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A note on Sassenfeld matrices

Ramona Baumann and Thomas P. Wihler

Ramona Baumann ist Masterstudentin an der Universität Bern und schreibt momentan ihre Masterarbeit über numerische Approximation der Poissongleichung mit unstetigen Randbedingungen unter der Betreuung von Professor Thomas Wihler.

Thomas Wihler ist Professor am Mathematischen Institut der Universität Bern und leitet dort die Forschungsgruppe für Angewandte und Numerische Mathematik. Nach dem Abschluss des Studiums und Doktorats in Mathematik an der ETH Zürich hat er mehrjährige Lehr- und Forschungsaufenthalte an der University of Minnesota (USA) sowie an der McGill University in Montreal (Kanada) absolviert, bevor er 2008 nach Bern kam. Schwerpunkt seiner aktuellen Forschung ist die numerische Lösung von partiellen Differentialgleichungen.

In 1951, H. Sassenfeld published a paper [3] (in German) in the context of iterative numerical methods for the solution of linear systems of equations. As part of his work, he introduced a class of matrices, subsequently referred to as *Sassenfeld matrices*, for which he was able to establish convergence of the famous Gauß–Seidel iteration scheme (see, e.g., [1, §11.2]). Since that time Sassenfeld’s convergence criterion has found its way into various texts in numerical mathematics. The goal of this note is to present a short discussion about Sassenfeld matrices in the context of the Gauß–Seidel method, and, in particular, to provide an elementary proof of the fact that such matrices are invertible.

To set the stage, for $m \geq 2$, we consider an $m \times m$ square matrix $A \in \mathbb{R}^{m \times m}$, with non-vanishing diagonal entries. Denoting, for $1 \leq i, j \leq m$, by a_{ij} the entry of A in row i and

Bei der Lösung von mathematischen Problemen in Anwendungen treten oftmals riesige lineare Gleichungssysteme mit Millionen von Unbekannten auf. Selbst mithilfe von Rechnerunterstützung ist die Komplexität des klassischen Gaußschen Eliminationsverfahrens dann meistens viel zu gross, um eine rasche Lösung zu erlauben. Alternativ werden iterative numerische Methoden, zu denen auch das berühmte Gauß–Seidel Verfahren gehört, erfolgreich eingesetzt. In der vorliegenden Arbeit erinnern die Autoren an ein hinreichendes Konvergenzkriterium für die Gauß–Seidel Iteration von Helmut Sassenfeld aus dem Jahr 1951 und machen einige ergänzende Beobachtungen.

column j (analogous notation is used for other matrices), and by

$$l_{ij} = \begin{cases} a_{ij} & \text{if } i > j \\ 0 & \text{otherwise} \end{cases}, \quad d_{ij} = \begin{cases} a_{ii} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad u_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{otherwise} \end{cases},$$

the corresponding entries below, on, and above the diagonal of A , respectively, we obtain the splitting $a_{ij} = l_{ij} + d_{ij} + u_{ij}$, or, equivalently, in matrix form $A = L + D + U$. Then, if A is invertible, the Gauß–Seidel method for solving a linear system of equations

$$Ax = b, \quad (1)$$

where $b \in \mathbb{R}^m$ is a given right-hand side vector, is given iteratively by the lower triangular system

$$(D + L)x_n = -Ux_{n-1} + b, \quad n \geq 1, \quad (2)$$

with an initial guess $x_0 \in \mathbb{R}^m$. Since the diagonal entries of A are nonzero, we note that the above system (2) is uniquely solvable for any $n \geq 1$.

In order to discuss the convergence of the iterates $\{x_n\}_{n \geq 0}$ from (2) to the exact solution x of the original linear system (1), we define a sequence of numbers s_1, \dots, s_m by means of the recursive relation

$$s_i = \frac{1}{|a_{ii}|} \left(\sum_{1 \leq j < i} s_j |a_{ij}| + \sum_{i < j \leq m} |a_{ij}| \right) = \frac{1}{|d_{ii}|} \left(\sum_{1 \leq j < i} s_j |l_{ij}| + \sum_{i < j \leq m} |u_{ij}| \right),$$

for $i = 1, \dots, m$. We call $\mu(A) = \max_{1 \leq i \leq m} s_i \geq 0$ the *Sassenfeld index* of A . Furthermore, we say that $A \in \mathbb{R}^{m \times m}$ is a *Sassenfeld matrix* if $\mu(A) < 1$.

Before stating our first result, we define the ∞ -norm of a vector $v \in \mathbb{R}^m$ with components $\{v_i\}_{i=1}^m$ by $\|v\|_\infty = \max_{1 \leq i \leq m} |v_i|$. Furthermore, for any matrix $M \in \mathbb{R}^{m \times m}$ the induced matrix norm is given by $\|M\|_\infty = \sup_{v \in \mathbb{R}^m, \|v\|_\infty=1} \|Mv\|_\infty$. We notice the estimate $\|Mv\|_\infty \leq \|M\|_\infty \|v\|_\infty$.

Proposition 1. *Within the above framework there holds the bound*

$$\|(D + L)^{-1}U\|_\infty \leq \mu(A).$$

Proof. Consider an arbitrary vector $v \in \mathbb{R}^m$ with $\|v\|_\infty = 1$, and define

$$w = (D + L)^{-1}Uv.$$

Then, there holds $(D + L)w = Uv$, and, thus, $w = D^{-1}(-Lw + Uv)$. Writing this equation componentwise, we have that

$$w_i = \frac{1}{a_{ii}} \left(- \sum_{1 \leq j < i} a_{ij} w_j + \sum_{i < j \leq m} a_{ij} v_j \right), \quad 1 \leq i \leq m.$$

Then, taking moduli on either side of the above equality, and recalling that $\|v\|_\infty = 1$, leads to

$$|w_i| = \frac{1}{|a_{ii}|} \left| - \sum_{1 \leq j < i} a_{ij} w_j + \sum_{i < j \leq m} a_{ij} v_j \right| \leq \frac{1}{|a_{ii}|} \left(\sum_{1 \leq j < i} |a_{ij}| |w_j| + \sum_{i < j \leq m} |a_{ij}| \right).$$

For $i = 1$, this implies that $|w_1| \leq s_1 \leq \mu(A)$.

Furthermore, if, for some $i \in \{2, \dots, m\}$, there holds that $|w_j| \leq s_j$, for all $j = 1, \dots, i - 1$, then we derive the bound

$$|w_i| \leq \frac{1}{|a_{ii}|} \left(\sum_{1 \leq j < i} |a_{ij}| s_j + \sum_{i < j \leq m} |a_{ij}| \right) = s_i \leq \mu(\mathbf{A}).$$

By induction, we conclude that $|w_i| \leq \mu(\mathbf{A})$ for any $i = 1, \dots, m$. Therefore, we infer that $\|\mathbf{w}\|_\infty \leq \mu(\mathbf{A})$, and the proof is complete. \square

The following two results follow immediately from the above proposition.

Theorem 2. *If a matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$, with $m \geq 2$, or its transpose, \mathbf{A}^\top , is a Sassenfeld matrix, then \mathbf{A} is invertible.*

Proof. Suppose that \mathbf{A} is a non-invertible Sassenfeld matrix. Then, there exists a vector $\mathbf{v}^* \in \mathbb{R}^m$, with $\|\mathbf{v}^*\|_\infty = 1$, such that $\mathbf{A}\mathbf{v}^* = \mathbf{0} \in \mathbb{R}^m$. This implies that

$$(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}\mathbf{v}^* = -\mathbf{v}^*.$$

Hence, by means of Proposition 1, we conclude that

$$\mu(\mathbf{A}) \geq \|(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}\|_\infty \geq \|(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}\mathbf{v}^*\|_\infty = \|\mathbf{v}^*\|_\infty = 1,$$

which constitutes a contradiction to the assumption $\mu(\mathbf{A}) < 1$. We close the proof by remarking that \mathbf{A} is invertible if and only if \mathbf{A}^\top is invertible. \square

Remark 3. We note that Theorem 2 is sharp in the sense that there are non-invertible matrices $\mathbf{A} \in \mathbb{R}^{m \times m}$ with $\mu(\mathbf{A}) = 1$. To see this, for $\alpha \geq 1$, let us consider the matrix

$$\mathbf{A}_\alpha = \begin{pmatrix} 1 & 0 & \cdots & 0 & \alpha \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ 1 & 0 & \cdots & 0 & \alpha \end{pmatrix} \in \mathbb{R}^{m \times m},$$

for which there holds that $\mu(\mathbf{A}_\alpha) = \alpha \geq 1$ (and, in particular, $\mu(\mathbf{A}_1) = 1$). Furthermore, for $\alpha \rightarrow \infty$, this example illustrates that non-invertible matrices may have arbitrarily large Sassenfeld indices; incidentally, replacing the entry 1 in the bottom left corner of \mathbf{A}_α by 0, we see that the same observation applies to invertible matrices. Finally, we remark that a matrix features Sassenfeld index 0 if and only if it is lower triangular (with non-vanishing diagonal entries). \diamond

Theorem 3. *If $\mathbf{A} \in \mathbb{R}^{m \times m}$ is a Sassenfeld matrix, then the Gauß–Seidel scheme (2) converges to the unique solution \mathbf{x} of (1) for any starting vector $\mathbf{x}_0 \in \mathbb{R}^m$.*

Proof. For $n \geq 0$ we signify by $\mathbf{e}_n = \mathbf{x} - \mathbf{x}_n$ the difference between the exact solution \mathbf{x} of (1) and the n th iterate \mathbf{x}_n of the Gauß–Seidel iteration (2). Then, for $n \geq 1$, we observe the identities

$$\begin{aligned} \mathbf{x} &= -(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}\mathbf{x} + (\mathbf{D} + \mathbf{L})^{-1} \mathbf{b}, \\ \mathbf{x}_n &= -(\mathbf{D} + \mathbf{L})^{-1} \mathbf{U}\mathbf{x}_{n-1} + (\mathbf{D} + \mathbf{L})^{-1} \mathbf{b}. \end{aligned}$$

Taking the difference yields $\mathbf{e}_n = -(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}\mathbf{e}_{n-1}$. Therefore, due to Proposition 1, we arrive at

$$\|\mathbf{e}_n\|_\infty = \|(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}\mathbf{e}_{n-1}\|_\infty \leq \|(\mathbf{D} + \mathbf{L})^{-1}\mathbf{U}\|_\infty \|\mathbf{e}_{n-1}\|_\infty \leq \mu(\mathbf{A}) \|\mathbf{e}_{n-1}\|_\infty.$$

Recalling that $\mu(\mathbf{A}) < 1$, and iterating the above bound, results in

$$\|\mathbf{e}_n\|_\infty \leq \mu(\mathbf{A})^n \|\mathbf{e}_0\|_\infty \rightarrow 0,$$

as $n \rightarrow \infty$. This shows the convergence of the Gauß–Seidel scheme (2). \square

Remark 4. The above Theorem 3 provides a sufficient assumption for the convergence of the Gauß–Seidel iteration scheme (2). We emphasize, however, that this criterion is by no means necessary. For instance, for $\beta \geq 1$, consider the matrix $\mathbf{B}_\beta \in \mathbb{R}^{m \times m}$ with entries

$$(b_\beta)_{ij} = \begin{cases} 1 & \text{for } i = j \\ \beta & \text{for } i = 1, j = 2, \\ 0 & \text{otherwise} \end{cases}, \quad 1 \leq i, j \leq m.$$

Here, it holds $\mu(\mathbf{B}_\beta) = \beta \geq 1$, and the Gauß–Seidel method converges in two steps for any right-hand side vector $\mathbf{b} \in \mathbb{R}^m$ and any initial guess $\mathbf{x}_0 \in \mathbb{R}^m$. \diamond

In order to close this note we provide a simple (pseudocode) MATLAB [2] function for computing the Sassenfeld index $\mu(\mathbf{A})$ of a quadratic matrix with nonzero diagonal entries.

```

1: function  $\mu =$  SASSENFELDINDEX(A)
2:   A = abs(A); ▷ remove signs from all matrix entries
3:   [m, n] = size(A);
4:   if isempty(A) || ne(m, n) || nnz(~diag(A)) then ▷ check whether matrix is admissible
5:     disp('matrix not admissible'); return ;
6:   end if
7:   s = ones(1, m); ▷ initialize Sassenfeld vector  $s = (s_1, \dots, s_m)$ 
8:   for i = 1 : m do
9:     s(i) = 0;
10:    s(i) = dot(s, A(i, :))/A(i, i); ▷ compute Sassenfeld index of A
11:  end for
12:   $\mu =$  max(s);
13: end function

```

References

- [1] G.H. Golub and Ch.F. Van Loan, *Matrix computations*, 4th ed., Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2013.
- [2] C. Moler, *Numerical Computing with MATLAB*, Society for Industrial and Applied Mathematics (SIAM), 2004.
- [3] H. Sassenfeld, *Ein hinreichendes Konvergenzkriterium und eine Fehlerabschätzung für die Iteration in Einzelschritten bei linearen Gleichungen*, Z. Angew. Math. Mech. **31** (1951), 92–94.

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