

**Zeitschrift:** Elemente der Mathematik  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 70 (2015)  
**Heft:** 2

**Artikel:** Buffon's problem with a pivot needle  
**Autor:** Bäsel, Uwe  
**DOI:** <https://doi.org/10.5169/seals-630622>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 15.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

---

---

## Buffon's problem with a pivot needle

---

---

Uwe Bäsel

Uwe Bäsel is professor for machine elements and mechanism theory at the Faculty of Mechanical and Energy Engineering at the Hochschule für Technik, Wirtschaft und Kultur (HTWK) Leipzig, with interests in applied mathematics, especially geometry and kinematics.

The classical Buffon needle problem asks for the probability that a needle of length  $\ell$  thrown at random onto a plane lattice  $\mathcal{R}_d$  of parallel lines at a distance  $d \geq \ell$  apart will hit one of these lines. This problem was stated and solved by Buffon in his *Essai d'Arithmétique Morale*, 1777 (see, e.g., [5, pp. 71–72], [6, pp. 501–502]). If an arbitrary convex body  $\mathcal{C}$  with maximum width  $\leq d$  is used in this experiment, then the hitting probability is given by  $u/(\pi d)$ , where  $u$  denotes the perimeter of  $\mathcal{C}$ . This is the result of Barbier in 1860 [1, pp. 274–275], [6, p. 507]. If  $\mathcal{C}$  is a needle (line segment), then  $u = 2\ell$ . If  $\mathcal{C}$  is an ellipse, then there are elliptic integrals in the formulas of the hitting probabilities, see Duma and Stoka [3].

We consider a needle  $\mathcal{N}_{a,b}$  consisting of two line segments  $C'A'$ ,  $C'B'$  of lengths  $a := |C'A'|$  and  $b := |C'B'|$ , connected in a pivot point  $C'$  (see Fig. 1), and assume  $a + b \leq d$ .

Beim klassischen Buffonschen Problem wird eine Nadel betrachtet, die auf ein ebenes Gitter äquidistanter Geraden geworfen wird, wobei die Nadellänge maximal so groß wie der Abstand zwischen benachbarten Gittergeraden ist. Es wird nach der Wahrscheinlichkeit gefragt, dass die Nadel eine der Geraden trifft. Dieses Problem ist nach Georges-Louis Leclerc, Comte de Buffon (1707–1788) benannt, der es formulierte und 1777 in seinem *Essai d'Arithmétique Morale* löste. Heutzutage existiert eine große Anzahl von Arbeiten, die dieses Problem verallgemeinern. Wurfobjekt in der vorliegenden Arbeit ist eine Gelenknadel, die aus zwei gelenkig miteinander verbundenen Schenkeln besteht, wobei die Gesamtlänge der Nadel wiederum maximal so groß wie der Abstand zwischen benachbarten Gittergeraden sein soll. Eine derartige Nadel kann eine Gittergerade in einem oder zwei Punkten treffen, wofür die entsprechenden Wahrscheinlichkeiten berechnet werden. Diese Wahrscheinlichkeiten enthalten überraschenderweise das vollständige elliptische Integral zweiter Art, was sich im Fall verschieden langer Schenkel nicht weiter vereinfachen lässt.

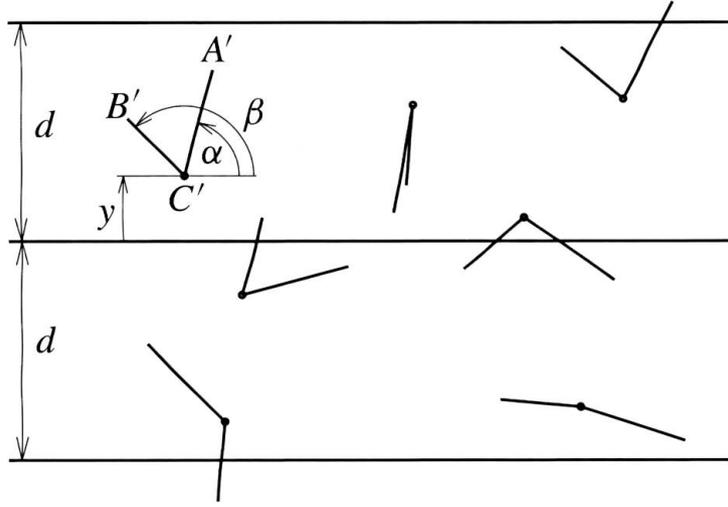


Figure 1 Lattice  $\mathcal{R}_d$  and randomly thrown needle  $\mathcal{N}_{a,b}$

The *random throw* of  $\mathcal{N}_{a,b}$  onto  $\mathcal{R}_d$  is defined as follows: The  $y$ -coordinate of the point  $C'$  is a random variable uniformly distributed in  $[0, d]$ . The angles  $\alpha$  and  $\beta$  between the lines of  $\mathcal{R}_d$ , and segments  $C'A'$  and  $C'B'$ , respectively, are random variables uniformly distributed in  $[0, 2\pi]$ . All three random variables are stochastically independent.

The probability of the event that  $\mathcal{N}_{a,b}$  hits two lines of  $\mathcal{R}_d$  at the same time is equal to zero, even in the case  $a + b = d$ . The expectation  $\mathbb{E}(n)$  of the random variable  $n = \text{number of intersection points between } \mathcal{N}_{a,b} \text{ and } \mathcal{R}_d$  is given by  $\mathbb{E}(n) = 2(a + b)/(\pi d)$ , cf. [4].

Here we are asking for the probabilities  $p(i)$ ,  $i \in \{0, 1, 2\}$ , of the events that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  in exactly  $i$  points. We denote by  $A$  and  $B$  the events that segments  $C'A'$  and  $C'B'$ , respectively, hit one line of  $\mathcal{R}_d$ .

The following theorem states the result of this paper.

**Theorem.** *If  $a + b \leq d$ , then the probabilities  $p(i)$  that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  in exactly  $i$  points are given by*

$$p(0) = 1 - \frac{(a + b)(\pi + 2E(k))}{\pi^2 d}, \quad p(1) = \frac{4(a + b)E(k)}{\pi^2 d},$$

$$p(2) = \frac{(a + b)(\pi - 2E(k))}{\pi^2 d},$$

where

$$E(k) = E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

is the complete elliptic integral of the second kind with  $k^2 = 4ab/(a + b)^2$ .

*Proof.* We observe that the angle  $\phi := \angle(C'A', C'B')$  is a random variable uniformly distributed in  $[0, 2\pi]$ . Due to the result of Barbier, the conditional probability  $P(A \cup B \mid \phi)$  of  $A \cup B$  for fixed value of  $\phi \in [0, 2\pi]$  is given by  $u(\phi)/(\pi d)$ , where  $u(\phi)$  is the perimeter

of the convex hull of  $\mathcal{N}_{a,b}$ . ( $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  if and only if its convex hull hits  $\mathcal{R}_d$ .) Using the law of total probability, the probability that  $\mathcal{N}_{a,b}$  hits  $\mathcal{R}_d$  is given by

$$\begin{aligned} P(A \cup B) &= \int_0^{2\pi} P(A \cup B | \phi) \frac{d\phi}{2\pi} = \frac{1}{2\pi^2 d} \int_0^{2\pi} u(\phi) d\phi \\ &= \frac{1}{2\pi^2 d} \int_0^{2\pi} [a + b + c(\phi)] d\phi = \frac{a + b + \bar{c}}{\pi d}, \end{aligned}$$

where  $c := |A'B'|$ , and

$$\bar{c} := \frac{1}{2\pi} \int_0^{2\pi} c(\phi) d\phi = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + b^2 - 2ab \cos \phi} d\phi.$$

Using  $\cos \phi = 2 \cos^2(\phi/2) - 1$ , we have

$$\begin{aligned} \bar{c} &= \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(a+b)^2 - 4ab \cos^2 \frac{\phi}{2}} d\phi \\ &= \frac{a+b}{2\pi} \int_0^{2\pi} \sqrt{1 - \frac{4ab}{(a+b)^2} \cos^2 \frac{\phi}{2}} d\phi. \end{aligned}$$

For abbreviation we put  $k^2 = 4ab/(a+b)^2$ . From the inequality  $\sqrt{ab} \leq (a+b)/2$  between the geometric and the arithmetic mean, one finds  $k^2 \leq 1$ , hence  $0 \leq k \leq 1$  with  $k = 1$  only for  $a = b$ . With the substitution  $\chi = \phi/2$  we get

$$\begin{aligned} \bar{c} &= \frac{a+b}{\pi} \int_0^{\pi} \sqrt{1 - k^2 \cos^2 \chi} d\chi = \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2 \cos^2 \chi} d\chi \\ &= \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \chi} d\chi = \frac{2(a+b)E(k)}{\pi}. \end{aligned}$$

It follows that

$$\begin{aligned} P(A \cup B) &= \frac{a+b+\bar{c}}{\pi d} = \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ P(A \cap B) &= P(A) + P(B) - P(A \cup B) = \frac{2a}{\pi d} + \frac{2b}{\pi d} - \frac{a+b+\bar{c}}{\pi d} \\ &= \frac{a+b-\bar{c}}{\pi d} = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}, \end{aligned}$$

and

$$\begin{aligned} p(0) &= 1 - P(A \cup B) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d}, \\ p(1) &= P(A \cup B) - P(A \cap B) = \frac{a+b+\bar{c}}{\pi d} - \frac{a+b-\bar{c}}{\pi d} = \frac{2\bar{c}}{\pi d} \\ &= \frac{4(a+b)E(k)}{\pi^2 d}, \\ p(2) &= P(A \cap B) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}. \end{aligned}$$

□

This is the result from [2, pp. 57–58]. There it was obtained as special case of the more general result in Corollary 4.2 [2, p. 56].

**Remark 1.** If the angle  $\phi$  is constant, then we have

$$P(A \cup B) = \frac{a + b + c}{\pi d} \quad \text{and} \quad P(A \cap B) = \frac{a + b - c}{\pi d}$$

with  $c = \sqrt{a^2 + b^2 - 2ab \cos \phi}$ . This yields

$$p(0) = 1 - \frac{a + b + c}{\pi d}, \quad p(1) = \frac{2c}{\pi d}, \quad p(2) = \frac{a + b - c}{\pi d},$$

see Santaló [5, pp. 77–78].

**Remark 2.** If  $a = b$ , we have  $k = 1$ ,  $E(1) = 1$ , and therefore

$$p(0) = 1 - \frac{2a(\pi + 2)}{\pi^2 d}, \quad p(1) = \frac{8a}{\pi^2 d}, \quad p(2) = \frac{2a(\pi - 2)}{\pi^2 d}.$$

If  $a \neq 0$  and  $b = 0$ , then  $k = 0$  and  $E(0) = \pi/2$ , and therefore  $P(A \cup B) = P(A) = 2a/(\pi d)$ . This is the result of the classical Buffon needle problem.

## Acknowledgment

I wish to thank the anonymous referee for careful reading and valuable advice that helped me to improve this paper.

## References

- [1] J.-É. Barbier: Note sur le problème de l'aiguille et le jeu du joint couvert, *Journal des mathématiques pures et appliquées*, 2d ser., **5** (1860), 273–286.
- [2] U. Bäsel: *Geometrische Wahrscheinlichkeiten für nichtkonvexe Testelemente*, Doctoral Thesis, FernUniversität Hagen, 2008. <http://deposit.fernuni-hagen.de/1011/>
- [3] A. Duma, M. Stoka: Hitting probabilities for random ellipses and ellipsoids, *J. Appl. Prob.* **30** (1993), 971–974.
- [4] J.F. Ramaley: Buffon's Noodle Problem, *Amer. Math. Monthly* **76** (1969), 916–918.
- [5] L.A. Santaló, *Integral Geometry and Geometric Probability*, Addison-Wesley, London, 1976.
- [6] E. Seneta, K.H. Parshall, F. Jongmans: Nineteenth-century developments in probability: J.J. Sylvester, M.W. Crofton, J.-É. Barbier and J. Bertrand, *Arch. Hist. Exact Sci.* **55** (2001), 501–524.

Uwe Bäsel  
 Hochschule für Technik, Wirtschaft  
 und Kultur (HTWK) Leipzig  
 Fakultät Maschinenbau und Energietechnik  
 PF 30 11 66  
 D-04251 Leipzig, Germany  
 e-mail: uwe.baesel@htwk-leipzig.de