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On the theorem of the three perpendiculars

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1 Introduction

In geometry text books, the following theorem is usually known as the theorem of the three perpendiculars.

Theorem 0 (In 3-dimensional Euclidean space). *Assume the point x is outside the plane Π and the line Λ is included in Π; if xy is orthogonal to Π, with $y \in \Pi \setminus \Lambda$, and yz is orthogonal to Λ, with $z \in \Lambda$, then xz is orthogonal to Λ.*

Die jahrhundertelangen Bemühungen das fünfte Postulat Euklids – das Parallelenaxiom – zu beweisen, waren, wie wir heute wissen, von vornherein zum Scheitern verurteilt. Sie führten schliesslich im 19. Jahrhundert zur Entdeckung und Entwicklung der nichteuklidischen Geometrie. Seither wird nach Gemeinsamkeiten und Unterschieden zwischen den verschiedenen, durch ihre jeweiligen Axiomensysteme definierten Geometrien, gesucht. Die Frage nach der Existenz und der Eindeutigkeit von Parallelen weist auf einen wesentlichen Unterschied der Modelle. Im Gegensatz dazu ist Orthogonalität in der euklidischen, der hyperbolischen und der elliptischen Geometrie anzutreffen. Der aus der euklidischen Geometrie bekannte Satz über die drei Lote im dreidimensionalen Raum wird in der vorliegenden Arbeit im Rahmen der sphärischen und der hyperbolischen Geometrie diskutiert. Dabei warten einige Überraschungen auf die Leserschaft.

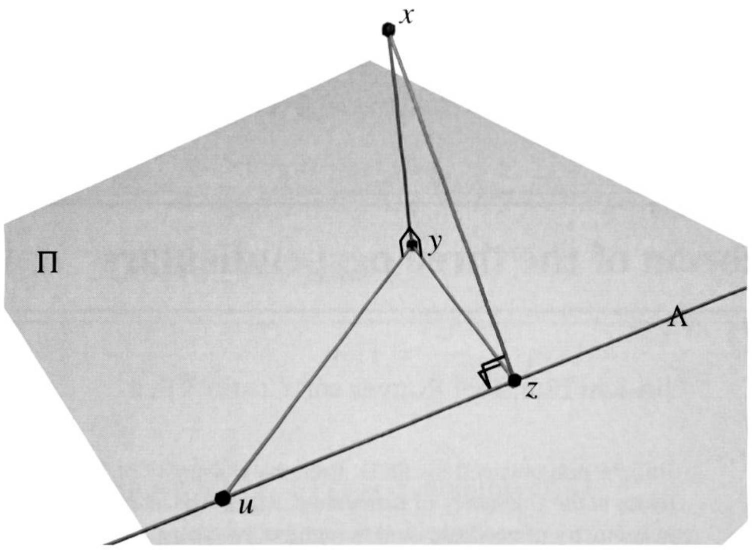

Fig. 1 Equivalence between Theorem 0 and Variant I.

In a very short note [1], H.N. Gupta observed (see Figure 1) that Theorem 0 is equivalent to

Variant I (In n-dimensional Euclidean space). *Let x, y, z, u be four pairwise distinct points. If the triangles xyz, xyu, yzu are right at y, y and z respectively, then the triangle xzu is right at z.*

For the completeness of this note, we give a short classical proof of this fact.

Proof. Let v be the symmetric point of u with respect to z. Since uzy and vzy are right at z and $d(u, z) = d(v, z)$, the triangles uzy and vzy are congruent and $d(u, y) = d(v, y)$. Since xy is orthogonal to the 2-space through y, z and u, the triangle xyv is right at y. It follows that the triangles xyv and xyu are congruent and $d(x, v) = d(x, u)$. Hence the triangles xzv and xzu are congruent, whence $\measuredangle xzv = \measuredangle xzu$. Moreover, $\measuredangle xzv + \measuredangle xzu = \pi$, whence $\measuredangle xzv = \measuredangle xzu = \pi/2$. □

H.N. Gupta noted that Variant I does not emphasize any plane Π or line Λ, and so implies that Theorem 0 holds for spaces of any dimensions. He noticed that the proof of Variant I holds as well for hyperbolic spaces, but he mentioned nothing about the spherical case. Instead, he focused on the fact that the triangles mentioned in Variant I can be exchanged: *whenever three of the four mentioned triangles are right, the fourth one is also right.* This last statement does not hold for spheres; see Remark 3.

In order to state Theorem 0 in other spaces, one has to replace Euclidean lines with geodesics and Euclidean planes with totally geodesic surfaces. Consequently, in the statement of Variant I, Euclidean triangles have to be replaced with geodesic triangles.

It is straightforward to see that the above proof of Variant I also remains valid in $\mathbb{H}^3$ and $\mathbb{S}^3$. In this note we give two new proofs for Theorem 0 following two distinct and more descriptive approaches; it is just a pretext to play with standard models of the sphere, the hyperbolic space and their subspaces. We are convinced that other methods of proof can be imagined, for example by the use of isometries. We believe that Theorem 0 remains valid in other spaces; it may be interesting to find them.

2 Preliminaries

Throughout this section and the next one, Ω will denote an Euclidean space, a sphere, or a hyperbolic space, of arbitrary dimension n. In the last section, Ω will be a space a little more general.

By definition, a *p-space* in Ω is a p-dimensional totally geodesic complete submanifold. A *geodesic* is always supposed to be maximal, or, in other words, to be a 1-space. A geodesic through x and y is denoted by xy.

With these definitions, we can give the following statement, more general than Variant I.

Variant II (In $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$)**.** *Assume the point x is outside the p-space Π and the q-space Λ is included in Π ($n > p > q > 0$). If xy is orthogonal to Π, with $y \in \Pi \setminus \Lambda$, and yz is orthogonal to Λ, with $z \in \Lambda$, then xz is also orthogonal to Λ.*

Proof. Notice first that the proof of Variant I holds in Ω. A geodesic xz is orthogonal to a q-space $\Lambda \ni z$ if and only if all triangles xzu, $u \in \Lambda$, are right at z. This fact and Variant I yield the conclusion. □

For the next proofs, we need explicit models. For hyperbolic spaces, there exist several standard models; we chose the one which is formally similar to the standard model of spheres.

If Ω is a sphere of dimension n, then it is assumed to be the unit sphere in the space $\mathbb{R}^{n+1}$ endowed with the canonical inner product $\langle\ ,\ \rangle$.

Similarly, if Ω is the n-dimensional hyperbolic space, it is assumed to be embedded in $\mathbb{R}^{1,n}$ as

$$\big\{(x_0, \dots, x_n) \in \mathbb{R}^{1,n} |\ \langle x, x\rangle = 1, x_0 > 0\big\}.$$

Recall that $\mathbb{R}^{1,n}$ is the linear space $\mathbb{R}^{n+1}$ endowed with the pseudo-Euclidean inner product

$$\langle x, y\rangle = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n,$$

where x_i denotes the ith coordinate of x, the index i starting from 0.

For the uniformity of the presentation, the same bracket notation stands for the two distinct inner products, according to the case.

For any subset P of Ω, $\mathrm{Sp}\,(P)$ stands for the linear subset of $\mathbb{R}^{n+1}$ spanned by P; in particular, $\mathrm{Sp}\,(\Omega) = \mathbb{R}^{n+1}$.

The formula expressing the distance between two points x and y of Ω is:

$$d\,(x, y) = \arccos \langle x, y\rangle \text{ if } \Omega \text{ is a sphere, and}$$
$$d\,(x, y) = \operatorname{arcosh} \langle x, y\rangle \text{ if } \Omega \text{ is a hyperbolic space.}$$

Another parallelism is the description of p-spaces: $\Phi \subset \Omega$ is a p-space if and only if there exists a linear subspace $F \subset \mathrm{Sp}\,(\Omega)$ such that $\Phi = F \cap \Omega$. Of course, $F = \mathrm{Sp}\,(\Phi)$.

3 The projection approach

This approach begins with the observation that Theorem 0 is an obvious corollary of its following variant (choose $E = \mathbb{R}^3$, $F = \Pi$, $G = \Lambda$)

Variant III. *Let $(E, \langle\cdot,\cdot\rangle)$ be a pseudo-Euclidean vector space. Let F be a proper subspace of E, and G a proper subspace of F. Assume that the restrictions of the inner product to $F \times F$ and $G \times G$ are non-degenerate. Let f and g be the orthogonal projections onto F and G respectively. Then $g = g \circ f$.*

Proof. Let $H \overset{\text{def}}{=} G^{\perp} \cap F$. We have $M = F^{\perp} \oplus H \oplus G$, and if $x = x_{F^{\perp}} + x_H + x_G$, with $x_{F^{\perp}} \in F^{\perp}$, $x_H \in H$, $x_G \in G$, then $f(x) = x_H + x_G$, $g(x) = x_G$. □

In a Euclidean space E, there are two equivalent ways to define the projection f onto the linear subspace F. The point $f(x)$ is at the same time the only point $y \in F$ such that the line xy is orthogonal to F (*i.e.*, the *orthogonal* projection), and the unique closest point to x among the points of F (*i.e.*, the *metrical* projection). Metrical and orthogonal projections can both be defined in any Riemannian manifold (even in more general spaces), onto any closed submanifold. It is well known that a metrical projection is always an orthogonal projection; this follows from the first variation formula. It is also clear that any point admits at least one metrical projection. In the (pseudo-) Euclidean case, the two notions coincide, so we will simply call those maps *(pseudo-)Euclidean projections*.

In the case of hyperbolic spaces, any point has at most one orthogonal projection on any p-space. This is an obvious consequence of the fact that the sum of the angles of any triangle is less than π. Therefore, the metrical and orthogonal projections coincide and are single-valued, as in the Euclidean case.

In the case of spherical spaces, however, the situation is slightly more complicated.

Lemma 1. *Let Φ be a p-space of $\Omega \simeq \mathbb{S}^n$ $(0 < p < n)$ and $x \in \Omega \setminus \Phi$. Then $y \in \Phi$ is an orthogonal projection of x onto Φ if and only if either y or $-y$ is a metrical projection of x onto Φ.*

Proof. Assume $y \in \Phi$. The point y is an orthogonal projection of x if and only if $\mathrm{Sp}(xy) = \mathbb{R}y \oplus U$, $\mathrm{Sp}(\Phi) = \mathbb{R}y \oplus V$, with $y \perp U$, $y \perp V$ and $U \perp V$. This proves that y is an orthogonal projection if and only if $-y$ is so. Assume now that y is an orthogonal projection of x, and that $d(x, y) \leq \pi/2$ (interchange y and $-y$ if necessary). Let z be a metrical projection of x. Put $a = d(x, y)$, $b = d(x, z)$ and $c = d(y, z)$, and notice that, by the choice of y, $\cos a$ and $\cos b$ are non-negative. The spherical triangle xyz is right at y, therefore $\cos b = \cos a \cos c \leq \cos a$. On the other hand, since z is a metrical projection, $\cos b \geq \cos a$, whence $a = b$ and y is also a metrical projection. □

Lemma 2. *Let Φ be a p-space of $\Omega \simeq \mathbb{S}^n$ $(0 < p < n)$ and $x \in \Omega$.*

1. *If $x \in \mathrm{Sp}(\Phi)^{\perp}$ then any $y \in \Phi$ is a metrical projection of x onto Φ.*
2. *If $x \notin \mathrm{Sp}(\Phi)^{\perp}$ then x has a unique metrical projection $\phi(x)$. The map $\phi : \Omega \setminus \mathrm{Sp}(\Phi)^{\perp} \to \Phi$ satisfies $\phi = s \circ f$, where $f : \mathrm{Sp}(\Omega) \to \mathrm{Sp}(\Phi)$ is the Euclidean projection and $s : \mathrm{Sp}(\Omega) \setminus \{0\} \to \Omega$ is the radial projection.*

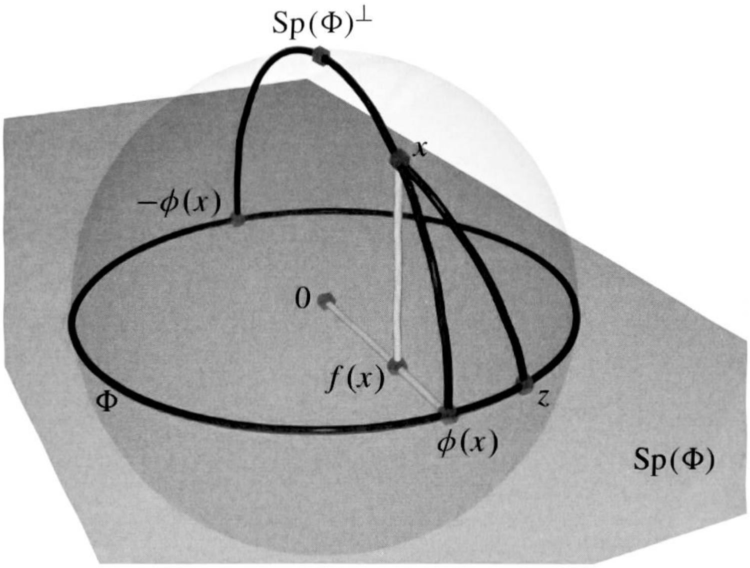

Fig. 2 Metrical projection on spheres.

Proof. 1. This is clear from the fact that $d(x, y) = \pi/2$ for any $y \in \Phi$.

2. Let z be an arbitrary point in Φ; we have

$$\cos d(x, z) = \langle x, z \rangle = \langle f(x), z \rangle = \| f(x) \| \cos d(z, s \circ f(x)),$$

whence the unique global minimum of $d(x, \cdot)\,|_\Phi$ is $s \circ f(x)$ (see Figure 2). □

Remark 3. *Let Φ be the 2-sphere through y, z, and u. If $x \in Sp(\Phi)^\perp$ then the triangles xyz, xzu, and xyu are right at y, z and u, but yzu is not right in general, see Figure 3. Therefore, as stated in the introduction, the triangles mentioned in Variant I cannot be exchanged in the case of spheres.*

A result similar to the second part of Lemma 2 holds in the hyperbolic case (see Figure 4).

Lemma 4. *Let Φ be a p-space of $\Omega \simeq \mathbb{H}^n$ $(0 < p < n)$. Any $x \in \Omega$ admits a unique metrical projection $\phi(x)$ onto Φ. Moreover, the map $\phi : \Omega \to \Phi$ satisfies $\phi = s \circ f$, where $f : \mathrm{Sp}(\Omega) \to \mathrm{Sp}(\Phi)$ is the pseudo-Euclidean projection, and $s : \{x \in \mathrm{Sp}(\Omega) \mid \langle x, x \rangle > 0,\, x_0 > 0\} \to \Omega$ is the radial projection.*

Proof. The proof is similar to the proof of Lemma 2, with hyperbolic cosines instead of cosines. The only difficulty is to prove that

$$D \overset{\text{def}}{=} \{x \in \mathrm{Sp}(\Omega) \mid \langle x, x \rangle > 0, x_0 > 0\}$$

is stable under f; *i.e.*, $f(D) \subset D$. First notice that the proof reduces to the three-dimensional case. We can assume without loss of generality that $\mathrm{Sp}(\Phi)$ has equation $x_0 = ax_1$

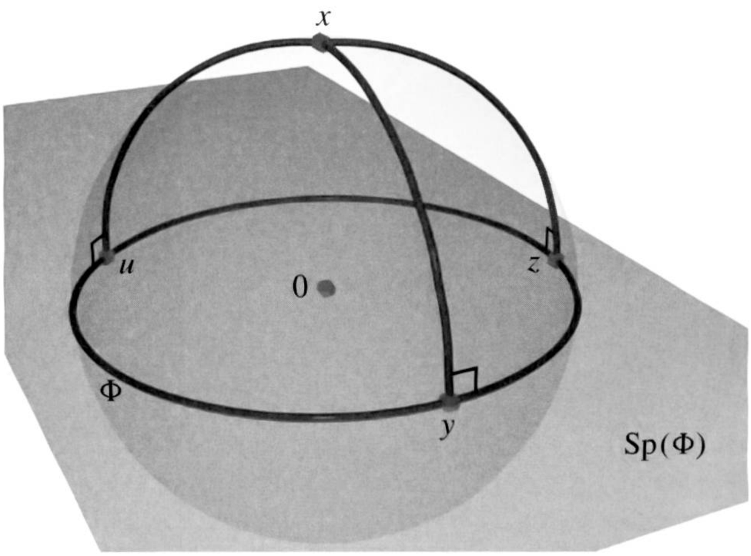

Fig. 3 The triangles mentioned in Variant I cannot be exchanged in the case of spheres. Note that Φ is actually a 2-dimentional sphere, where the points u, y, z form an arbitrary triangle.

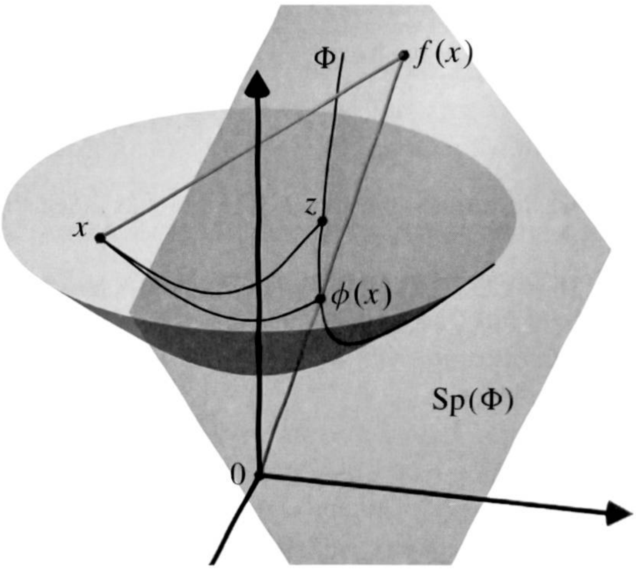

Fig. 4 Metrical projection on hyperbolic spaces.

$(a > 1)$. By straightforward computation, the matrix of f in the canonical basis is

$$\frac{1}{a^2-1}\begin{pmatrix} a^2 & -a & 0 \\ a & -1 & 0 \\ 0 & 0 & a^2-1 \end{pmatrix}.$$

It follows that

$$\langle f(x), f(x) \rangle = \frac{(x_0 - a x_1)^2}{a^2-1} + \langle x, x \rangle,$$

whence $\langle f(x), f(x)\rangle > 0$ whenever $\langle x, x\rangle > 0$. Moreover, $x \in D$ implies $ax_0 - x_1 > 0$, and consequently the first coordinate of $f(x)$ is positive. □

By Lemmas 1 and 2, the spherical and hyperbolic variants of Theorem 0 are a consequence of

Variant IV. *Let Φ be a p-space of $\Omega \simeq \mathbb{H}^n$ or $\Omega \simeq \mathbb{S}^n$ and Γ be a q-space included in Φ $(0 < q < p < n)$. Let $\phi : D_\phi \to \Phi$ and $\gamma : D_\gamma \to \Gamma$ be the metrical projections onto their respective images, where $D_\phi = \Omega \setminus \mathrm{Sp}(\Phi)^\perp$ and $D_\gamma = \Omega \setminus \mathrm{Sp}(\Gamma)^\perp$ if $\Omega \simeq \mathbb{S}^n$ and $D_\phi = D_\gamma = \Omega$ otherwise. Then $\gamma = \gamma \circ \phi$.*

Proof. Define $s : \{x \in \mathrm{Sp}(\Omega) \mid \langle x, x\rangle > 0\} \to \Omega$ by $s(x) = x/\sqrt{\langle x, x\rangle}$. By Lemmas 2 and 4, $\phi = s \circ f$ and $\gamma = s \circ g$, where f and g are the pseudo-Euclidean projections onto $\mathrm{Sp}(\Phi)$ and $\mathrm{Sp}(\Gamma)$ respectively. From the definition of s and the linearity of g, we get $s \circ g \circ s = s \circ g$. From this fact and Variant III we get

$$\gamma \circ \phi = s \circ g \circ s \circ f = s \circ g \circ f = s \circ g = \gamma .$$

□

4 The constant angle approach

This section is devoted to our second method of proof. This method has a *local character* and therefore yields a more general statement.

A *space form* is a complete Riemannian manifold which is locally isometric to $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$. There exist many such spaces; simple examples are the standard projective space and flat tori.

Lemma 5. *Let Φ and Δ be two distinct 2-spaces in a three-dimensional space form Ω.*

i. *$\gamma \overset{\text{def}}{=} \Phi \cap \Delta$ is a geodesic.*
ii. *The angle between Φ and Δ is constant along γ.*

Proof. The statements are local; therefore it is sufficient to prove them for $\mathbb{R}^3$, $\mathbb{H}^3$, or $\mathbb{S}^3$. The Euclidean case is clear, so we can assume $\Omega \simeq \mathbb{S}^3$ or $\Omega \simeq \mathbb{H}^3$.

i. Put $D = \mathrm{Sp}(\Delta)$ and $F = \mathrm{Sp}(\Phi)$. Then $\gamma = \Omega \cap D \cap F$ is a d-space, with $d = \dim(D \cap F) - 1$. Since Φ and Δ are 2-spaces, $\dim(D) = \dim(F) = 3$. By hypothesis we have $D \neq F$, whence $D + F = \mathrm{Sp}(\Omega)$. Now

$$\dim(D \cap F) = \dim(D) + \dim(F) - \dim(D + F) = 2.$$

ii. Let z be a point of γ. Let $u, v \in \mathrm{Sp}(\Omega)$ be unit normal vectors to $\mathrm{Sp}(D)$ and $\mathrm{Sp}(F)$ respectively. Note that, in the case $\Omega \simeq \mathbb{H}^3$, D and F cannot be tangent to the isotropic cone, so $\mathrm{Sp}(D) = u^\perp$ and $\mathrm{Sp}(F) = v^\perp$. Let $n_D(z)$ (resp. $n_F(z)$) be a vector of $T_z\Omega$ normal to $T_z\Delta$ (resp. $T_z\Phi$). Obviously, $T_z\Omega = z^\perp$ and

$$T_z\Delta = D \cap T_z\Omega = u^\perp \cap z^\perp = (\mathbb{R}u + \mathbb{R}z)^\perp .$$

Hence $n_D(z) \in (\mathbb{R}u + \mathbb{R}z) \cap z^\perp = \mathbb{R}u$.

It follows that $\measuredangle(\mathbb{R}n_D(z), \mathbb{R}n_F(z)) = \measuredangle(\mathbb{R}u, \mathbb{R}v)$ does not depend on $z \in \gamma$. □

If the constant angle between two 2-spaces of a 3-dimensional space form is $\pi/2$, then the two spaces are said to be orthogonal.

Proof of Theorem 0 *for space forms.* Denote by Ω a 3-dimensional space form, and let Γ be the 2-space containing x, y and z. Let $u \in T_z\Omega$ be a vector normal to Γ. Since $\Gamma \supset xy \perp \Pi$, Γ and Π are orthogonal, whence $T_z\Lambda = \left(T_z(yz)^{\perp} \cap T_z\Pi\right)$, $\left(T_z(yz)^{\perp} \cap T_z\Pi\right) \perp \left(T_z(yz)^{\perp} \cap T_z\Gamma\right)$, and $\left(T_z(yz)^{\perp} \cap T_z\Gamma\right) = \left(T_z(yz) + \mathbb{R}u\right)^{\perp}$. Hence $T_z\Lambda = \mathbb{R}u$, *i.e.*, $T_z\Lambda \perp T_z\Gamma \supset T_z(xy)$. □

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