

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 70 (2015)
Heft: 1

Artikel: On Beckner's inequality for Gaussian measures
Autor: Gwynne, Ewain / Hsu, Elton P.
DOI: <https://doi.org/10.5169/seals-630611>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.11.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On Beckner's inequality for Gaussian measures

Ewain Gwynne and Elton P. Hsu¹

Ewain Gwynne obtained his Bachelor's degree from Northwestern University in 2013 and is currently a mathematics graduate student at Massachusetts Institute of Technology. His main mathematical interests are analysis and probability theory.

Elton Hsu is a Professor of Mathematics at Northwestern University. His mathematical interests include stochastic analysis and its applications to analysis.

The standard Gaussian measure on euclidean space \mathbb{R}^n ,

$$\gamma(dx) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx,$$

has many fascinating properties, among them the Poincaré inequality

$$\|f\|_2 \leq \|\nabla f\|_2 \quad \text{for} \quad \int_{\mathbb{R}^n} f d\gamma = 0$$

¹The first author was supported by an undergraduate research grant from Northwestern University. The second author was supported in part by the Simons Foundation Collaborative Grant and by the Institute of Applied Mathematics of the Chinese Academy of Sciences and the University of Science and Technology of China during his visits to these institutions.

Das Gaußsche Maß auf \mathbb{R}^n besitzt zahlreiche schöne Eigenschaften. Einige davon tauchen im Zusammenhang mit verschiedenen Normen bei Ungleichungen auf. Die Poincaré-Ungleichung und die logarithmische Sobolev-Ungleichung von Gross sind zwei prominente Beispiele. 1989 bewies Beckner eine L^p -Ungleichung für $1 \leq p < 2$, welche zwischen den beiden genannten Ungleichungen interpoliert: Die Poincaré-Ungleichung erhält man für $p = 1$, die Ungleichung von Gross für $p \rightarrow 2$. Die Autoren der vorliegenden Arbeit benutzen nun die Tatsache, dass das Gaußsche Maß als Wärmeleitungskern auftritt, um mit Hilfe der klassischen Wärmeleitungshalbgruppe Beckners Ungleichung neu zu beweisen und sie gleichzeitig auf den Fall $p > 2$ auszuweiten.

and Gross's [6] logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} f^2 \log |f| d\gamma - \|f\|_2^2 \log \|f\|_2 \leq \|\nabla f\|_2^2.$$

Beckner [4] has proved the functional inequality

$$\|f\|_2^2 - \|f\|_p^2 \leq (2 - p)\|\nabla f\|_2^2, \quad 1 \leq p < 2. \quad (1)$$

For $p = 1$, inequality (1) is equivalent to the Poincaré inequality, as can be seen for bounded f by adding a sufficiently large constant C so that $f + C$ is nonnegative, and for a general f by approximation by bounded functions. Furthermore, if we divide both sides of (1) by $2 - p$ and let $p \rightarrow 2$, the left side tends to the left side of the logarithmic Sobolev inequality. Thus Beckner's inequality interpolates between the Poincaré inequality and the logarithmic Sobolev inequality.

Beckner's original proof of (1) is based on the explicit spectral decomposition of the Ornstein–Uhlenbeck operator in terms of Hermite polynomials and Nelson's [9] hypercontractivity inequality for the Ornstein–Uhlenbeck semigroup. Apparently unaware of Beckner's work at the time, Latała and Oleszkiewicz [7] proved an extension of Beckner's inequality for measures $ce^{-|x_1|^r - \dots - |x_n|^r} dx$ with $1 \leq r \leq 2$. However, in the Gaussian case $r = 2$ the inequality (1) was derived from the logarithmic Sobolev inequality and the hypercontractivity of the Ornstein–Uhlenbeck semigroup, via an argument similar to that in Beckner [4]. Many other authors also studied Beckner's inequality and its generalizations in various directions; see, e.g., Arnold, Bartier, and Dolbeault [1]; Arnold, Markowich, Toscani, and Unterreiter [2]; Barthe and Roberto [3]; Chafai [5]; Ledoux [8]; and Wang [11]. But none of these works includes a proof of (1) which does not rely on ideas or results comparable in difficulty to the logarithmic Sobolev inequality or its consequence the hypercontractivity. In addition, most of these works prove Beckner's inequality in a much broader setting than that in which Beckner originally derived it, which can make it difficult for a reader without substantial background in the field to discern the beauty and simplicity of the original inequality. This situation makes it desirable and instructive to search for a more direct proof of Beckner's inequality. In this note, we shall demonstrate this possibility by proving the following slight extension of Beckner's inequality by an elementary argument based on the classical heat semigroup.

Theorem. *Let $q \geq 2$ and $1 \leq p \leq q$. Then if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth function such that f and each of its partial derivatives belong to $L^q(\mathbb{R}^n)$, we have*

$$\|f\|_q^2 - \|f\|_p^2 \leq (q - p)\|\nabla f\|_q^2. \quad (2)$$

Remark 1. We state the inequality here for smooth functions for expository purposes, but an elementary approximation argument shows that it is also valid for functions f in the Sobolev space $W^{1,q}(\mathbb{R}^n)$.

The basic tool for our proof is the classical heat semigroup $\{P_s\}$ defined by

$$P_s f(x) = \frac{1}{(2\pi s)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/2s} dy.$$

Note that if f is bounded and continuous, then $P_s f \rightarrow f$ as $s \rightarrow 0$, and if $f \in L^1(\gamma)$, then

$$P_1 f(0) = \int_{\mathbb{R}^n} f d\gamma.$$

Furthermore, it is easy to verify from the definition that the heat semigroup has the following properties:

$$P_s P_t = P_{s+t}, \quad \partial_s P_s = \frac{1}{2} \Delta P_s = \frac{1}{2} P_s \Delta, \quad \nabla P_s = P_s \nabla.$$

Here ∇ and Δ are the usual gradient and Laplace operator on \mathbb{R}^n , respectively. Aside from these elementary properties, the only other tool we will need for the proof of our main result (2) is Hölder's inequality for a Borel measure ν on \mathbb{R}^n :

$$\int_{\mathbb{R}^n} fg d\nu \leq \left(\int_{\mathbb{R}^n} |f|^p d\nu \right)^{1/p} \left(\int_{\mathbb{R}^n} |g|^q d\nu \right)^{1/q} \quad (3)$$

for $f \in L^p(\mathbb{R}^n, \nu)$, $g \in L^q(\mathbb{R}^n, \nu)$, and exponents $p, q \in [1, \infty]$ such that $1/p + 1/q = 1$. By replacing f with $|f|$ and then approximating $|f|$ by smooth positive functions bounded away from 0 and ∞ , it is enough to show the inequality (2) for a smooth function f such that $0 < c \leq f \leq C$. For $0 \leq s \leq 1$, consider the function

$$\phi_s(x) = \left[P_s (P_{1-s} f^p)^{q/p}(x) \right]^{2/q}. \quad (4)$$

We can write the left side of (2) as

$$\|f\|_q^2 - \|f\|_p^2 = \phi_1(0) - \phi_0(0) = \int_0^1 \partial_s \phi_s(0) ds.$$

The idea of considering such a function in the context of functional inequalities can be traced back to Neveu [10].

The technical part of our proof is a straightforward computation of the derivative of (4) with respect to s , which will lead to a convenient expression for this derivative (see (7) below). From this, we will repeatedly apply Hölder's inequality to get the simple upper bound

$$\partial_s \phi_s(0) \leq (q - p) \left(\int_{\mathbb{R}^n} |\nabla f|^q d\gamma \right)^{2/q},$$

from which our desired inequality (2) follows immediately by integrating with respect to s from 0 to 1.

From the definition (4) of ϕ_s we have

$$\partial_s \phi_s = \partial_s \left[P_s g_s^{q/p} \right]^{2/q} = \frac{2}{q} a_s \partial_s (P_s g_s)^{q/p},$$

where, to simplify the notation hereafter, we have introduced the functions

$$g_s = P_{1-s} f^p \quad \text{and} \quad a_s = \left(P_s g_s^{q/p} \right)^{2/q-1}.$$

The derivative $\partial_s (P_s g_s)^{q/p}$ can be easily calculated and we obtain

$$\partial_s \phi_s = \frac{2}{q} a_s (\partial_s P_s) g_s^{q/p} + \frac{2}{p} a_s P_s \left(g_s^{q/p-1} \partial_s g_s \right). \quad (5)$$

Using the relation $\partial_s P_s = (1/2) P_s \Delta$, we may rewrite the first term on the right side as $(1/q) a_s P_s \Delta \left(g_s^{q/p} \right)$, which equals

$$\frac{1}{p} \left(\frac{q}{p} - 1 \right) a_s P_s \left(g_s^{q/p-2} |\nabla g_s|^2 \right) + \frac{1}{p} a_s P_s \left(g_s^{q/p-1} \Delta g_s \right) \quad (6)$$

by the identity

$$\Delta (h^{q/p}) = \frac{q}{p} \left(\frac{q}{p} - 1 \right) h^{q/p-2} |\nabla h|^2 + \frac{q}{p} h^{q/p-1} \Delta h$$

applied with $h = g_s$. From $\partial_s P_{1-s} = -(1/2) \Delta P_{1-s}$ we have $\partial_s g_s = -(1/2) \Delta g_s$, so the second term in the sum (6) exactly cancels the second term in (5). In the remaining term, we use the fact that P_{1-s} commutes with ∇ to write $\nabla g_s = p P_{1-s} (f^{p-1} \nabla f)$. This gives

$$\partial_s \phi_s = (q - p) a_s P_s \left(g_s^{q/p-2} |P_{1-s} (f^{p-1} \nabla f)|^2 \right). \quad (7)$$

Note that P_{1-s} is an integral with respect to a (probability) measure, so we can use Hölder's inequality (3) with the exponents $p/(p-1)$ and p to get

$$|P_{1-s} (f^{p-1} \nabla f)| \leq P_{1-s} (f^{p-1} |\nabla f|) \leq (P_{1-s} f^p)^{(p-1)/p} (P_{1-s} |\nabla f|^p)^{1/p}.$$

Thus, by (7),

$$\partial_s \phi_s \leq (q - p) a_s P_s \left(g_s^{q/p-2/p} (P_{1-s} |\nabla f|^p)^{2/p} \right). \quad (8)$$

The case $q = 2$ is covered by trivial modifications to what follows, so in the remainder of the proof we assume $q > 2$. A second application of Hölder's inequality with the exponents $q/(q-2)$ and $q/2$ yields

$$P_s \left(g_s^{q/p-2/p} (P_{1-s} |\nabla f|^p)^{2/p} \right) \leq \left(P_s g_s^{q/p} \right)^{1-2/q} \left(P_s (P_{1-s} |\nabla f|^p)^{q/p} \right)^{2/q}.$$

The first factor on the right side is exactly a_s^{-1} , which cancels the factor a_s in (8). We thus have

$$\partial_s \phi_s \leq (q - p) \left(P_s (P_{1-s} |\nabla f|^p)^{q/p} \right)^{2/q}. \quad (9)$$

Since $1 \leq p \leq q$, another application of Hölder's inequality gives

$$P_{1-s} |\nabla f|^p \leq (P_{1-s} |\nabla f|^q)^{p/q}.$$

This together with the semigroup property $P_s P_{1-s} = P_1$ gives

$$\left(P_s (P_{1-s} |\nabla f|^p)^{q/p} \right)^{2/q} \leq (P_s P_{1-s} |\nabla f|^q)^{2/q} = \left(\int_{\mathbb{R}^n} |\nabla f|^q d\gamma \right)^{2/q}.$$

The last equality holds after evaluating at $x = 0$. It follows from (9) that

$$\partial_s \phi_s(0) \leq (q - p) \left(\int_{\mathbb{R}^n} |\nabla f|^q d\gamma \right)^{2/q}.$$

Integrating from $s = 0$ to $s = 1$ yields the desired inequality (2).

We conclude this note with a few more remarks.

Remark 2. The constant $q - p$ on the right side of our new inequality (2) cannot be improved. This can be seen by taking $f(x) = e^{tx_1}$ for $t > 0$, calculating both sides explicitly, and letting $t \rightarrow 0$.

Remark 3. The condition $q \geq 2$ in (2) is essential. Indeed, an inequality of the form

$$\|f\|_q^2 - \|f\|_p^2 \leq C(q - p)\|\nabla f\|_q^2 \quad (10)$$

cannot hold in the parameter range $1 \leq p < q < 2$ with any constant C . Replacing f by $1 + \epsilon f$ in (10) and comparing the coefficients of ϵ^2 in the Taylor expansions of both sides, we see that (10) would lead to

$$\|f\|_2^2 - \left(\int_{\mathbb{R}^n} f d\mu \right)^2 \leq \|\nabla f\|_q^2.$$

Taking again $f(x) = e^{tx_1}$, this time with a very large t , we see easily that this inequality cannot hold if $q < 2$.

Remark 4. However, the function

$$\theta(q, p) = \frac{\|f\|_q^2 - \|f\|_p^2}{1/p - 1/q}$$

is increasing in both arguments whenever $1 \leq p < q$ (see Latała and Oleszkiewicz [7]). This fact together with the original Beckner's inequality (1) implies

$$\|f\|_q^2 - \|f\|_p^2 \leq \frac{2}{q}(q - p)\|\nabla f\|_2^2, \quad \text{for } 1 \leq p \leq q \leq 2.$$

Remark 5. In Section 3.1 of [8], Ledoux used a nonlinear partial differential equation to prove a version of (1) for the invariant probability measures of a Markov semigroup whose generator satisfies a curvature-dimension inequality. In the Gaussian case, his inequality reduces to a sharpened form of (1), with the right side multiplied by $(n - 1)/n$ and the parameter p allowed to increase to $2n/(n - 1)$.

References

- [1] A. Arnold, J. Bartier, and J. Dolbeault, Interpolation between logarithmic Sobolev and Poincaré inequalities, *Commun. Math. Sci.*, **5**, no. 4 (2007), 971–979.
- [2] A. Arnold, P. Markowich, G. Toscani, and A. Unterreiter, On logarithmic Sobolev inequalities and the rate of convergence to equilibrium for Fokker–Planck type equations, *Comm. Partial Differential Equations*, **26**, no. 1-2 (2001), 43–100.

- [3] F. Barthe and C. Roberto, Sobolev inequalities for probability measures on the real line, *Studia Math.*, **159** (2003), 481–497.
- [4] W. Beckner, A Generalized Poincaré Inequality for Gaussian Measures, *Proceedings of the American Mathematical Society*, **105**, no. 2 (1989), 397–400.
- [5] D. Chafaï, Entropies, convexity, and functional inequalities: On Phi-entropies and Phi-Sobolev inequalities, *J. Math. Kyoto Univ.*, **44**, no. 2 (2004), 325–363.
- [6] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061–1083.
- [7] R. Latała and K. Oleszkiewicz, Between Sobolev and Poincaré, *Geometric Aspects of Functional Analysis, Lecture Notes in Math.*, **1745** (2000), Springer, 147–168.
- [8] M. Ledoux, The geometry of Markov diffusion generators, *Ann. Fac. Sci. Toulouse*, **IX** (2000), 305–366.
- [9] E. Nelson, The free Markoff field, *J. Funct. Anal.*, **12** (1973), 211–227.
- [10] J. Neveu, Sur l’espérance conditionnelle par rapport à un mouvement brownien, *Ann. Insti. Henri Poincaré, B.*, **12** (1976), 105–110.
- [11] F.Y. Wang, A Generalization of Poincaré and Log-Sobolev Inequalities, *Potential Analysis*, **22**, no. 1 (2005), 1–15.

Ewain Gwynne and Elton P. Hsu
Department of Mathematics
Northwestern University
Evanston, IL 60208, USA
e-mail: ewaingwynne@gmail.com
ehsu@math.northwestern.edu