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Elemente der Mathematik

Johann Bernoulli and the cycloid: A theorem for posterity

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1 Preamble

"Vous sçavez que l'envie est la Suivante de la Gloire[1]. [You know that Envy is the follower of Glory.]"

(Johann Bernoulli to Johannes Scheuchzer (1684–1738), April 1, 1721)

Johann Bernoulli (1667–1748) became, after Newton's scientific retirement, Leibniz' death and the premature decease of his brother Jakob (1654–1705), the world's leading mathematician in the first decades of the 18th century, a period of dramatic advances in science due to the newly discovered calculus of Newton and Leibniz. He was the "master" of the young Leonhard Euler (1707–1783) who, in turn, became "the master of us all" (in Laplace's words[2]). Fully aware of his value, Gabriel Cramer (1704–1752) edited his

[1] `http://www.ub.unibas.ch/bernoulli/index.php/Hauptseite`

[2] According to Libri (*Journal des Savants*, 1846, p. 51).

Johann Bernoulli (1667–1748), dessen Geburtstag sich heuer zum 350ten mal jährt, war mit seinem Bruder Jakob der dritte Entdecker der Differential- und Integralrechnung und hatte mit seiner Forschungs- und Lehrtätigkeit dieser erst den endgültigen Durchbruch verschafft. Darüber hinaus wurde er, besonders über seinen Schüler Leonhard Euler, zu einer der einflussreichsten Persönlichkeiten der Mathematikgeschichte. Alle vier Bände seiner 1742 zu Lebzeiten erschienenen *Opera omnia* sind mit einem Bildchen geschmückt, wo ein "neidischer" Hund gegen ein an einem Baum hängendes Zykloidenbildchen anbellt. Auch auf Johanns Konterfei, in noblen Gewändern sitzend, hält er stolz das Bild einer Zykloide "in die Kamera". Dieser Artikel will über sein Werk dieser Vorliebe für die Zykloide nachgehen.

collected works [2], whose printing ended in 1743 under the supervision of Marc-Michel Bousquet (1696–1762) in Lausanne[3].

The title page of Bernoulli's *Opera Omnia* bears a surprising vignette (see Fig. 1, right), where Johann's rivals are depicted as a dog barking at a mathematical picture, nailed to a tree out of its reach. This picture represents a cycloid with the words *Supra invidiam*, which can be translated by "beyond envy[4]". Also in his engraved portrait on the left page, where his self-confidence leaves no doubt, we see him holding a piece of paper again containing a drawing of a cycloid.

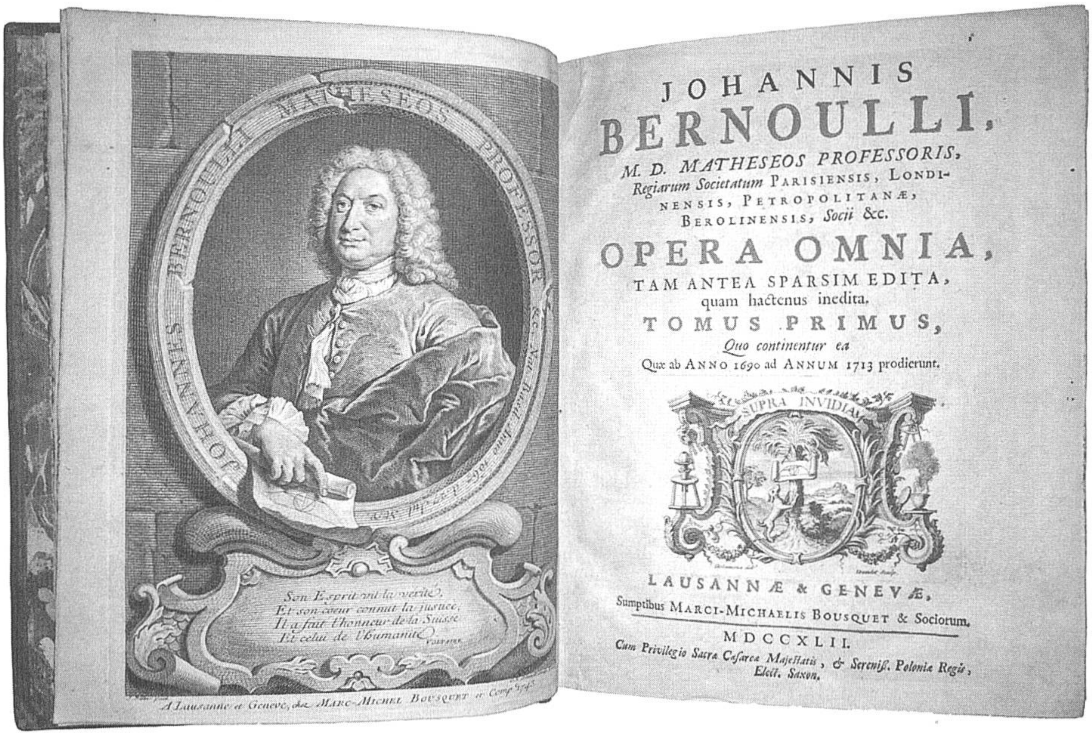


Figure 1 Frontispiece and title page of the first volume of Johann Bernoulli's *Opera Omnia* [2] (Bousquet, 1742, private collection).

Curiously, the same vignette reappears on the title page of Euler's masterpiece *Methodus inveniendi lineas curvas*, published in 1744 again by Bousquet (see Fig. 2, left). Bousquet had visited Berlin in March 1743 and brought the four volumes of Johann's *Opera Omnia* as a gift for the King of Prussia Frédéric II. On this occasion, Euler presented him with the recently completed manuscript of his *Methodus*[5]. Finally, as if this were not enough, Bousquet again used the same vignette, this time reversed, in his edition of the correspondence between Leibniz and Johann Bernoulli published in 1745 (see Fig. 2, right).

[3] About the history of the publication of this book, see [12] and [11].

[4] In a letter from June 22, 1718 to Johannes Scheuchzer, Johann Bernoulli declares explicitly that this was his motto. One can think that it corresponds to his position in all the disputes he had during his life, in particular with his oldest brother Jakob. These letters can be consulted on the web site of the University of Basel mentioned in note 1.

[5] According to Carathéodory in his introduction to volume XXIV of Euler's *Opera Omnia*.

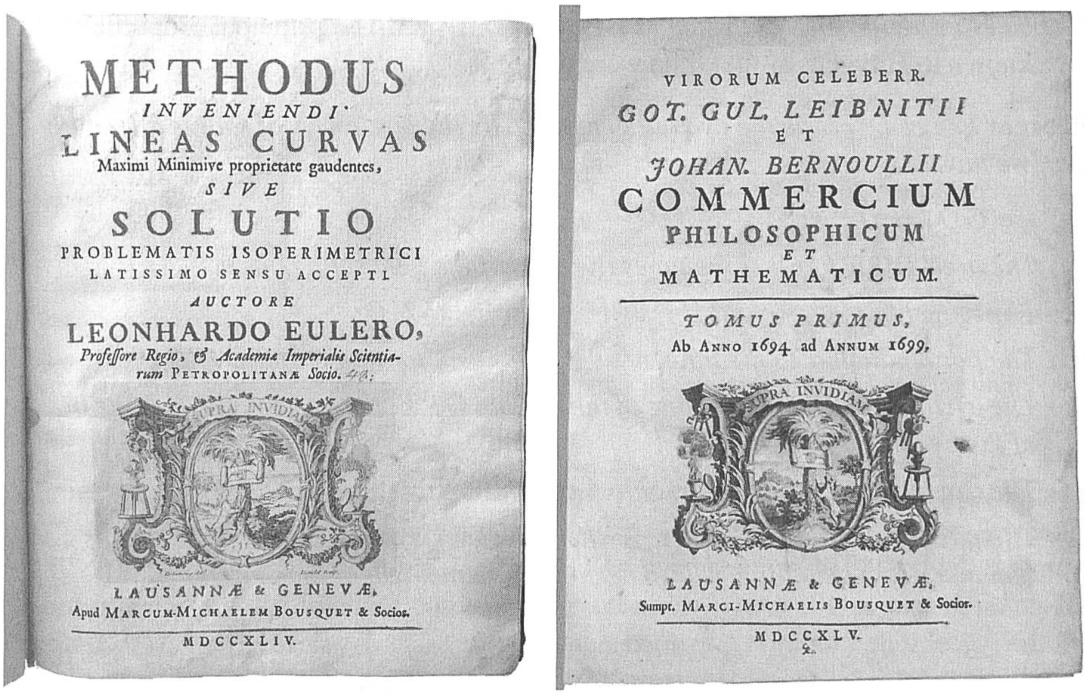


Figure 2 Title page of Euler's *Methodus* and title page of the Leibniz–Bernoulli Correspondence [9] (Bousquet, 1744 and 1745 respectively, private collection).

In this year 2017 marking the 350th anniversary of Johann Bernoulli's birth, we review some of his results on the cycloid and discuss the mathematical origin of the cycloid on the vignette.

2 New proofs of earlier results on the cycloid

> "Le P. Mersenne apprit à Descartes (...) la découverte de Roberval. (...) et c'est ici le commencement des querelles nombreuses que cette Hélène des géomètres causa parmi eux. [Father Mersenne taught Roberval's discovery to Descartes. (...) and this was the beginning of the many quarrels that this Helen of geometers caused among them.]"
>
> (Jean Étienne Montucla (1725–1799), [19, II, p. 55])

According to Evangelista Torricelli[6] (1608–1647) the cycloid was invented in 1599 by Galileo Galilei (1564–1642) as the curve generated by a point P of a generating circle GA which rolls on a straight line DE (see Fig. 3). For several decades, its geometric properties (areas, tangents, arc length, etc.) remained a challenge to the mathematicians of the 17th century (Roberval, Descartes, Fermat, Pascal, etc.). One of the first published great studies of this curve[7] is due to Christiaan Huygens (1629–1695) and is contained in

[6] Torricelli writes in his *Opera geometrica* (1644) that "this line was named cycloid by our predecessors, principally by Galileo 45 years ago" [Vocata est à prædecessoribus nostris. Præcipue à Galileo iam supra 45. annum, huiusmodi linea *adb*. Cyclois (...).], *De dimensione parabolæ, Appendix de dimensione cycloidis*, [24, p. 85].

[7] In 1659, John Wallis (1616–1703) publishes his *Tractatus Duo*, the first of which is devoted to the cycloid. Huygens writes that Wallis "tasche a toute force de maintenir l'honneur de sa nation" ["Strives with all his might to maintain the honor of his nation"] [17, III. p. 57].

his book *Horologium oscillatorium* [18], printed in 1673. The principal properties of this curve known to Huygens at that time were these:

Theorem 1. *Let DGE be the cycloid generated by the circle GA of radius a (see Fig. 3). Then, we have*

(a) $\mathrm{arc}(GO) = OP$;

(b) *The area $DGEAD$ is three times the area of the generating circle GA;*

(c) *The tangent to the cycloid HP at P is parallel to GO;*

(d) *The perpendicular to the tangent at P is tangent to the cycloid DFE;*

(e) *The cycloid FQE is the evolute of the cycloid GPE, the cycloid GPE is the involute of the cycloid FQE;*

(f) *The arc length GPE is is equal to $4a$;*

(g) *The pair of cycloids in Figure 3, when reversed, constitutes an isochronous pendulum, namely a pendulum whose period is independent of the amplitude.*

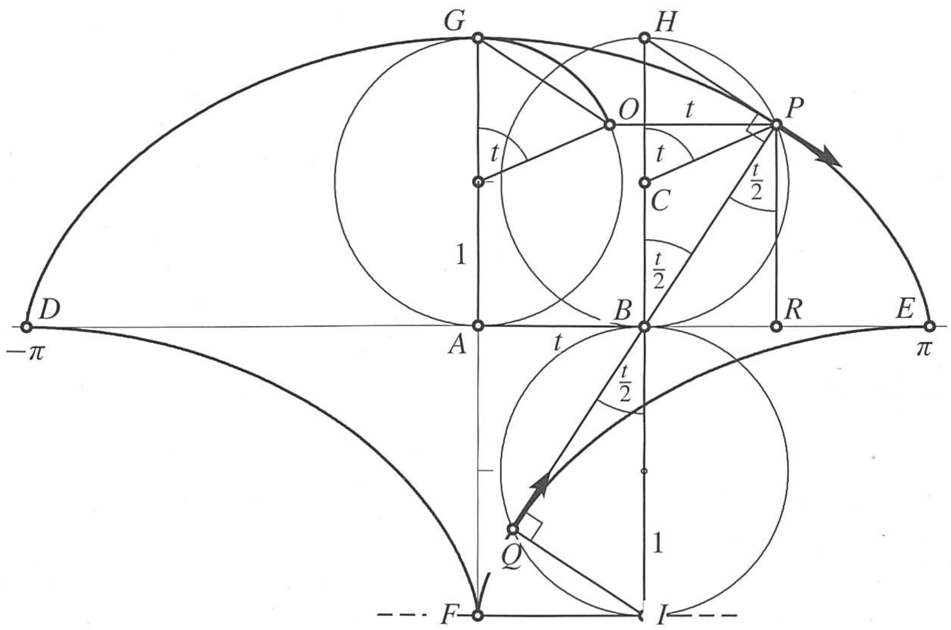


Figure 3 Principal properties of the cycloid ($a = 1$).

Proof. Huygens' proofs fill a large part of his book. We shall see below how Johann Bernoulli gained more and more insight and elegant proofs for these results. For the moment, we just indicate Huygens' proof of (a) and (b) from a manuscript written in summer 1658. For the proof of (a), observe that after the circle has rolled from A to B, it has rotated by the same amount, hence $t = AB = \mathrm{arc}(HP) = \mathrm{ang}(PCH)$. By parallelism, $AB = OP$ and $\mathrm{arc}(HP) = \mathrm{arc}(GO)$, hence (a) is true. For the proof of (b), Huygens decomposes the sickle-shaped region $BGMDANKEB$ (see Fig. 4) with "indivisibles". If our corresponding thin slices are chosen symmetrically, i.e., such that $FH = HL$, then their common length $EG+NM$ is always equal to $\mathrm{arc}(BG)+\mathrm{arc}(BM) = \mathrm{arc}(BD) = a\pi$.

Hence all the slices together fill the rectangle $ADHQ$, whose area is equal to the area of the circle. □

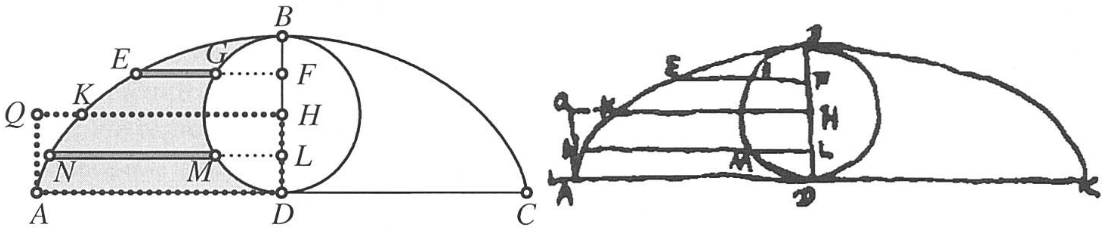


Figure 4 Huygens' proof of the area formula for the cycloid: a modern and an original drawing from 1658 [17, XIV, p. 348].

Johann's *Lectiones mathematicæ*. To investigate the properties of the cycloid, Johann has a new tool, the differential and integral calculus, and an eager student whom he met during his trip to Paris, the very noble Guillaume François Antoine de l'Hospital, marquis de Sainte-Mesme et du Montellier, comte d'Antremonts, seigneur d'Oucques et autres lieux (1661–1704). De l'Hospital was a "good geometer for the common geometry" but knew "nothing in differential calculus which he barely knows by name[8]". So Johann introduced him into this new calculus with much enthusiasm during the years 1691/1692 and continued their correspondence [6] until de l'Hospital's death. The second part of these lectures, on integral calculus, were later published in Johann's *Opera Omnia*[9].

Computation of slopes. In the *Lectio* XVII Johann evaluates quickly and in a masterly manner the differentials for the cycloid: he has already taught the marquis how to use them to determine the tangent to the cycloid[10]. From the two shaded similar triangles (see Fig. 5, left) we have by Thales' theorem[11]

$$a : z : y-a = ds : -dy : dz \quad \Rightarrow \quad \frac{ds}{dy} = -\frac{a}{z}, \quad \frac{dz}{dy} = \frac{a-y}{z}. \tag{1}$$

Hence, since $x = z + s$, we have

$$\frac{dx}{dy} = \frac{dz}{dy} + \frac{ds}{dy} = -\frac{y}{z} = -\frac{y}{\sqrt{2ay-y^2}} = -\sqrt{\frac{y}{2a-y}} \quad \text{or} \quad \frac{dy}{dx} = -\sqrt{\frac{2a-y}{y}}. \tag{2}$$

The formula $\frac{dy}{dx} = -\frac{z}{y}$ proves Theorem 1 (c). For what is for us today the second derivative, we obtain from the chain rule

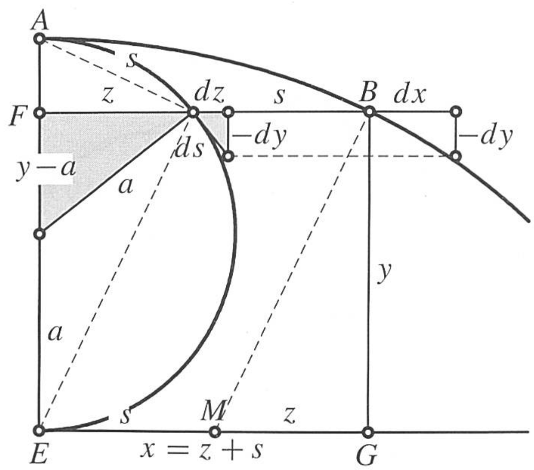
$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{\frac{2a-y}{y}}}\cdot\frac{-y-2a+y}{y^2}\cdot\left(-\sqrt{\frac{2a-y}{y}}\right) = \frac{-a}{y^2}. \tag{3}$$

[8] Letter of Johann Bernoulli to Pierre Rémond de Montmort, May 21, 1718.

[9] [2, III, pp. 385–558]. There is a German translation [4].

[10] [5, pp. 21–22].

[11] For a precise meaning of what we call "Thales' theorem" or "Eucl. VI.2" we refer to, e.g., [21].



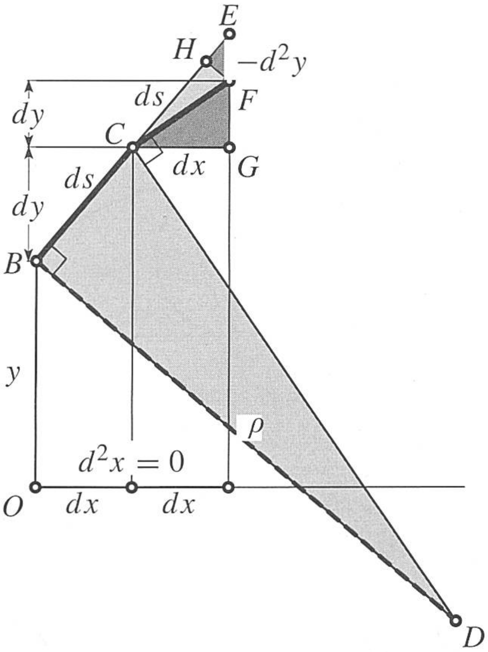
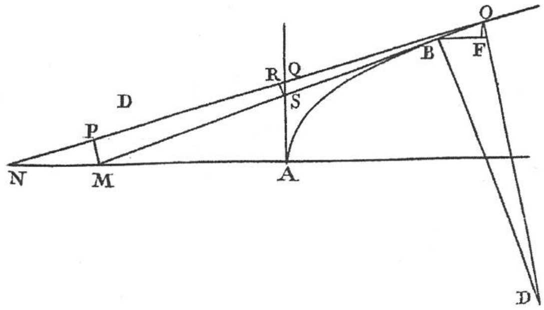


Figure 5 Slopes for the cycloid (left) and radius of curvature (right).



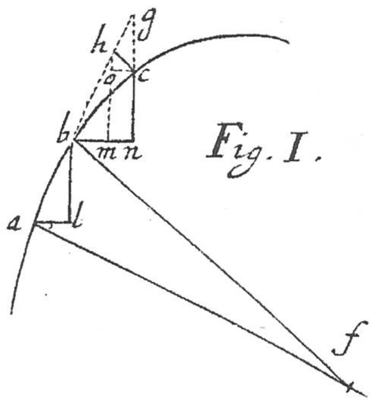


Figure 6 Drawings for the proof of (4) by Johann in [2, III, p. 438] (left) and by Jakob in [1, I, pp. 577–578] (right).

Radius of curvature. The treatment of curvature was discovered by both Bernoulli brothers around 1692 (see Fig. 6, left and right). Johann explained how to obtain the formula for the radius of curvature in the *Lectio* XVI [2, III, p. 437] and its application to the cycloid in the *Lectio* XVII [2, III, pp. 438–439]:

Theorem 2.

(a) *The radius of curvature for a curve* $y(x)$ *is given by*

$$\rho = \frac{(dx^2+dy^2)\sqrt{dx^2+dy^2}}{-dx\,d^2y} = \frac{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}. \tag{4}$$

(b) *For the cycloid in Figure* 5 (*left*) *the radius of curvature at* B *satisfies*

$$\rho = 2 \cdot BM\,. \tag{5}$$

Proof. (a) We explain the idea and simplify the proof of the two Bernoulli brothers by using a picture somewhat similar to that of Jakob rather than that of Johann (see Fig. 5, right). We represent the curve by the polygon BCF based on a grid of abcissa with equal distances dx. The idea[12] is to construct the center of curvature D as the intersection of two neighboring perpendiculars to the curve, i.e., to BC and CF. If the upper dy is smaller than the lower dy, we have a negative second difference d^2y, which creates curvature. If CHE is the straight extension of BC, there appear two pairs of similar triangles $CBD \sim FHC$ (light grey) and $CGE \sim FHE$ (dark grey), hence

$$\frac{\rho}{ds} = \frac{ds}{HF} \quad \text{and} \quad \frac{dx}{ds} = \frac{HF}{-d^2y}\,.$$

Multiplying the two equalities eliminates HF and gives (4) since $ds = \sqrt{dx^2 + dy^2}$.

(b) For the proof of (5) we have from (2) that

$$1 + \frac{dy^2}{dx^2} = 1 + \frac{2a - y}{y} = \frac{2a}{y}\,. \tag{6}$$

By inserting (6) and (3) into (4) we obtain

$$\rho = \sqrt{\frac{8a^3}{y^3}} \cdot \frac{y^2}{a} = \sqrt{8ay} = 2 \cdot \sqrt{2ay} = 2 \cdot \sqrt{y^2 + z^2} = 2 \cdot BM\,. \qquad \square$$

Rectification of curves. The next fundamental result about the cycloid uses the idea of the paragraph *De Rectificatione curvarum ope sua Evolutionis*[13] and is explained in the last couple of lines of the *Lectio* XIX [2, III, pp. 445–446]:

Theorem 3.

(a) *For the arc length of the cycloid* (*see Fig.* 7) *we have*

$$\ell = \text{arc } AC = 2 \cdot AD = 2 \cdot CG = 2u\,. \tag{7}$$

(b) *For the "dimensione spatii curvilinei* AGC*" we have*

$$\mathcal{V} = \text{spatium } AGC = \text{segm. } AD = \mathcal{U}\,. \tag{8}$$

[12] It appeared later that Newton used the same idea in his manuscript *Methodus Fluxionum*, written in 1671 but published tardily in 1736 [20, pp. 65–66].

[13] "On the rectification of curves by means of their involute".

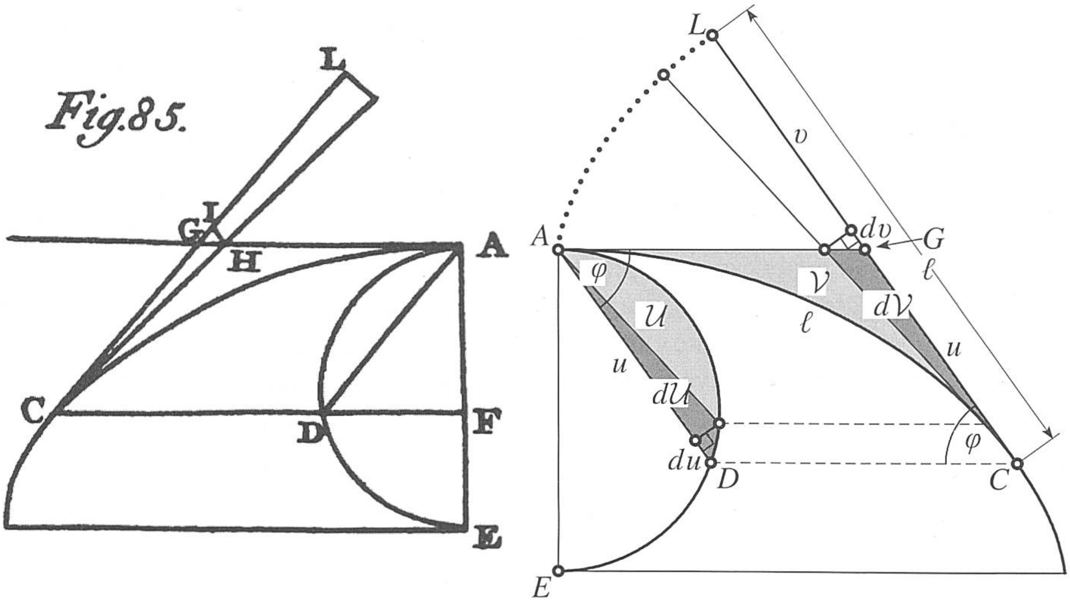


Figure 7 The “Fig. 85.” from *Lectio* XIX (left) and the proof of Theorem 3 (right).

Proof. (a) We extend the tangent CG until L by an unknown length v such that $u+v=\ell$ (Fig. 7, right). We consider the quantities $u, v, \mathcal{U}, \mathcal{V}$ as functions, for example of the angle φ, which is the same at A and C by Theorem 1 (c). If φ decreases by an infinitely small quantity $d\varphi$ (the same quantity at A and C), we obtain a pair of small right-angled triangles (one of these is GIH on the left picture), which have the same angles and one pair of equal legs. Hence the second pair of legs is also equal and so $du = dv$ “hujus integrale” is $u = v$, i.e., $\ell = 2u$. Johann writes this last conclusion, analytically, as follows: if $AF = x$ and a is the radius of the circle, then

$$ADF \sim HGI \Rightarrow GI = \frac{DF\left[=\sqrt{2ax-x^2}\right]\cdot HG\left[=ds=\frac{adx}{\sqrt{2ax-x^2}}\right]}{AD\left[=\sqrt{2ax}\right]} = \frac{a\,dx}{\sqrt{2ax}}$$

$$\Rightarrow GL = \sqrt{2ax} = AD = CG.$$

(b) Again, since the angles $d\varphi$ in both dark triangles are equal as are also the legs, we have $d\mathcal{U} = d\mathcal{V}$, which proves (b) after integration. □

The dotted curve described by L, when φ decreases, has $v+u=2v$ as radius of curvature and thus satisfies Theorem 2 (b) everywhere. Therefore it must again be a cycloid. This proves Theorem 1 (d), (e), (f) all together.

3 The caustic of the cycloid

> “Regula quam dedimus ad determinandas curvas Causticas non solum succedit in geometricis, sed etiam se ad mechanicas extendit. [The rule we have given for the determination of caustic curves was successful not only for geometric curves, but extends also to mechanical ones.]”
>
> (Johann Bernoulli, [2, III, p. 472])

A caustic is the envelope of families of light rays which have been reflected in a curve or surface. Particularly famous is the caustic of a circle (see Fig. 8, left). Publication about this curve began in 1682 by Ehrenfried Walther von Tschirnhaus (1651–1708) with a couple of incorrect mathematical statements[14]. Johann showed in a paper from 1691 "Per vulgarem Geometriam Cartesianam" that this curve has an equation of degree 6 (and not of degree 4 as Tschirnhaus had asserted) and gave in *Lectio* XXVI–XXVIII detailed properties culminating in the result that it is an epicycloid[15].

In case of the cycloid, there is a nice result that Jakob learns from his brother with "astonishment[16]":

Theorem 4. *The caustic of a cycloid is again a cycloid, composed of two branches half as large as the reflecting one (see Fig. 8, right).*

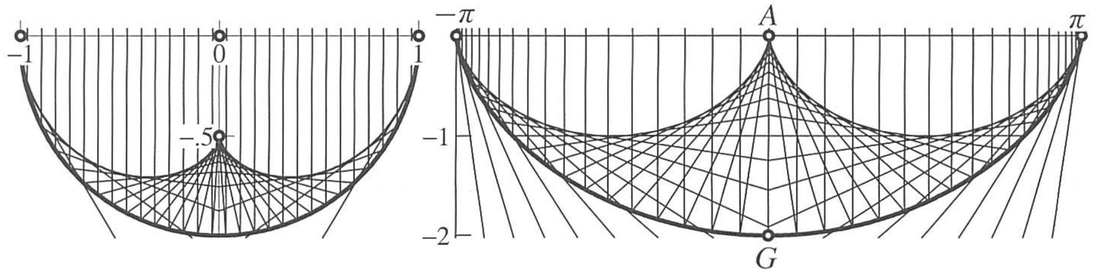


Figure 8 The caustic of a circle (left) and of a cycloid (right).

Proof. Consider a curve ABC determined by the coordinates $AG = x$ and $GB = y$ (see Fig. 9, left) reflecting a bundle of light rays arriving along GB and focussing in H. Around March 1692, Johann obtained for the distance BH the formula

$$BH = -\frac{dx^2 + dy^2}{2\,d^2y} = -\frac{1 + \frac{dy^2}{dx^2}}{2\frac{d^2y}{dx^2}} \tag{9}$$

and gives a long proof in *Lectio* XXVI [2, III, pp. 464–466]. In order to simplify the proof, we draw, inspired by the pictures in Figure 6, a pair of parallel rays moving up from G at distance dx and their reflections BH (reflected in BC) and CH (reflected in CE; see Fig. 10, left).

The second mirror CE is rotated by a small angle (in grey) due to $-d^2y$. Because reflected light rays rotate *twice* as fast as the mirror, this second ray CH has rotated *two* grey angles. We therefore use the angle bisector to divide triangle BHC into two parts and obtain the similar triangles BHN and DCE. Since in the infinitesimal limit $NB = \frac{CB}{2} = \frac{ds}{2}$ (a consequence of Eucl. VI.3), we have by Thales' Theorem

$$\frac{BH}{ds/2} = \frac{ds}{-d^2y}$$

[14] For details and their corrections by Huygens we refer to M. Mattmüller's commentaries in vol. 5 of *Werke von Jacob Bernoulli*, pp. 348–349.

[15] [2, I, pp. 52–59], [2, III, *Demonstratio* and Fig. 106, p. 470].

[16] "(...) omnia non sine stupore perlegere potui" [1, I, p. 503].

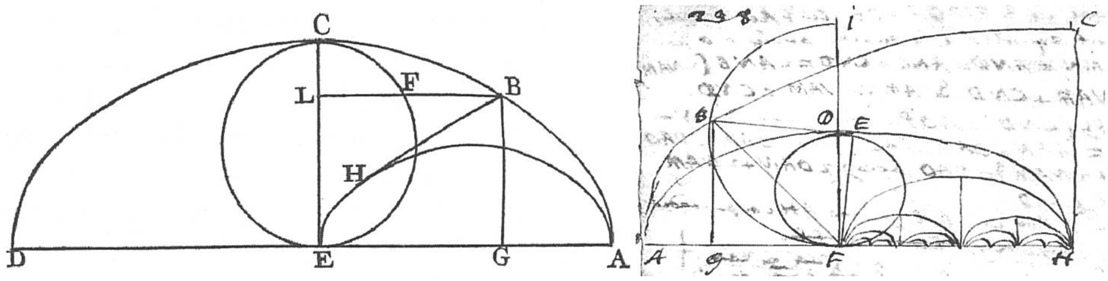


Figure 9 Johann's drawing from [2, III, p. 482] (left) and Jakob's original drawing from 1692 (*Meditationes* CXCII, Universitätsbibliothek, Basel, Mscr. L I a 3, f. 238, `http://www.e-manuscripta.ch`).

which, since $ds^2 = dx^2 + dy^2$, is formula (9).

For the cycloid, inserting (6) and (3), this formula simplifies to

$$BH = BG = y. \tag{10}$$

Johann concludes from this that the caustic is again a cycloid[17].

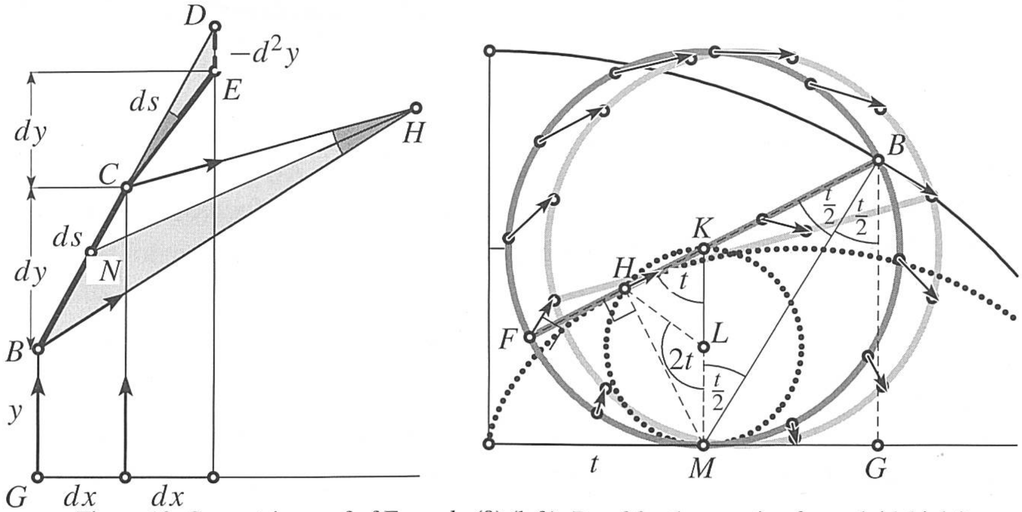


Figure 10 Geometric proof of Formula (9) (left); Proof for the caustic of a cycloid (right).

We can, however, avoid all these calculations by considering the rolling circle, which creates the cycloid, together with its diameter BF (see Fig. 10, right). From the equal angles $\frac{t}{2}$ left and right of BM, we see that this diameter coincides with the reflected ray BH. Neighboring rays appear when the circle rolls on, thereby rotating at every instant around the base point M, and intersect in that point where the velocity is *parallel* to FB, i.e., in the perpendicular projection H of M onto FB. The equality of triangles MHB and MGB then proves (10). Furthermore, H lies on the Thales circle with center L and radius $\frac{a}{2}$, rolling at the same horizontal speed, because the angle HLM is twice the angle FKM (by Eucl. III.20). Therefore H describes a small cycloid as stated. □

[17] [2, III, pp. 478–480]. Johann communicates this property in a (now lost) letter to his brother, dated March 15, 1692 [8, p. 144]. Jakob publishes this result the same year [1, I, pp. 506–507].

4 The brachystochrone problem

"Tous ceux qui sçavent au moins les Nouvelles des Sciences, ont entendu parler du celebre Problême de *la plus vîte Descente*. [All those who at least know the news of Science, have heard about the famous problem of descent in shortest time.]"

(Bernard Le Bouyer de Fontenelle (1657–1757), [16, I, p. 51])

At the end of an article published in the *Acta Eruditorum* of June 1696, Johann Bernoulli suggests the following problem:

"Datis in plano verticali duobus punctis A & B, assignare Mobili M viam AMB, per quam gravitate sua descendens, & moveri incipiens a puncto A, brevissimo tempore perveniat ad alterum punctum B. [Given two points A and B in a vertical plane, find the path AMB along which a moving point M, descending under gravitation and starting to move at A, arrives at the other point B in shortest time.]" [2, I, p. 161]

He adds that the curve is well known to geometers and fixes a time limit of six months for submitting a solution. Leibniz decided[18] that the limit should be extended until Easter of the following year, so that others, perhaps not yet experts in the new calculus, might try to find a solution. In the same spirit, Johann wrote to de l'Hospital on June 30:

"Je voudrois que quelques uns de vos Geometres qui se vantent de posseder de si excellentes methode de *maximis et minimis*, s'y attachassent, car voylà un exemple, qui leur donnera de la besoigne et peutetre plus que leur methode ne pourra faire. [I would like that some of your Geometers, who are so proud of their excellent methods de *maximis et minimis*, should also attack this problem, because this example will give them a lot of work, and perhaps more than their methods can achieve.]" [6, p. 321]

At the end of the deadline, the texts of Johann, Jakob, Leibniz, de l'Hospital, Tschirnhaus and Newton are published in the *Acta Eruditorum* of May 1697:

Theorem 5. *The brachystochrone curve is a cycloid.*

Johann's indirect solution. This solution ([2, I, pp. 187–193], see Fig. 11) was another of Johann's strokes of genius: he applies "une merveilleuse identité de nôtre courbe avec la courbure du rayon de lumière[19]". Namely, recalling the research of Fermat, Leibniz and Huygens, he uses the fact that a light ray, passing through two different media and obeying the Snellius–Descartes law of refraction

$$\frac{v_1}{\sin\alpha_1} = \frac{v_2}{\sin\alpha_2} \qquad (v_1, v_2 \text{ speeds of light}) \tag{11}$$

connects two given points A and B in the shortest possible time.

If now the light ray crosses *several* layers of material at varying speed (see Fig. 12 (a)), it would satisfy equation (11) all along the curve, hence

$$\frac{v}{\sin\alpha} = a \text{ (constant)} \quad \text{or} \quad v^2(dx^2+dy^2) = a^2\,dy^2 \quad \text{or} \quad dy = \frac{v}{\sqrt{a^2-v^2}}\,dx \tag{12}$$

[18] *Journal des Sçavans*, 19 November 1696, pp. 451–455.

[19] Letter of Johann Bernoulli to the Marquis de l'Hospital, March 30, 1697, [3, p. 347].

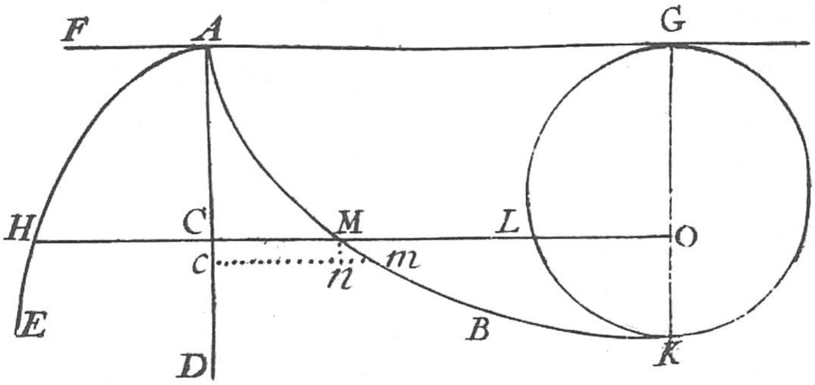


Figure 11 Johann's figure for the brachystochrone problem [2, I, p. 202].

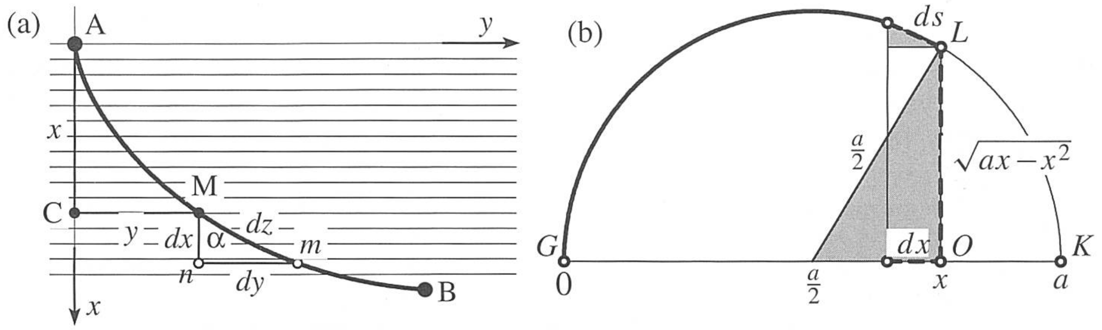


Figure 12 (a) The brachystochrone problem; (b) Integration of the differential equation (13).

(because $\sin\alpha = \frac{dy}{dz} = \frac{dy}{\sqrt{dx^2+dy^2}}$). The same should be true for our moving body, for which we know, since Galileo, that the velocity (represented by the curve AHE to the left of Fig. 11) forms a parabola as function of the altitude x, hence is proportional to $\sqrt{x}$. We normalize the constant of gravity to have $v = \sqrt{ax}$, so that (12) becomes

$$dy = \sqrt{\frac{x}{a-x}}\,dx.$$

This is precisely the differential equation (2) for a cycloid generated by a circle of radius $a/2$. However, Johann establishes the link with the cycloid as follows:

$$dy = \sqrt{\frac{x}{a-x}}\,dx = \frac{x\,dx}{\sqrt{ax-x^2}} = \frac{a\,dx}{2\sqrt{ax-x^2}} - \frac{(a-2x)dx}{2\sqrt{ax-x^2}}\,. \tag{13}$$

Here, the last operation cleverly produced the factor $a - 2x$, which for us is the inner derivative of the denominator. So the second term can easily be integrated (from A to C, hence from 0 to x) and gives $\sqrt{ax - x^2}$, which is the distance LO in Figures 11 and 12 (b) for the circle of radius $\frac{a}{2}$. The first term, from Figure 12 (b) and Thales' theorem, is the arc length ds of this circle, so that its integral becomes the arc GL. Hence integrating equation (13) between A and M of gives

$$CM = \textstyle\int dy = \mathrm{arc}(GL) - LO\,. \tag{14}$$

This relation holds for any circle of radius $\frac{a}{2}$ tangent to AG. If we place it so that $AG = CM + ML + LO = \text{arc}(GL) + \text{arc}(LK)$, then equation (14) leads to

$$ML = \text{arc}(LK)\,. \tag{15}$$

This is the characterization of Theorem 1 (a) for a cycloid.

Johann's solution "d'une maniere directe & extraordinaire". Johann had this "extraordinary" idea for his second proof already in 1697 but, following the advice of Leibniz, did not publish it for more than 20 years[20]. Two centuries later, Constantin Carathéodory (1873–1950), in an appendix to his thesis (Göttingen 1904 with H. Minkowski) wrote "diese höchst eleganten Betrachtungen [sind] die erste vollkommen befriedigende, strenge Lösung eines Variationsproblems[21]".

Proof. Imagine a horizontal line ALL' and a fixed narrow sector LDL' with fixed distance $LD = a$ and fixed inclination α (see Fig. 13, left). We ask: for which point C at unknown distance $CL = x$, does a body starting at A, and arriving at C with speed $v = \sqrt{2gx\sin\alpha}$ (g gravitational acceleration), cross this sector on an orbit CC' in shortest possible time? We imagine the angle $d\alpha$ to be infinitely small ("infiniment aigu"), so that we can take the orbit to be a small straight line making an angle $\beta + \frac{\pi}{2}$ with the line DL. The crossing time is thus

$$dt = ds \cdot \frac{1}{v} = \frac{(a+x)\,d\alpha}{\cos\beta} \cdot \frac{1}{\sqrt{2gx\sin\alpha}}\,. \tag{16}$$

We see that β must be zero, i.e., the crossing is perpendicular to DL and that, neglecting constants, we have to minimize

$$\frac{a+x}{\sqrt{x}} = \frac{a}{\sqrt{x}} + \sqrt{x} \quad \text{which leads to} \quad x = a.$$

Our intuition tells us that for $d\alpha$ tending to zero, the intersection D of two neighboring perpendiculars is the center of curvature. The condition $x = a$ means that the base line AL divides the radius of curvature in the middle, hence the solution curve should be a cycloid by (5).

Synthetic solution. We draw the entire fan of perpendiculars of a cycloid (Fig. 13, right), indicating the crossing time (16) for each sector by shades of grey (white = fast, dark = slow). We clearly see that any curve other than the cycloid enters somewhere into slower regions and has somewhere angles β different from zero, hence needs more time for the entire trajectory. □

Final solution. In order to complete his solution for fixed given points A and B, Johann draws an arbitrary cycloid AS (see Fig. 14, left) and uses the fact that all cycloids starting at A are similar. Hence, the intersection point R of the line AB with this cycloid determines

[20] [2, I, pp. 197–198], [2, II, pp. 266–269 (1718)]. Johann writes in 1718: "L'incomparable MR. LEIBNIZ (...) trouva cette méthode directe d'une beauté si singulière, qu'il me conseilla de ne la pas publier (...)."
[21] [10, pp. 69–70].

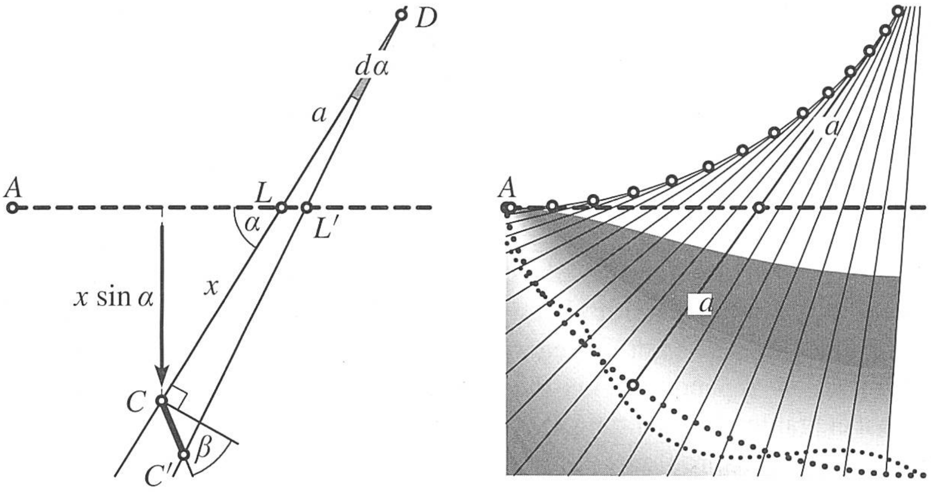


Figure 13 Johann's "extraordinary" solution of the brachystochrone problem.

the similarity factor AB/AR. An anonymous author from England had submitted, without further explanation, the same drawing (see Fig. 14, right) and had been designated by "from the lion's claw[22]".

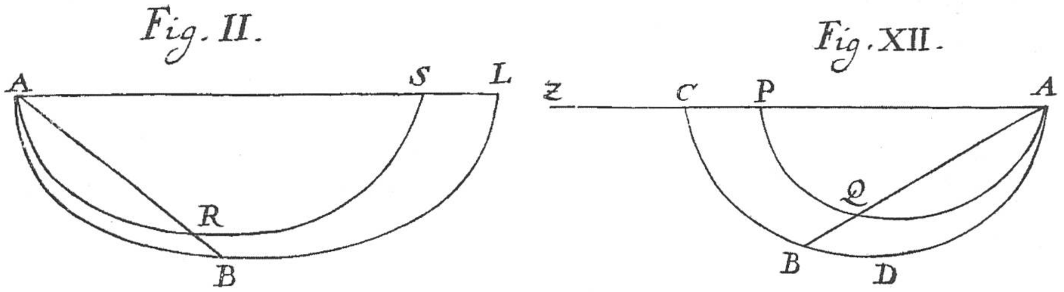


Figure 14 Johann's final solution of the brachystochrone problem (left) and Newton's solution (right) in the *Acta Eruditorum* (May 1697).

5 The isochronous pendulum

"Quod si vero *Hugeniana*, licet legitima, sed ob multarum propositionum farraginem & perplexitatem non arrisit; laudo propositum succinctiorem tradendi, modo tradidisset genuinam. [But Huygens' proof, although correct, did not please him [Philippe de La Hire] because of the jumble of several propositions; I agree to give it more succinctly, but it should appear accurate.]" (Johann Bernoulli, [2, I, p. 248])

Before Huygens, time measurements were very rudimentary and the precision of pendulum clocks suffered from the fact that the period of oscillation increased with increasing amplitudes. A spectacular discovery of Huygens'[23], useful for the invention of accurate pendulum clocks, was item (g) in Theorem 1 above, namely that the period of a pendulum moving on a reversed cycloid is independent of the amplitude.

[22] "*ex ungue leonem*" [2, I, p. 196].

[23] [18, Prop. XXV, p. 57].

After reading an erroneous proof of this theorem by Philippe de La Hire (1640–1718), Johann Bernoulli published a very short proof in the *Acta Eruditorum* of June 1698 [2, I, pp. 248–249].

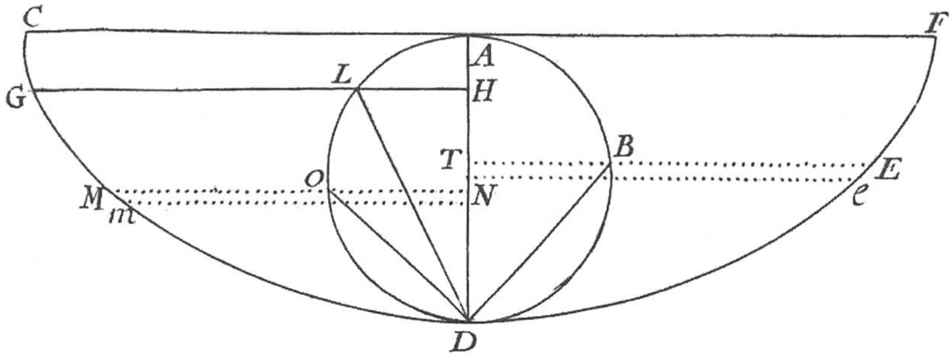


Figure 15 Johann's drawing for the isochronous oscillations of the cycloidal pendulum [2, I, p. 254].

Proof. Suppose that two bodies start sliding simultaneously, one at G, the other at F (see Fig. 15). We divide the arcs CD and FD in the same number of equidistant infinitely small parts, a pair of these being Mm corresponding to Ee. We denote the ratio

$$\frac{\operatorname{arc}(GD)}{\operatorname{arc}(FD)} = q \qquad \text{so that also} \qquad Mm = q \cdot Ee\,, \quad \operatorname{arc}(MD) = q \cdot \operatorname{arc}(ED)\,. \tag{17}$$

Therefore, "per naturam Cycloidis" (see Theorem 3 (a)), we also have

$$LD = q \cdot AD\,, \quad OD = q \cdot BD \quad \text{so that} \quad LD^2 = q^2 \cdot AD^2\,, \quad OD^2 = q^2 \cdot BD^2\,. \tag{18}$$

If a is the diameter of the circle $ABDOL$, then

$$AD = \frac{AD^2}{a}\,, \qquad HD = \frac{LD^2}{a}\,, \qquad TD = \frac{BD^2}{a}\,, \qquad ND = \frac{OD^2}{a} \tag{19}$$

by Thales' Theorem. Thus we have by (19) and (18)

$$HN = HD - ND = \frac{LD^2 - OD^2}{a} = q^2 \cdot \frac{AD^2 - BD^2}{a} = q^2\,(AD - TD) = q^2 \cdot AT\,.$$

Finally, "per naturam gravium descendentium" the velocities of the bodies at the positions M and E are proportional to $\sqrt{HN}$ and $\sqrt{AT}$ respectively, therefore

$$\text{velocity for } Mm \;=\; q \cdot \text{velocity for } Ee.$$

Since $Mm = q \cdot Ee$ (see (17)), this equality shows that the two bodies take precisely the same time to travel along the two intervals. Since this happens everywhere, the "descensus per DF & DG sunt isochroni". □

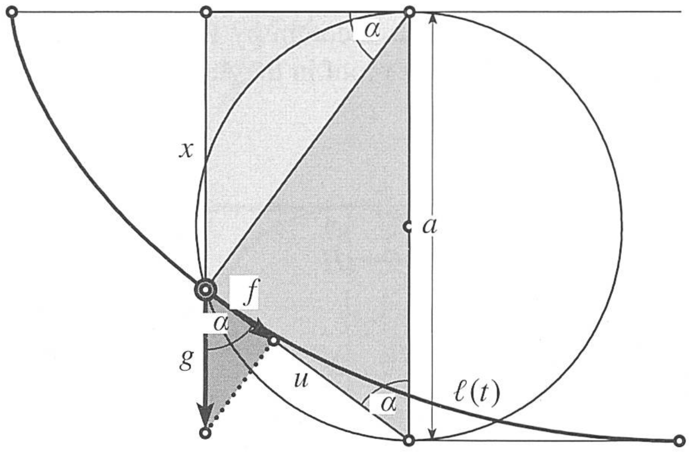


Figure 16 Short proofs for the brachystochrone and the isochronous property.

Short proofs. New insight arose from Euler's work (in particular [15, E112] from 1747 and [13, E62] from 1743). For a mass point (of mass $m=1$, see Fig. 16) sliding on a curve with distance $\ell(t)$ from the base point, accelerated by gravitation g, we obtain a force f in direction of the curve for which we have, using two similar triangles,

$$\frac{f}{g}=\frac{u}{a} \qquad \text{hence} \qquad f=\frac{g}{2a}\cdot\ell \qquad (\text{because } \ell=2u\,;\ \text{Theorem 3 (a)}).$$

Therefore the movement is governed by[24]

$$\ddot{\ell}+\frac{g}{2a}\cdot\ell=0 \quad\Rightarrow\quad \ell=A\cdot\cos\sqrt{\frac{g}{2a}}\cdot t \qquad (\text{for } \ell(0)=A,\ \dot{\ell}(0)=0).$$

Because cos reaches zero for $\sqrt{\frac{g}{2a}}\cdot t=\frac{\pi}{2}$, the time of descent to the base point is $\frac{\pi}{2}\cdot\sqrt{\frac{2a}{g}}$, independent of the amplitude A.

From another pair of similar triangles (see again Fig. 16), we find that

$$x=a\cdot\sin^2\alpha, \quad \text{i.e.,} \quad \frac{\sqrt{ax}}{\sin\alpha}=a,$$

which, together with Galileo's law for the velocity, is the condition (12) for the brachystochrone.

Thus the three similar triangles of Figure 16 prove both famous properties of the cycloid simultaneously.

6 Squarable areas bounded by the cycloid

"(...) toutte la facilité qu'il [Jakob] pretend faire voir en cela, ne sert qu'à relever vostre solution, & à faire admirer davantage que dans une courbe aussi examinée que celle-là, on ne luy en eust

[24] "While physicists call these "Newton's equations", they occur nowhere in the work of Newton or of anyone else prior to 1747." (C. Truesdell, *Essays in the History of Mechanics*, 1968).

cru que deux quarrables avant vous. [(...) all the ease he claims to display in this, serves only to enhance your solution and make us admire more the fact that before you we believed that this curve, the object of so much study, could have only two squarable areas.]"

(Pierre Varignon (1654–1722) to Joh. Bernoulli, April 5, 1700, [7, p. 236])

In the introduction to a memoir published in two versions (Latin and French) in 1699[25], Johann begins by writing that this year is, as we have already mentioned, the centenary of Galileo's invention of this curve according to Torricelli. We know that Galileo tried without success to determine the area under the cycloid and that around 1637 several geometers had found this area to be three times that of the generating circle. Later, a challenge by a certain Amos Dettonville (who was in fact Blaise Pascal, 1623–1662) published in June 1658[26] again encouraged research about this curve. A month later, Huygens found a new proof for the area formula (see Fig. 4) and, by comparing surfaces of spheres and cylinders in space, found a segment of the cycloid whose area does not involve the quadrature of the circle ([17, II, pp. 348–351], see Fig. 17, left). This result is the outcome of his attempt to solve Pascal's first problem: given any point Z on the cycloid, determine the dimension of the surface CZY [22, II, p. 319]. Huygens writes about Pascal's problems:

"Ils me semblent si difficiles pour la pluspart que je doubte fort si celuy mesme qui les a proposez les pourroit tous resoudre, et voudrois bien qu'il nous en eust assuré dans ce mesme imprimé. Autrement il est fort aisé d'inventer des problemes impossibles (...). [These problems seem to me so difficult, that I have strong doubts that the proposer himself was able to solve them all, and I am sorry that he did not inform us about this. Otherwise it is easy to invent impossible problems (...).]"

(Huygens to Ismaël Boulliau, July 25, 1658, [17, II, pp. 200–201])

Huygens' result was then mentioned by Pascal in his *Histoire de la roulette*[27] by saying that the "Dutchman" Huygens had discovered it, but also the "Englishman" Wren at nearly the same time. Huygens published it eventually (without proof) in his *Horologium oscillatorium* [18, p. 69] in 1673. The next year, Leibniz presented him with his *Quadrature arithmétique du cercle*, where (again without proof) another squarable segment of the cycloid is mentioned (see Fig. 17, right): the area of the segment AG of the cycloid is equal to the area of the triangle AFB. During the following twenty-five years no other such quadratures were found and people thought that such discoveries were impossible[28].

After having found an infinity of such squarable segments or sectors of the cycloid, Johann sent his article to the *Académie des sciences* with the words:

[25] Cycloidis primariæ Segmenta innumera Quadraturam recipientia; aliorumque ejusdem spatiorum quadrabilium determinatio: post varias illius fortunas nunc primum detecta a Joh. Bernoullio (Quadrature of innumerable segments of the ordinary cycloid and determination of other squarable areas now discovered for the first time after varied attempts by Joh. Bernoulli), *Acta Eruditorum* (*A. E.*), July 1699, pp. 316–320, [2, I, pp. 322–327] or [8, pp. 393–399]. The French version is presented to the Académie des sciences in Paris on July 11, 1699 [3]. Jakob and Johann were elected foreign associates of the Academy of sciences on February 14, 1699.

[26] It is reported that Pascal was searching a terribly difficult geometric problem in order to divert his spirit from painful tooth akes [22, II, p. 1254].

[27] "(...) M. Huygens, Hollandais, qui a le premier produit que la portion de la roulette retranchée par l'ordonnée à l'axe, menée du premier quart de l'axe du côté du sommet, est égale à un espace rectiligne donné. Et j'ai trouvé la même chose dans une lettre de M. Wren, Anglais, écrite presque en même temps." [22, II, p. 353]

[28] Johann Bernoulli attributes such an opinion to Tschirnhaus [2, I, p. 326] or [3, p. 135]. It seems that it is in September 1696 that he denies for the first time this assertion [9, I, p. 202].

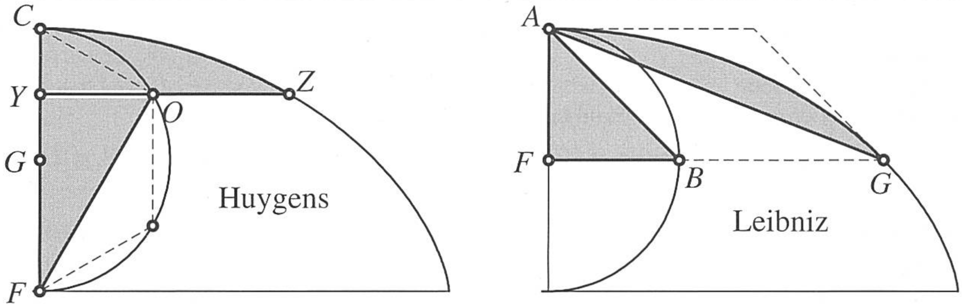


Figure 17 Left: if $CY = \frac{1}{4}CF$, then the area CZY equals the triangle FOY (Huygens). Right: if F is the centre of the generating circle, then the segment AG equals the triangle ABF (Leibniz).

> "(...) comme selon toutes les apparences, ce sera la derniere observation qu'on aura faite dans ce siecle au sujet de nôtre cycloïde, il est juste qu'après une durée de cent ans, qu'elle a continuellement exercé les Mathématiciens de toute l'Europe, elle retourne maintenant porter ce dernier éclat en France où elle a pris son premier lustre. [Because by all appearances, this will be the last observation made in this century about our cycloid, it is fair that after a period of a hundred years in which it has continuously exercised Mathematicians from all over Europe, it now returns to France, where it began to gleam, to shine with this new result.]" [3, p. 135]

We start the presentation of Johann's results with the following lemma, which he claims to be "une propriété de la cycloïde déjà connuë[29]":

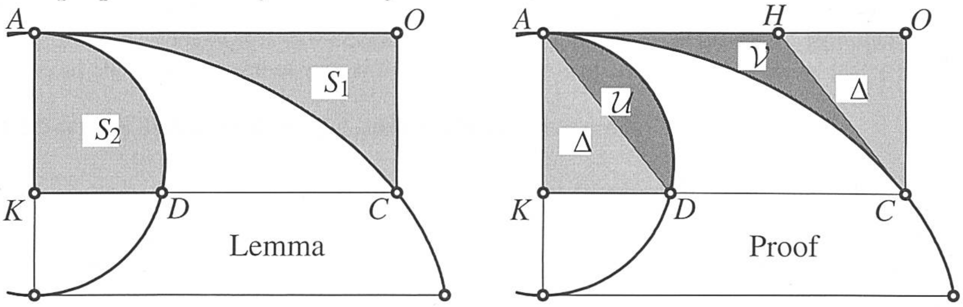


Figure 18 Figures for Lemma 1.

Lemma 1. *The surfaces S_1 and S_2 in Figure 18 (left) have the same area.*

Proof. This is in fact Theorem 3 (b), if the identical triangles AKD and COH are attached to $\mathcal{U}$ and $\mathcal{V}$ respectively (see Fig. 18, right). □

Here are the three new results contained in Johann's paper:

Theorem 6. *If $AK = IH$ (where H is the center of the generating circle) and points B and D lie on opposite branches of the cycloid (see Fig. 19 (a)), then*

$$\textit{area of segment } BCDB = \textit{sum of areas of triangles } LFI \textit{ and } MFK.$$

Theorem 7. *If $AK = IH$ and points B and D lie on the same branch of the cycloid (see Fig. 19 (b)), then*

$$\textit{area of segment } BCDB = \textit{difference of areas of triangles } LFI \textit{ and } MFK.$$

[29] [3, p. 137]. Newton also gave a demonstration of this lemma [20, p. 91].

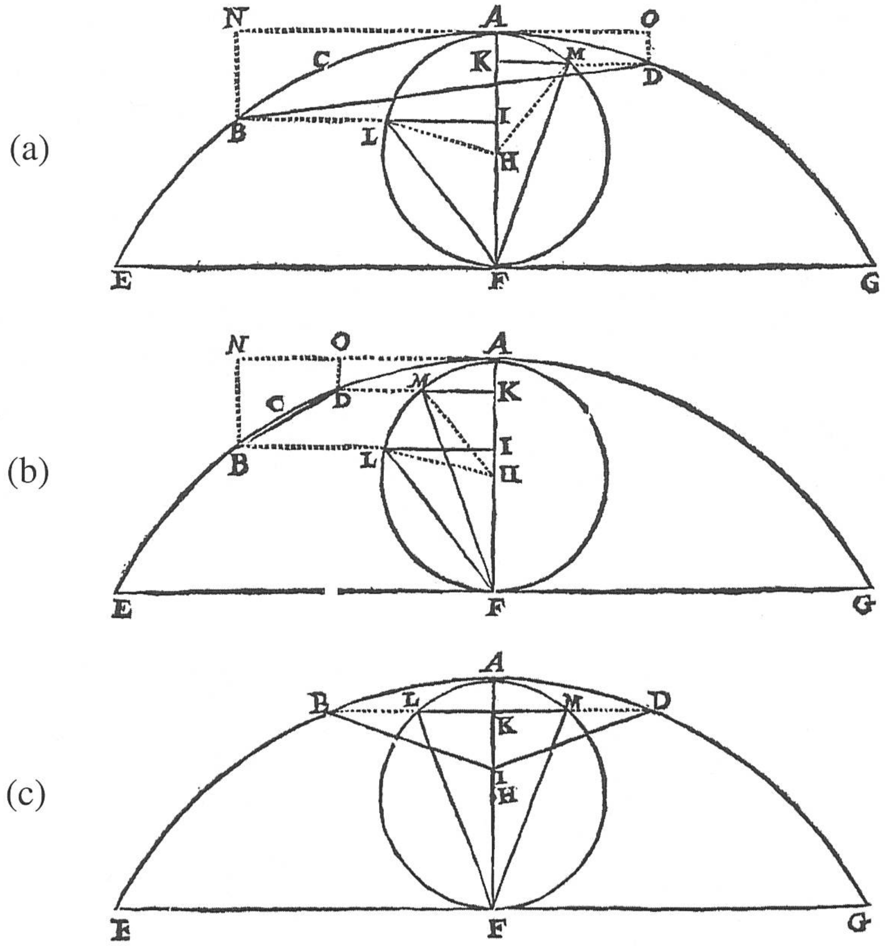


Figure 19 Johann's three theorems about squarable cycloidal areas (the figures are taken from [3]).

Theorem 8. *If* $AK = IH$ *and points* B *and* D *lie symmetrically with respect to* A (*see Fig.* 19 (c)), *then*

$$\textit{area of sector } IBADI = \textit{area of isosceles triangle } LFM.$$

Proof of Theorem 6. Because of the hypothesis $AK = IH$ we have $\frac{NB+OD}{2} = \frac{a}{2}$, where a is the radius of the circle AF. Thus the area of the trapezium $BDON$ is $NO \cdot \frac{a}{2}$. To make the proof as clear as possible, we surmount Johann's picture with the triangle $NOZ = T_1 + T_2 + T_2' + T_1'$ with $ZA = a$ (see Fig. 20) so that we have

$$\text{area trapezium } BDON = \text{area triangle } NOZ = T_1 + T_2 + T_2' + T_1'. \tag{20}$$

We now insert

$$T_1 = S_2 + T_3\,, \quad T_1' = S_2' + T_3'\,, \quad T_2 = T_4\,, \quad T_2' = T_4' \tag{21}$$

(the first two equalities follow from Theorem 1 (a) and the last two from the fact that we have two pairs of triangles with same base and same altitude) in (20) and we obtain

$$\text{Trap.}_{BDON} = S_2 + T_3 + T_4 + T_4' + S_2' + T_3'.$$

We then replace S_2 by S_1 and S_2' by S_1' by using Lemma 1 and have

$$\text{Segm.}_{BADB} = \text{Trap.}_{BDON} - S_1 - S_1' = T_4 + T_3 + T_4' + T_3'. \qquad \square$$

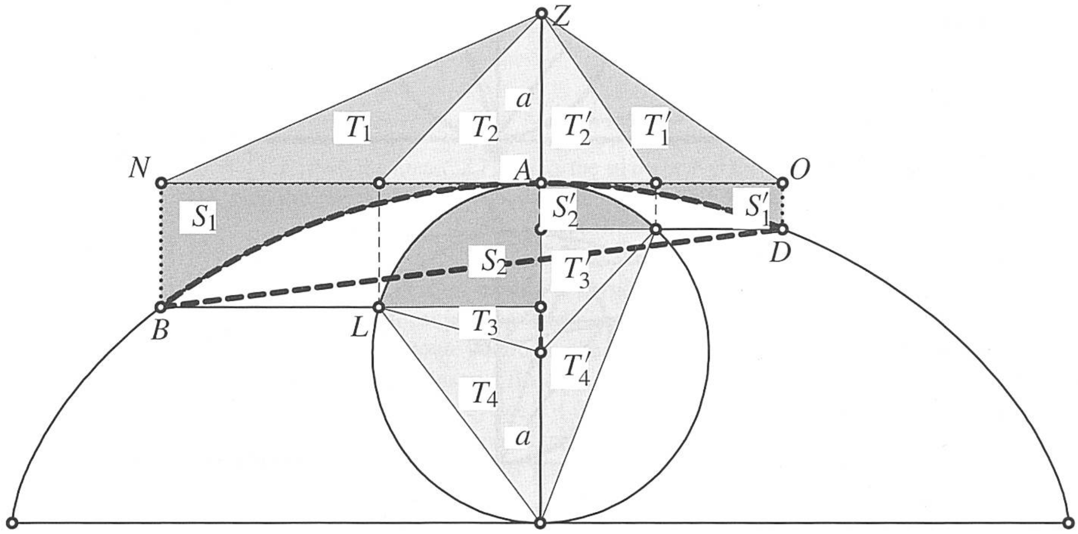


Figure 20 Proof of Johann's first result on squarable segments of the cycloid.

Proof of Theorem 7. Here also $\frac{NB+OD}{2} = \frac{a}{2}$ but now $NO = NA - AO$ and thus

$$\text{Trap.}_{BDON} = T_1 + T_2 - T_2' - T_1'.$$

Therefore, we obtain by using (21) and $S_2 = S_1$, $S_2' = S_1'$ as above

$$\text{Segm.}_{BDB} = \text{Trap.}_{BDON} - S_1 + S_1' = T_4 + T_3 - T_4' - T_3'. \qquad \square$$

Proof of Theorem 8. We move point I horizontally to a point b on the cycloid, so that the areas of the triangles IBD and bBD are the same, hence also the areas of the gray sectors $IBAD$ and $bBAD$ (see Fig. 21). The latter is the difference of the segment of Theorem 6 and that of Theorem 7 and therefore its area is equal to $2(T_4' + T_3')$. $\square$

Theorem 6 for $K = I$ is the result of Huygens and for $I = A$ (and therefore $K = H$) that of Leibniz.

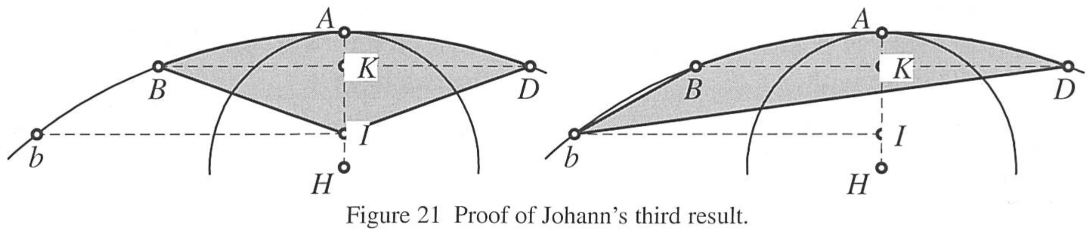


Figure 21 Proof of Johann's third result.

At the end of the article he sent to Paris, Johann Bernoulli states that, whenever the "démonstration synthetique" of his general results "aura eu le bonheur de plaire à l'Académie", he would also forward his analytic calculations, which were at the origin of his discovery. But he did not carry out this project.

7 Epilogue

> "J'espere, Monsieur, que (...) vous ne communiquerez rien à mon frere de tout ce que je viens de vous montrer, car cela luy donneroit une ouverture à la solution du probleme (...). [I hope, sir, that (...) you will communicate nothing to my brother of all that I have just shown you, for that would open his way to the solution of the problem (...).]"
>
> (Johann Bernoulli to de l'Hospital, January 22, 1701, [6, p. 373])

Which of the many theorems presented by Johann on the cycloid is the one for which he seems particularly proud and on which he decided to base his reputation in an allegorical way? Is this yet another result[30]? Let us take a closer look at the pictures of Figure 1 (see Fig. 22, left). This same drawing also appears on another portrait (see Fig. 22, right) and on an oil painting version of it preserved at the University of Basel.

We recognize clearly the picture for Theorem 6 (see Fig. 19 (a)) in his paper of 1699 but with a small mistake due to the engraver[31]. Precisely hundred years after the discovery of the cycloid by Galileo, after a century of efforts by the most eminent mathematicians, Johann was able to generalize two results of two of them and to achieve a nice discovery in a long standing tradition.

At the end of the Latin version of the same paper, Johann states without proof that if $HA = a$, $HK = x = \frac{a}{8} + \frac{a\sqrt{41}}{8}$, $\mathrm{arc}(ML) = \mathrm{arc}(AM)$ (notations of Fig. 23), then

$$\textit{zona cycloidalis } IKDB = \text{triangles } HAL + IAL - HAM - KAM\,. \tag{22}$$

Jakob Bernoulli's reaction. His brother's discovery was apparently a bitter pill for Jakob[32]. Two months later he publishes an article[33] in which he does not give a word of mention to Johann's result, and starts by writing:

> "Omnia, quæ circa Quadraturas spatiorum cycloidalium inveniri possunt, una Cycloidis proprietate dudum detecta nituntur, & ex ea tam aperte fluunt, ut Viri celeberrimi Hugenius & Leibnitius, qui duo ejus segmenta quadrarunt, non potuissent non pari facilitate cætera omnia segmenta & sectores quadrabiles reperire, si animum intendere voluissent. [Everything that can be found concerning the quadrature of cycloidal areas depends on a newly discovered property of the cycloid from which this follows so easily that the very famous Huygens and Leibniz, who both obtained a squarable segment, could not have found with such ease other squarable segments or sectors if they had wanted to do.]" [1, II, p. 871]

This "newly discovered property" of the cycloid is Lemma 1 above. It allows Jakob to calculate, with the notations of the Figure 23 and some elementary geometry, the areas

$$\begin{aligned}
\text{Zona } AKD = \text{Zona } AKDO - \text{Zona } AKM \quad &= (1-x)(p+s) - \frac{1}{2}s + \frac{1}{2}px \\
\text{Zona } AMD = \text{Zona } AKDO - 2\cdot\text{Zona } AKM &= (1-x)(p+s) - s + px\,.
\end{aligned}$$

[30] On iterated involutes of the circle, see Ph. Henry, G. Wanner, *Jost Bürgi, Johann Bernoulli and the Euler Numbers*, in preparation.

[31] The point M on Figure 19 (a) is not the intersection of the circle with BD!

[32] Concerning the relationship between the two brothers, see [23].

[33] *A. E.*, September 1699, pp. 427–428, [1, II, pp. 871–873], also in [2, I, pp. 328–329] or [8, pp. 400–403].

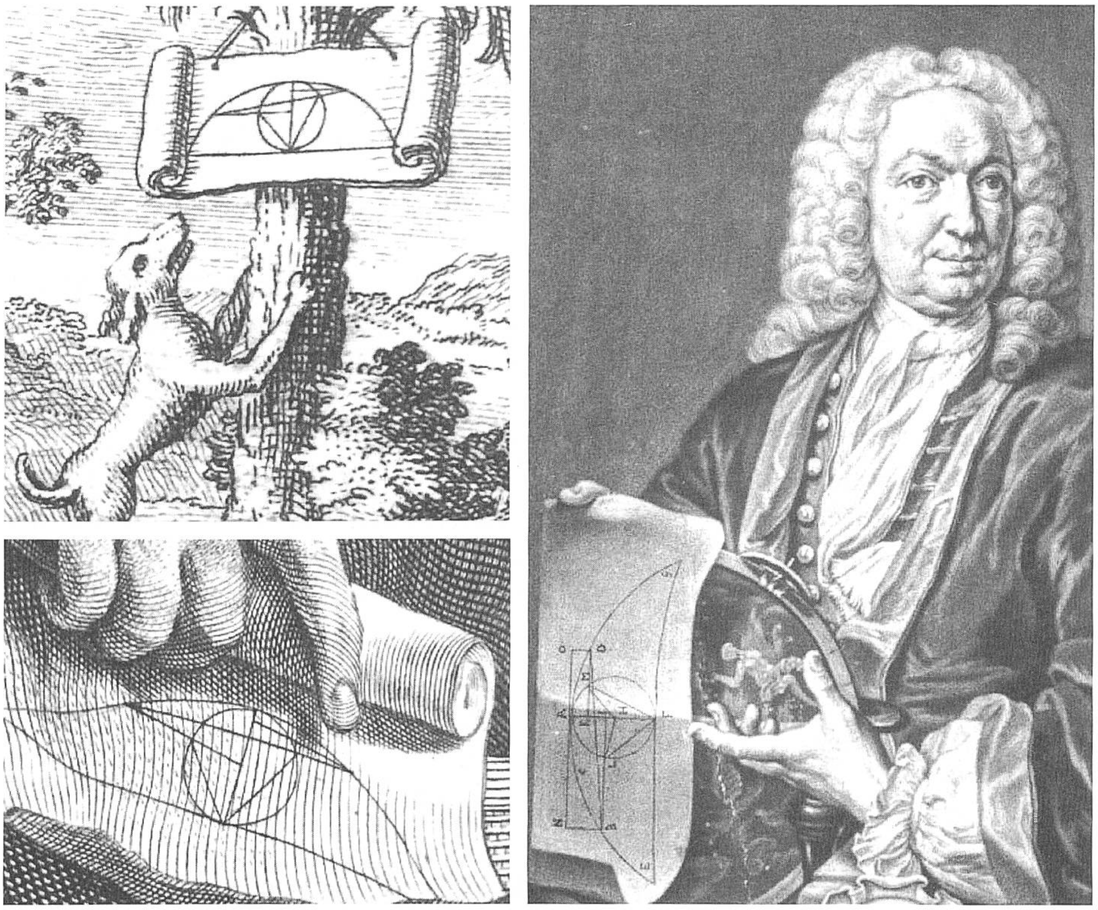


Figure 22 Left: details of the frontispiece and the vignette. Right: Johann's portrait in the Matriculation Register of the Rectorate of the University of Basel (Universitätsbibliothek, Basel, AN II 4a, f. 187v, `http://www.e-codices.unifr.ch`).

In the same way, he finds the areas of the analogous "zona"

$$\begin{aligned}\text{Zona } AIB &= \text{Zona } AIBN - \text{Zona } AIL &&= (1-z)(q+t) - \frac{1}{2}t + \frac{1}{2}qz\\ \text{Zona } ALB &= \text{Zona } AIBN - 2\cdot\text{Zona } AIL &&= (1-z)(q+t) - t + qz\,.\end{aligned}$$

By subtracting, we obtain[34]

$$\begin{aligned}\text{Zona } IKDB \quad &= q - \tfrac{1}{2}qz - p + \tfrac{1}{2}px + \Big[t\left(\tfrac{1}{2}-z\right) - s\left(\tfrac{1}{2}-x\right)\Big]\\ \text{Zona } LMDB = 2\cdot\text{Zona } ADB = \quad & \qquad q - p \qquad + \qquad [sx - tz]\,.\end{aligned} \tag{23}$$

The first terms of the expressions in (23) are "purely rectilinear[35]" because they are sums

[34]The relation Zona $LMDB = 2\cdot$Zona ADB was stated by Jakob without further comment and allowed him to declare ADB to be a "sector quadrabilis" of the cycloid. By Lemma 1, we have Zona $AB = ANB - [ANBD = AIL] = \frac{q}{2} - \frac{tz}{2}$ and similarly Zona $AD = \frac{p}{2} - \frac{sx}{2}$. Therefore, we obtain Zona $ADB =$ Zona $AB -$Zona $AD = \frac{1}{2}\cdot$ Zona $LMDB$. It seems that this was known to Leibniz as a result of his "méthode de la métamorphose", communicated in some letters but never published.

[35][1, II, p. 872].

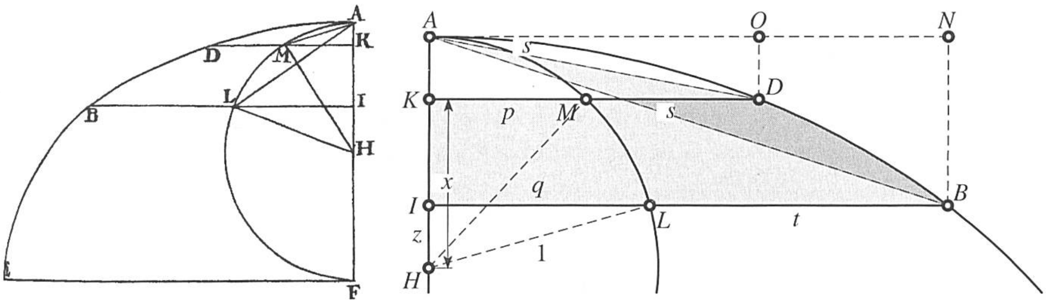


Figure 23 Johann's figure (*Acta Eruditorum*, 1699) and Jakob's notations.

and differences of four (or two) triangles. Therefore the terms in brackets, containing arc lengths (depending on π), should vanish. If we manage to find $HI = z$ and $HK = x$ such that

$$t\left(\tfrac{1}{2} - z\right) - s\left(\tfrac{1}{2} - x\right) = 0 \text{ in the first case and } sx - tz = 0 \text{ in the second,} \tag{24}$$

we would have found new squarable regions of the cycloid.

Jakob's idea is to assume that $t = ns$ for $n = 2, 3, 4\ldots$ and thus (24) becomes

$$z = \frac{n-1}{2n} + \frac{x}{n} \text{ in the first case and } z = \frac{x}{n} \text{ in the second.} \tag{25}$$

Since for $n = 2, 3, 4, \ldots$ we have the equations

$$z = 2x^2 - 1, \quad z = 4x^3 - 3x, \quad z = 8x^4 - 8x^2 + 1, \quad \ldots, \tag{26}$$

we find with (25) the following algebraic equations for the unknowns:

n	first case	second case
2	$0 = x^2 - \frac{x}{4} - \frac{5}{8}$	$0 = x^2 - \frac{x}{4} - \frac{1}{2}$
3	$0 = x^3 - \frac{5}{6}x - \frac{1}{12}$	$0 = x^3 - \frac{5}{6}x$
4	$0 = x^4 - x^2 - \frac{x}{32} + \frac{5}{64}$	$0 = x^4 - x^2 - \frac{x}{32} + \frac{1}{8}$

Johann's result (22) is the same as (23) (first case), since his value for $HK = x$ is the root of the quadratic equation corresponding to $n = 2$ of the first case[36].

The polynomials (26) are today known as Chebyshev polynomials of the first kind[37] and they allow us to calculate the maximal solutions numerically for any degree (see Fig. 24). Observe that for $n \to \infty$, the left "spatia" converge to that of Huygens and the right one to that of Leibniz.

Jakob concludes his article by writing:

[36]Johann will agree that this constitutes the "foundation" of his method [2, I, p. 331].

[37]Since $x = \cos\alpha$, $z = \cos(n\alpha)$, this is the definition of these polynomials $T_n(\cos\alpha) = \cos(n\alpha)$.

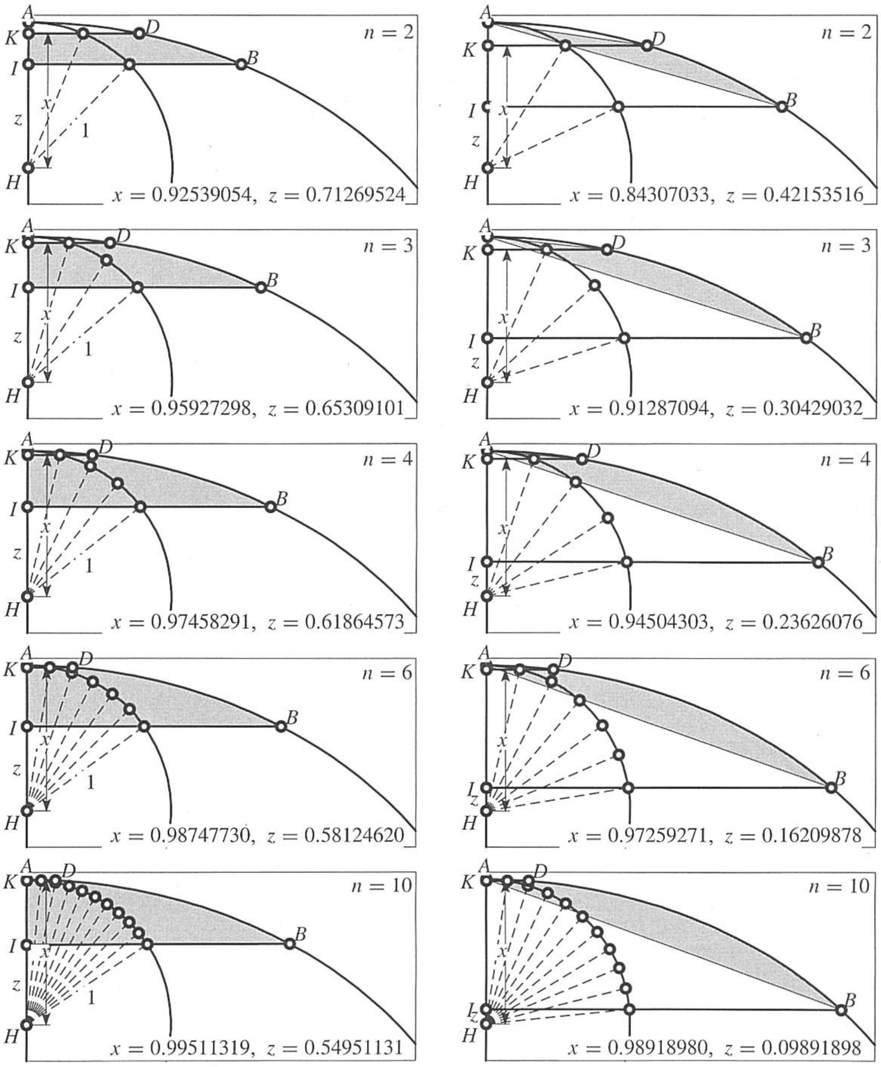


Figure 24 Jakob's "infinita spatia quadrabilia".

> "Methodum vero tam facilem haud alia fini pandere volui, quam ut Frater, exemplo meo, ad paria præstanda incitatus, mei quoque Problematis Isoperimetrici promissam analysin tandem aliquando nobis impertiat. [I developed this really easy method for no other reason than that my brother should follow my example and reveal after all his promises his solution of my problem about isoperimetric curves.]" [1, II, p. 873]

This last sentence should remind the reader that Jakob, who might be on the way to loosing a battle against Johann, was victorious in another battle, the one about the calculus of variations[38].

Jakob's second paper. After being challenged by Johann[39], Jakob publishes another paper[40] containing the following construction (without proof):

[38]For more information, see [8].

[39]*A. E.*, June 1700, pp. 266–271, [2, I, pp. 330–335] or [8, pp. 420–424].

[40]*A. E.*, December 1700, pp. 551–552, [1, II, pp. 892–994], also in [2, I, pp. 336] or [8, pp. 455–457].

Theorem 9. *Let AQH be a quarter of circle. To obtain geometrically a "quadrabilis" area* (22) *such that $s = \alpha t$ for a given $0 < \alpha < 1$,*

i) *put $AP = AH = 1$ and choose G such that $AG = \alpha$;*
ii) *let R be the midpoint of GP and draw a parallel to GH through R;*
iii) *construct the curve PSO as follows: for every horizontal line CEF, set $HT = \alpha\, CE$, draw the circle of center T and radius 1 which intersects the cycloid in V (the farthest point of A), draw the horizontal line VN and put $NO = HF$;*
iv) *the intersection S of the curve PSO with the parallel through R defines K (and thus M, D) and put $HI = KS$ to define I (and thus L, B).*

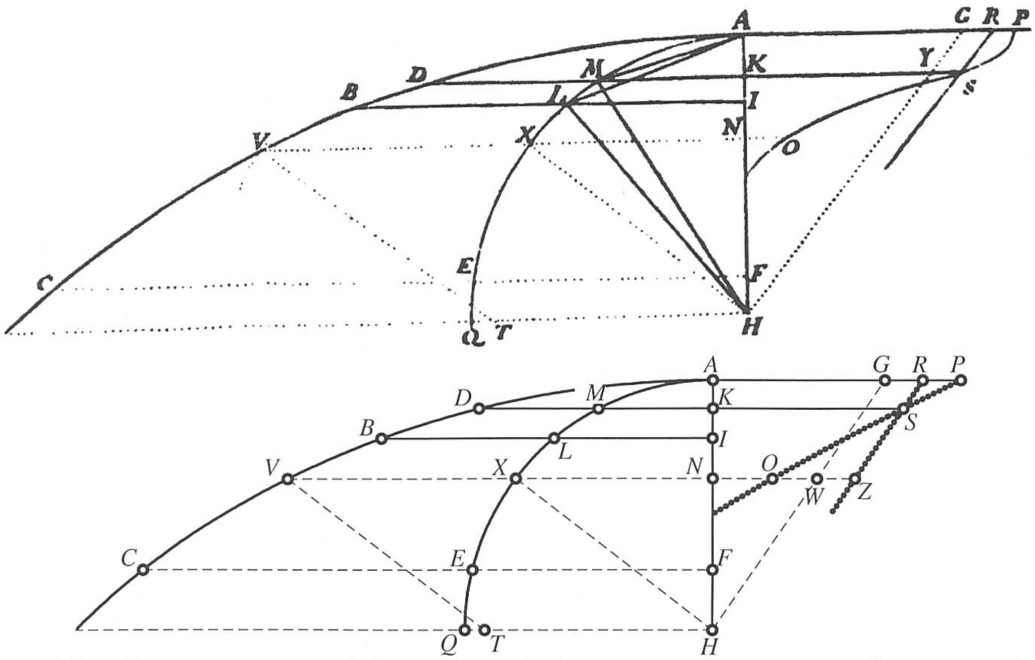


Figure 25 Jakob's construction: a facsimile [1, II, p. 894] (above) and a modern drawing (below, $\alpha = AG = 0.693147$, $z = IH = 0.767492$, $x = KH = 0.885909$).

Proof. By construction of T and of the parallelogram $THXV$, the pair of segments CE and VX are such that $VX = \alpha \cdot CE$, so that by Theorem 1 (a) the arcs XA and EA are also in the same ratio. With the notations $FH = z$ and $NH = x$ (as above), we have by construction (see Fig. 25)

$$NO = z \qquad \text{and} \qquad NZ = NW + WZ = \alpha x + \frac{1-\alpha}{2}.$$

If the line VX moves up and down, the points O and Z respectively describe a curve OSP and a straight line ZSR. At the point S of their intersection, the relation (25) (first case) with, for the present situation α instead of $\frac{1}{n}$, is satisfied. □

Johann's reply. Johann, not satisfied with this construction, writes:

> "Dico algebraïce; qui enim ad eas determinandas utitur curva quadam transcendente, sane is non plus præstitit, quam qui quadraturam circuli, ex supposita peripheriæ rectificatione, se invenisse gloriaretur. [I say algebraically; indeed, he who uses a certain transcendental curve to determine them (the squarable areas) does not accomplish any more than he who expects glory for having found the quadrature of the circle assuming the rectification of its circumference.]" [2, I, p. 389]

Johann is also not satisfied with the equations (26) because, according to him[41], in the three very simple cases given by his brother one does not distinguish the law of formation of the polynomials. He would have liked to invite his brother to find solutions based on the formula[42]

$$\frac{x}{a} = b^{n-1} - \frac{n-2}{1} b^{n-3} + \frac{(n-3)(n-4)}{1 \cdot 2} b^{n-5} - \frac{(n-4)(n-5)(n-6)}{1 \cdot 2 \cdot 3} b^{n-7} \ \&c$$

"cujus progressionis natura per se manifesta est[43]", where $a = 2\sin\alpha$, $b = 2\cos\alpha$, $x = 2\sin(n\alpha)$. While Jakob had given in (26) the first three Chebyshev polynomials of the first kind, Johann's formula is today written $\sin(n\alpha) = \sin\alpha \cdot U_{n-1}(\cos\alpha)$ with the general formula of the Chebyshev polynomials *of the second kind*[44].

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