Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 72 (2017)

Heft: 1

Artikel: Density property of certain sets and their applications

Autor: Sahoo, Manas R.

DOI: https://doi.org/10.5169/seals-735173

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 14.04.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

Elem. Math. 72 (2017) 9 – 14
0013-6018/17/010009-6
DOI 10.4171/EM/318

Elemente der Mathematik

Density property of certain sets and their applications

Manas R. Sahoo

Manas Ranjan Sahoo is an assistant professor at the School of Mathematical Sciences, National Institute of Science Education and Research in Jatni, India. He earned his Ph.D. in 2013 from the Tata Institute of Fundamental Research, Centre for Applicable Mathematics in Bangalore. His areas of interest reach from partial differential equations and Colombeau generalized functions over conservation laws to general measure theory.

1 Introduction

It is well known that the set $\mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$ is a dense set in $\mathbb{R}$ if q is irrational. Here we provide a proof using the Engel expansion. Let $p, q \in \mathbb{R}^+ \setminus \{1\}$ be fixed. In [1], the authors proved that a set of the form $\{\pm p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number. Here we give a different proof. The authors in [1] also proved that, if $\frac{\ln p}{\ln q}$ is an irrational number and f is a continuous function on $\mathbb{R} \setminus \{0\}$, then $\int_x^{px} f(t)dt$ and $\int_x^{qx} f(t)dt$ are constant functions of x if and only if $f(t) = \frac{c}{t}$, where c is a real number. We extend this result to the class of integrable functions. In this paper we also obtain an equivalent characterization of irrational numbers. Using this characterization we

Friedrich Engel hat 1913 vorgeschlagen, eine reelle Zahl $q > 0$ durch eine unendliche Reihe der Form

$$q = \sum_{n=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_n}$$

darzustellen, wobei p_n eine nicht fallende Folge natürlicher Zahlen ist. Diese Engel-Entwicklung ist eindeutig und stellt genau dann eine rationale Zahl q dar, wenn die Folge der p_n ab einem bestimmten Index konstant ist. (Das entsprechende geometrische Endstück der Reihe lässt sich dann auch als Stammbruch schreiben und man erhält eine Ägyptische Darstellung von q.) Da zum Beispiel die Eulersche Zahl die Engel-Entwicklung $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ besitzt, kann man daraus sofort auf die Irrationalität von e schliessen. In der vorliegenden Arbeit wird diese Methode in Verbindung gebracht mit der Dichtheit gewisser Mengen in $\mathbb{R}$ und einem Problem der Masstheorie.

show for certain types of numbers that they are irrational: For example we show that e and $q^{1/n}$ (q is a prime number and $2 \le n \in \mathbb{N}$) are irrational numbers. See [4, 5, 6] for similar results.

2 Series representation of irrational numbers and density properties

For the sake of completeness we give below a proof of the Engel expansion.

Theorem 2.1. *For any irrational number* $0 < q < 1$, *there exist natural numbers* $p_i \ge 2,\ i = 1, 2, \ldots$ *with* $p_i \le p_{i+1}$ *such that*

$$q = \sum_{i=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_i}. \tag{2.1}$$

Proof. Since $0 < q < 1$, there exists a natural number $p_1 \ge 2$ such that $(p_1 - 1)q < 1 < p_1 q < 2$. Now set $\alpha_0 = q,\ p_0 = 2$ and define $\alpha_1 = p_1 q - 1$, hence $0 < \alpha_1 < 1$. Choose an integer p_2 such that

$$(p_2 - 1)\alpha_1 < 1 < p_2 \alpha_1 < 2.$$

Define $\alpha_2 = p_2 \alpha_1 - 1$. The above inequality $(p_1 - 1)q < 1 < p_1 q$ implies $p_1 q - 1 < p_1 q - (p_1 - 1)q$ which yields $\alpha_1 < \alpha_0$. Moreover the above inequalities give $(p_1 - 1)\alpha_0 < p_2 \alpha_1$. This implies $p_1 - 1 < p_2$, since $\alpha_1 < \alpha_0$. Hence $p_1 \le p_2$ as p_1 and p_2 are integers. By induction, we construct $p_n \in \mathbb{N}$ and α_n satisfying the properties

$$\begin{aligned} &(p_{n+1} - 1)\alpha_n < 1 < p_{n+1}\alpha_n < 2, \\ &\alpha_n = p_n \alpha_{n-1} - 1 \ \text{ and} \\ &p_n \le p_{n+1}. \end{aligned} \tag{2.2}$$

To see (2.2), let us assume we are given $p_i,\ 1 \le i \le k+1$ and α_i, $1 \le i \le k$, satisfying

$$\begin{aligned} &(p_{i+1} - 1)\alpha_i < 1 < p_{i+1}\alpha_i < 2 \\ &\alpha_i = p_i \alpha_{i-1} - 1 \ \text{ and } \ p_i \ge p_{i-1}. \end{aligned}$$

Then we construct p_{k+2} and α_{k+1} as follows: Take $\alpha_{k+1} = p_{k+1}\alpha_k - 1$. Choose p_{k+2} such that $(p_{k+2} - 1)\alpha_{k+1} < 1 < p_{k+2}\alpha_{k+1} < 2$. As for the previous analysis, the above inequalities give $\alpha_{k+1} < \alpha_k$ and $p_{k+1} - 1 < p_{k+2}$. This implies $p_{k+1} \le p_{k+2}$, which proves the statement (2.2).

Equation (2.2) yields:

$$\begin{aligned} &\frac{1}{p_n} < \alpha_{n-1} < \frac{2}{p_n} \\ &\quad \Longrightarrow \left|\alpha_{n-1} - \frac{1}{p_n}\right| < \frac{1}{p_n} \\ &\quad \Longrightarrow \left|p_{n-1}\alpha_{n-2} - 1 - \frac{1}{p_n}\right| < \frac{1}{p_n} \\ &\quad \Longrightarrow \left|\alpha_{n-2} - \frac{1}{p_{n-1}} - \frac{1}{p_{n-1}p_n}\right| < \frac{1}{p_{n-1}p_n}. \end{aligned} \tag{2.3}$$

Continuing in this way by induction we get:

$$\left|q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right| < \frac{1}{p_1 p_2 \cdots p_n}.$$

Passing to the limit as n tends to infinity in equation (2.3), we get the expression (2.1). $\square$

The Engel Expansion in Theorem 2.1 is unique. In fact, the expansion of q is an ascending variant of continued fractions. q can be written in the following way:

$$q = \cfrac{1 + \cfrac{1 + \cfrac{1 + \cdots}{p_3}}{p_2}}{p_1}.$$

For example the canonical values $p_i, i = 1, 2, \ldots$, for $q = \sqrt{2} - 1$ are

$$(p_1, p_2, p_3, \ldots) = (3, 5, 5, 16, 18, 78, 102, 120, 144, \ldots)$$

and the canonical values for $q = \frac{\sqrt{5}-1}{2}$ are

$$(p_1, p_2, p_3, \ldots) = (5, 6, 13, 16, 16, 38, 48, 58, 104, \ldots).$$

Theorem 2.2. *Define* $\mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$, $q \in \mathbb{R}$. *Then the following statements are equivalent.*

1. *q is an irrational number.*
2. *There exist $z_n \in \mathbb{A}$, $n \in \mathbb{N}$ such that z_n tends to zero as n tends to infinity.*
3. *$\mathbb{A}$ is dense in $\mathbb{R}$.*

Proof. $1 \Longrightarrow 2$: Let q be an irrational number. Without loss of generality we can assume $0 < q < 1$. Then by the above theorem,

$$\begin{aligned} q &= \sum_{i=1}^{\infty} \frac{1}{p_1 p_2 \cdots p_i} \\ &\Longrightarrow \left|q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right| < \frac{2}{p_1 p_2 \cdots p_{n+1}} \qquad (2.4) \\ &\Longrightarrow \left|p_1 p_2 \cdots p_n \left(q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right)\right| < \frac{2}{p_{n+1}}. \end{aligned}$$

Also note that $p_i < p_{i+1}$ holds infinitely often, since otherwise q would be rational. So p_{n+1} tends to infinity as n tends to infinity. Let $s_n = -\left(\sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i}\right) p_1 p_2 \cdots p_n$,

$r_n = p_1 p_2 \cdots p_n$. Then r_n and s_n are integers. Moreover we have that $z_n = r_n + q s_n \in \mathbb{A}$ tends to zero as $n \to \infty$.

$2 \Longrightarrow 3$: Without loss of generality, we assume that all z_n are positive. Let $a, b \in \mathbb{R}$ and $a < b$. Since $\frac{1}{z_n}(b-a)$ tends to infinity, there exists N_0 such that $\frac{1}{z_{N_0}} a < t < \frac{1}{z_{N_0}} b$, for some integer t. This implies $a < t z_{N_0} < b$. Since $t z_{N_0} \in \mathbb{A}$, $\mathbb{A}$ is dense in $\mathbb{R}$.

$3 \Longrightarrow 1$: Let $\mathbb{A}$ be dense in $\mathbb{R}$. We have to show that q is irrational. Assume the opposite, i.e., that q is a rational number of the form $q = \frac{m_0}{n_0}$, $m_0, n_0 \in \mathbb{Z}$ and $n_0 \neq 0$. Clearly $n_0 \mathbb{A} \subset \mathbb{Z}$, so the distance between two elements of $\mathbb{A}$ is at least $\frac{1}{n_0}$. Hence $\mathbb{A}$ is not dense. ☐

Corollary 2.3. *Suppose $p, q \in \mathbb{R}^+ \backslash \{1\}$. Then $\mathbb{B} = \{\pm p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number.*

Proof. Consider the set $\bar{\mathbb{B}} = \{m \ln p + n \ln q : m, n \in \mathbb{Z}\}$:

$$\{m \ln p + n \ln q : m, n \in \mathbb{Z}\} = \ln q \left\{ m \frac{\ln p}{\ln q} + n : m, n \in \mathbb{Z} \right\}.$$

By Theorem 2.2, $\{m \frac{\ln p}{\ln q} + n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}$ iff $\frac{\ln p}{\ln q}$ is an irrational number. Hence $\bar{\mathbb{B}}$ is a dense subset of $\mathbb{R}$.

Now we will show that $\bar{\mathbb{B}}$ is dense in $\mathbb{R}$ iff $\mathbb{B}$ is dense in $\mathbb{R}$. Let $y > 0$. There exists a sequence $m_t \ln p + n_t \ln q$ which converges to $\ln y$ as t tends to ∞. Now by the mean value theorem

$$|p^{m_t} q^{n_t} - y| = |\exp(m_t \ln p + n_t \ln q) - \exp(\ln y)| = \exp(c(t)) |[(m_t \ln p + n_t \ln q) - \ln y]|,$$

where $c(t)$ is a point lying between $(m_t \ln p + n_t \ln q)$ and $\ln y$. Since $c(t)$ is bounded, $p^{m_t} q^{n_t}$ converges to y. So $\{p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $[0, \infty)$. Hence $\mathbb{B}$ is dense in $\mathbb{R}$. Similarly one can show the converse. This completes the proof of Corollary 2.3. ☐

3 Applications

Example 3.1. *If q is a prime number, then for any natural number $n \geq 2$, $q^{1/n}$ is an irrational number.*

Proof. Choose $m \in \mathbb{N}$ such that $m < q^{1/n} < m+1$ and hence $0 < q^{1/n} - m < 1$. Now consider the set

$$\mathbb{A} = \left\{ \sum_{i=0}^{n-1} c_i q^{i/n} : c_i \in \mathbb{Z} \right\}.$$

For any $k \in \mathbb{N}$, $z_k = (q^{1/n} - m)^k \in \mathbb{A}$ and tends to zero as k tends to infinity. So for $a, b \in \mathbb{R}$, there exist $t \in \mathbb{Z}$ and $n_0 \in \mathbb{N}$ such that $\frac{a}{z_{n_0}} < t < \frac{b}{z_{n_0}}$. This implies $a < t z_{n_0} < b$. As $t z_{n_0} \in \mathbb{A}$, $\mathbb{A}$ is dense in $\mathbb{R}$.

If $q^{1/n}$ would be rational, then $q = \frac{r}{s}$. Clearly $s^n \mathbb{A} \subset \mathbb{Z}$. So the distance between any two numbers of $\mathbb{A}$ would be at least $\frac{1}{s^n}$, which is a contradiction. Hence $q^{1/n}$ is an irrational number. ☐

Example 3.2. *Any number q of the form* (2.1), *with $p_i < p_{i+1}$ for infinitely many i, is an irrational number. In particular e is an irrational number.*

Proof. If q is of the form (2.1) and $p_i < p_{i+1}$ for infinitely many i, then, by (2.4),

$$z_n = p_1 p_2 \cdots p_n \left(q - \sum_{i=1}^{n} \frac{1}{p_1 p_2 \cdots p_i} \right) \text{ tends to zero.}$$

So there are elements $z_n \in \mathbb{A} = \{m + nq : m, n \in \mathbb{Z}\}$ which tend to zero as n tends to infinity. So by Theorem 2.2 the result follows. ☐

Example 3.3. *Let $\frac{\ln p}{\ln q}$ $(p, q \in \mathbb{R}^+ \setminus \{1\})$ be an irrational number and f be a locally integrable function on $\mathbb{R} \setminus \{0\}$. Then $\int_x^{px} f(t)dt$ and $\int_x^{qx} f(t)dt$ are constant functions of x if and only if $f(t) = \frac{c}{t}$, $c \in \mathbb{R}$.*

Proof. The sufficient part of the theorem is trivial. We prove the necessary part: Define the measure μ on the multipicative group $\mathbb{R}^+$ as follows: Let E be any Borel measurable set of $\mathbb{R}^+$. Define $\mu(E) = \int_E f(y)dy$, then we claim that μ is a Haar measure on $\mathbb{R}$.

$$\begin{aligned} \mu([a,b]) &= \int_a^b f(y)dy \\ \Longrightarrow \mu([pa,pb]) &= \int_{pa}^{pb} f(y)dy \\ &= \int_{pa}^{a} f(y)dy + \int_a^b f(y)dy + \int_b^{pb} f(y)dy. \end{aligned} \tag{3.1}$$

Now $\int_{pa}^{a} f(y)dy + \int_b^{pb} f(y)dy = 0$, since $\int_x^{px} f(y)dy$ is constant. This implies

$$\mu([pa, pb] = \int_a^b f(y)dy = \mu([a, b].$$

By approximation, we get $\mu(pE) = \mu(E)$ and hence $\mu(p^m E) = \mu(E)$ for $m \in \mathbb{Z}$. Following the same analysis as before we get, $\mu(p^m q^n E) = \mu(E)$. By Corollary 2.3, the set $\{p^m q^n : m, n \in \mathbb{Z}\}$ is a dense subset of $\mathbb{R}^+$. This implies $\mu(aE) = \mu(E)$ for any $a \in \mathbb{R}^+$ and Borel measurable set E. This proves that μ is a Haar measure.

Note that $\bar{\mu}(E) = \int_E \frac{1}{t} dt$ is a Haar measure on the multiplicative topological group $\mathbb{R}^+$. Applying Theorem 11.9 from [2, Chapter 9], $\mu = c\bar{\mu}$, for some $c \in \mathbb{R}$. This in turn gives $f(t) = \frac{c_1}{t}$ on $\mathbb{R}^+$.

Similarly, considering the same Haar measure concept on $\mathbb{R}^+$ as before with $f(t)$ replaced by $f(-t)$, we can discover $f(-t) = \frac{c_2}{t}, t > 0$. Now $c_1 = -c_2$, since $\int_{-1}^{-p} f(t)dt = \int_1^p f(t)dt$. Hence $f(t) = \frac{c}{t}, c \in \mathbb{R}$. ☐

Acknowledgement

The author wishes to express his sincere gratitude to the anonymous referee for his valuable suggestions and comments which improved the presentation of the paper.

References

[1] Tian-Xiao He, Peter J.-S. Shiue and Xiaoya Zha, *Some dense subsets of real numbers and their applications*, J. Adv. Math. Stud. **4** (2011), no. 2, 25–32.

[2] G.B. Folland, *Real Analysis, Modern Techniques and Their Applications*, A Wiley-Interscience Publication, John Wiley and Sons Inc., 2nd edition.

[3] R.A. Devore and G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, New York, 1991.

[4] T. Apostol, *Irrationality of the square root of two – a geometric proof*, Amer. Math. Monthly **107** (2000), 841–842.

[5] P.H. Diananda and A. Oppenheim, *Criteria for irrationality of certain classes of numbers. II*, Amer. Math. Monthly **62** (1955), 222–225.

[6] L.L. Pennisi, *Elementary proof that e is irrational*, Amer. Math. Monthly **60** (1953), 474.

Manas R. Sahoo
School of Mathematical Sciences
National Institute of Science Education and Research
Bhimpur-Padanpur, Jatni
Khurda-752050, Odisha, India
e-mail: `manas@niser.ac.in`