich Engel hat 1913 vorgeschlagen, eine reelle Zahl $q > 0$ durch eine unendliche Reihe der Form

$$q = \sum_{n=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_n}$$

darzustellen, wobei p_n eine nicht fallende Folge natürlicher Zahlen ist. Diese Engel-Entwicklung ist eindeutig und stellt genau dann eine rationale Zahl q dar, wenn die Folge der p_n ab einem bestimmten Index konstant ist. (Das entsprechende geometrische Endstück der Reihe lässt sich dann auch als Stammbruch schreiben und man erhält eine Ägyptische Darstellung von q.) Da zum Beispiel die Eulersche Zahl die Engel-Entwicklung $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ besitzt, kann man daraus sofort auf die Irrationalität von e schliessen. In der vorliegenden Arbeit wird diese Methode in Verbindung gebracht mit der Dichtheit gewisser Mengen in $\mathbb{R}$ und einem Problem der Masstheorie.

show for certain types of numbers that they are irrational: For example we show that e and $q^{1/n}$ (q is a prime number and $2 \leq n \in \mathbb{N}$) are irrational numbers. See [4, 5, 6] for similar results.

2 Series representation of irrational numbers and density properties

For the sake of completeness we give below a proof of the Engel expansion.

Theorem 2.1. *For any irrational number* $0 < q < 1$, *there exist natural numbers* $p_i \geq 2$, $i = 1, 2, \ldots$ *with* $p_i \leq p_{i+1}$ *such that*

$$q = \sum_{i=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_i}. \tag{2.1}$$

Proof. Since $0 < q < 1$, there exists a natural number $p_1 \geq 2$ such that $(p_1 - 1)q < 1 < p_1 q < 2$. Now set $\alpha_0 = q$, $p_0 = 2$ and define $\alpha_1 = p_1 q - 1$, hence $0 < \alpha_1 < 1$. Choose an integer p_2 such that

$$(p_2 - 1)\alpha_1 < 1 < p_2 \alpha_1 < 2.$$

Define $\alpha_2 = p_2 \alpha_1 - 1$. The above inequality $(p_1 - 1)q < 1 < p_1 q$ implies $p_1 q - 1 < p_1 q - (p_1 - 1)q$ which yields $\alpha_1 < \alpha_0$. Moreover the above inequalities give $(p_1 - 1)\alpha_0 < p_2 \alpha_1$. This implies $p_1 - 1 < p_2$, since $\alpha_1 < \alpha_0$. Hence $p_1 \leq p_2$ as p_1 and p_2 are integers. By induction, we construct $p_n \in \mathbb{N}$ and α_n satisfying the properties

$$\begin{aligned} &(p_{n+1} - 1)\alpha_n < 1 < p_{n+1}\alpha_n < 2, \\ &\alpha_n = p_n \alpha_{n-1} - 1 \text{ and} \\ &p_n \leq p_{n+1}. \end{aligned} \tag{2.2}$$

To see (2.2), let us assume we are given p_i, $1 \leq i \leq k+1$ and α_i, $1 \leq i \leq k$, satisfying

$$\begin{aligned} &(p_{i+1} - 1)\alpha_i < 1 < p_{i+1}\alpha_i < 2 \\ &\alpha_i = p_i \alpha_{i-1} - 1 \text{ and } p_i \geq p_{i-1}. \end{aligned}$$

Then we construct p_{k+2} and α_{k+1} as follows: Take $\alpha_{k+1} = p_{k+1}\alpha_k - 1$. Choose p_{k+2} such that $(p_{k+2} - 1)\alpha_{k+1} < 1 < p_{k+2}\alpha_{k+1} < 2$. As for the previous analysis, the above inequalities give $\alpha_{k+1} < \alpha_k$ and $p_{k+1} - 1 < p_{k+2}$. This implies $p_{k+1} \leq p_{k+2}$, which proves the statement (2.2).

Equation (2.2) yields:

$$\begin{aligned} &\frac{1}{p_n} < \alpha_{n-1} < \frac{2}{p_n} \\ &\Longrightarrow \left|\alpha_{n-1} - \frac{1}{p_n}\right| < \frac{1}{p_n} \\ &\Longrightarrow \left|p_{n-1}\alpha_{n-2} - 1 - \frac{1}{p_n}\right| < \frac{1}{p_n} \\ &\Longrightarrow \left|\alpha_{n-2} - \frac{1}{p_{n-1}} - \frac{1}{p_{n-1}p_n}\right| < \frac{1}{p_{n-1}p_n}. \end{aligned} \tag{2.3}$$

Continuing in this way by induction we get:

$$\left|q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right| < \frac{1}{p_1 p_2 \cdots p_n}.$$

Passing to the limit as n tends to infinity in equation (2.3), we get the expression (2.1). □

The Engel Expansion in Theorem 2.1 is unique. In fact, the expansion of q is an ascending variant of continued fractions. q can be written in the following way:

$$q = \cfrac{1 + \cfrac{1 + \cfrac{1 + \cdots}{p_3}}{p_2}}{p_1}.$$

For example the canonical values $p_i, i = 1, 2, \ldots$, for $q = \sqrt{2} - 1$ are

$$(p_1, p_2, p_3, \ldots) = (3, 5, 5, 16, 18, 78, 102, 120, 144, \ldots)$$

and the canonical values for $q = \frac{\sqrt{5}-1}{2}$ are

$$(p_1, p_2, p_3, \ldots) = (5, 6, 13, 16, 16, 38, 48, 58, 104, \ldots).$$

Theorem 2.2. *Define $\mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$, $q \in \mathbb{R}$. Then the following statements are equivalent.*

1. *q is an irrational number.*
2. *There exist $z_n \in \mathbb{A}$, $n \in \mathbb{N}$ such that z_n tends to zero as n tends to infinity.*
3. *$\mathbb{A}$ is dense in $\mathbb{R}$.*

Proof. $1 \Longrightarrow 2$: Let q be an irrational number. Without loss of generality we can assume $0 < q < 1$. Then by the above theorem,

$$\begin{aligned} q &= \sum_{i=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_i} \\ &\Longrightarrow \left|q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right| < \frac{2}{p_1 p_2 \cdots p_{n+1}} \qquad (2.4) \\ &\Longrightarrow \left|p_1 p_2 \cdots p_n \left(q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right)\right| < \frac{2}{p_{n+1}}. \end{aligned}$$

Also note that $p_i < p_{i+1}$ holds infinitely often, since otherwise q would be rational. So p_{n+1} tends to infinity as n tends to infinity. Let $s_n = -\left(\sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right) p_1 p_2 \cdots p_n$,

$r_n = p_1 p_2 \cdots p_n$. Then r_n and s_n are integers. Moreover we have that $z_n = r_n + q s_n \in \mathbb{A}$ tends to zero as $n \to \infty$.

$2 \Longrightarrow 3$: Without loss of generality, we assume that all z_n are positive. Let $a, b \in \mathbb{R}$ and $a < b$. Since $\frac{1}{z_n}(b-a)$ tends to infinity, there exists N_0 such that $\frac{1}{z_{N_0}} a < t < \frac{1}{z_{N_0}} b$, for some integer t. This implies $a < t z_{N_0} < b$. Since $t z_{N_0} \in \mathbb{A}$, $\mathbb{A}$ is dense in $\mathbb{R}$.

$3 \Longrightarrow 1$: Let $\mathbb{A}$ be dense in $\mathbb{R}$. We have to show that q is irrational. Assume the opposite, i.e., that q is a rational number of the form $q = \frac{m_0}{n_0}$, $m_0, n_0 \in \mathbb{Z}$ and $n_0 \neq 0$. Clearly $n_0 \mathbb{A} \subset \mathbb{Z}$, so the distance between two elements of $\mathbb{A}$ is at least $\frac{1}{n_0}$. Hence $\mathbb{A}$ is not dense. □

Corollary 2.3. *Suppose $p, q \in \mathbb{R}^+ \setminus \{1\}$. Then $\mathbb{B} = \{\pm p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number.*

Proof. Consider the set $\bar{\mathbb{B}} = \{m \ln p + n \ln q : m, n \in \mathbb{Z}\}$:

$$\{m \ln p + n \ln q : m, n \in \mathbb{Z}\} = \ln q \left\{ m \frac{\ln p}{\ln q} + n : m, n \in \mathbb{Z} \right\}.$$

By Theorem 2.2, $\{m \frac{\ln p}{\ln q} + n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number. Hence $\bar{\mathbb{B}}$ is a dense subset of $\mathbb{R}$.

Now we will show that $\bar{\mathbb{B}}$ is dense in $\mathbb{R}$ iff $\mathbb{B}$ is dense in $\mathbb{R}$. Let $y > 0$. There exists a sequence $m_t \ln p + n_t \ln q$ which converges to $\ln y$ as t tends to ∞. Now by the mean value theorem

$$|p^{m_t} q^{n_t} - y| = |\exp(m_t \ln p + n_t \ln q) - \exp(\ln y)| = \exp(c(t)) |[(m_t \ln p + n_t \ln q) - \ln y]|,$$

where $c(t)$ is a point lying between $(m_t \ln p + n_t \ln q)$ and $\ln y$. Since $c(t)$ is bounded, $p^{m_t} q^{n_t}$ converges to y. So $\{p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $[0, \infty)$. Hence $\mathbb{B}$ is dense in $\mathbb{R}$. Similarly one can show the converse. This completes the proof of Corollary 2.3. □

3 Applications

Example 3.1. *If q is a prime number, then for any natural number $n \geq 2$, $q^{1/n}$ is an irrational number.*

Proof. Choose $m \in \mathbb{N}$ such that $m < q^{1/n} < m + 1$ and hence $0 < q^{1/n} - m < 1$. Now consider the set

$$\mathbb{A} = \left\{ \sum_{i=0}^{n-1} c_i q^{i/n} : c_i \in \mathbb{Z} \right\}.$$

For any $k \in \mathbb{N}$, $z_k = (q^{1/n} - m)^k \in \mathbb{A}$ and tends to zero as k tends to infinity. So for $a, b \in \mathbb{R}$, there exist $t \in \mathbb{Z}$ and $n_0 \in \mathbb{N}$ such that $\frac{a}{z_{n_0}} < t < \frac{b}{z_{n_0}}$. This implies $a < t z_{n_0} < b$. As $t z_{n_0} \in \mathbb{A}$, $\mathbb{A}$ is dense in $\mathbb{R}$.

If $q^{1/n}$ would be rational, then $q = \frac{r}{s}$. Clearly $s^n \mathbb{A} \subset \mathbb{Z}$. So the distance between any two numbers of $\mathbb{A}$ would be at least $\frac{1}{s^n}$, which is a contradiction. Hence $q^{1/n}$ is an irrational number. □

Example 3.2. *Any number q of the form* (2.1), *with $p_i < p_{i+1}$ for infinitely many i, is an irrational number. In particular e is an irrational number.*

Proof. If q is of the form (2.1) and $p_i < p_{i+1}$ for infinitely many i, then, by (2.4),

$$z_n = p_1 p_2 \cdots p_n \left(q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i} \right) \text{ tends to zero.}$$

So there are elements $z_n \in \mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$ which tend to zero as n tends to infinity. So by Theorem 2.2 the result follows. □

Example 3.3. *Let $\frac{\ln p}{\ln q}$ $(p, q \in \mathbb{R}^+ \setminus \{1\})$ be an irrational number and f be a locally integrable function on $\mathbb{R} \setminus \{0\}$. Then $\int_x^{px} f(t)dt$ and $\int_x^{qx} f(t)dt$ are constant functions of x if and only if $f(t) = \frac{c}{t}$, $c \in \mathbb{R}$.*

Proof. The sufficient part of the theorem is trivial. We prove the necessary part: Define the measure μ on the multipicative group $\mathbb{R}^+$ as follows: Let E be any Borel measurable set of $\mathbb{R}^+$. Define $\mu(E) = \int_E f(y)dy$, then we claim that μ is a Haar measure on $\mathbb{R}$.

$$\begin{aligned} \mu([a, b]) &= \int_a^b f(y)dy \\ \Longrightarrow \mu([pa, pb]) &= \int_{pa}^{pb} f(y)dy \\ &= \int_{pa}^{a} f(y)dy + \int_a^b f(y)dy + \int_b^{pb} f(y)dy. \end{aligned} \tag{3.1}$$

Now $\int_{pa}^{a} f(y)dy + \int_b^{pb} f(y)dy = 0$, since $\int_x^{px} f(y)dy$ is constant. This implies

$$\mu([pa, pb] = \int_a^b f(y)dy = \mu([a, b].$$

By approximation, we get $\mu(pE) = \mu(E)$ and hence $\mu(p^m E) = \mu(E)$ for $m \in \mathbb{Z}$. Following the same analysis as before we get, $\mu(p^m q^n E) = \mu(E)$. By Corollary 2.3, the set $\{p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}^+$. This implies $\mu(aE) = \mu(E)$ for any $a \in \mathbb{R}^+$ and Borel measurable set E. This proves that μ is a Haar measure.

Note that $\bar{\mu}(E) = \int_E \frac{1}{t} dt$ is a Haar measure on the multiplicative topological group $\mathbb{R}^+$. Applying Theorem 11.9 from [2, Chapter 9], $\mu = c\bar{\mu}$, for some $c \in \mathbb{R}$. This in turn gives $f(t) = \frac{c_1}{t}$ on $\mathbb{R}^+$.

Similarly, considering the same Haar measure concept on $\mathbb{R}^+$ as before with $f(t)$ replaced by $f(-t)$, we can discover $f(-t) = \frac{c_2}{t}, t > 0$. Now $c_1 = -c_2$, since $\int_{-1}^{-p} f(t)dt = \int_1^p f(t)dt$. Hence $f(t) = \frac{c}{t}$, $c \in \mathbb{R}$. □

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