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**Autor:** Guo Shi, Yong / Gong, Xiao-Bing  
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## Linear functional equations involving Babbage's equation\*

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Yong-Guo Shi and Xiao-Bing Gong

Yong-Guo Shi obtained his Ph.D. from the Sichuan University in 2012. He is now an associate professor at the Neijiang Normal University, Sichuan, China. His interests are in discrete dynamical systems and functional equations.

Xiao-Bing Gong obtained his Ph.D. from the Sichuan University in 2013. He is an associate professor at the Neijiang Normal University, Sichuan, China. His interests are in discrete dynamical systems and functional equations.

### 1 Introduction

A functional equation is an equation whose unknowns are functions. Cauchy's functional equation [17]  $\varphi(x + y) = \varphi(x) + \varphi(y)$ , Schröder's equation [28, 39]  $\varphi \circ f = s\varphi$ , and Schilling's equation [7, 25]  $4q\varphi(qx) = \varphi(x + 1) + 2\varphi(x) + \varphi(x - 1)$  are examples of such equations.

Functional equations arise in many branches of mathematics, for example, dynamical systems [1, 19, 24, 43], functional analysis [42], geometry [8, 9], information theory [3], wavelet theory [20, 21], and special functions [27]. They also occur in other disciplines such as physics [22, 33], engineering [15, 16], economics [4, 23] and so on.

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Funktionalgleichungen bilden nicht nur ein reichhaltiges Forschungsthema, sondern sie sind auch beliebte Probleme bei Mathematikwettbewerben. Oft entspringen Funktionalgleichungen konkreten Anwendungen. Sucht man etwa für ein diskretes dynamisches System  $x \mapsto f(x)$  ein erstes Integral  $\phi$ , so entspricht dies gerade dem Auffinden einer nicht konstanten Lösung der Funktionalgleichung  $\varphi \circ f = \varphi$ . Die Autoren untersuchen in ihrer Arbeit eine Klasse von Funktionalgleichungen, welche mit der Babbage-Gleichung in Beziehung steht. Letztere fragt nach einer Funktion  $f$ , deren  $n$ -te Iterierte  $f^n$  die Identität ist. Insbesondere werden in der vorliegenden Arbeit explizite Lösungen für die Gleichung  $\varphi = \pm\varphi \circ f + g$  angegeben, wobei  $f$  eine Lösung der Babbage-Gleichung, und  $g$  eine gegebene Funktion ist.

The systematic study of functional equations did not begin until 1966 [2], although many great mathematicians have been studying them before, including Euler (1768), Cauchy (1821), Abel (1823), Darboux (1895), and Banach (1920) (cf. [27]). In the last five decades, the theory of functional equations has developed very rapidly and gradually became an independent field of mathematics. Functional equations also became a common topic in mathematics competitions, see the books [13, 30, 41], some problems and solutions in the journals *The American Mathematical Monthly* and *Mathematical Excalibur* [18], and the website “KöMaL” [34].

Apart from competition problems, a considerable number of interesting problems (see, e.g., [10, 14, 24]) involve the following single variable functional equation – Babbage’s equation

$$\varphi^n = \text{id}, \quad (1)$$

where  $\varphi^n$  denotes the  $n$ th iterate of a self-map  $\varphi$ , and  $\text{id}$  stands for the identity. Ch. Babbage [5, 6] studied its solutions in the reals. In 1916, J.F. Ritt [38] gave four types of real solutions. Later, the results on Babbage’s equation were generalized into many different directions, e.g., continuous solutions in [28, Theorem 15.2], meromorphic solutions in [28, pp. 291–292], also [40, Example 2], homeomorphic solutions on the unit circle in [26], and involutions on the plane in [31].

Motivated by the functional equation  $\varphi \circ f = \varphi$  for an integrable map  $f$  (see [24]) and the competition problem to determine the function  $\varphi : \mathbb{R} \setminus \{0, 1\} \rightarrow \mathbb{R}$  such that

$$\varphi(x) + \varphi\left(\frac{x-1}{x}\right) = 1 + x,$$

this paper investigates the single variable functional equations

$$\varphi = \pm \varphi \circ f + g, \quad (2)$$

where  $f, g$  are given and  $f$  is globally periodic with the prime period  $n$  (i.e.,  $f^i \neq \text{id}$  for  $1 < i < n$  and  $f^n = \text{id}$ ).

The general form of these equations above is

$$F(\varphi \circ f_1, \dots, \varphi \circ f_n, \text{id}) = 0, \quad (3)$$

where  $F, f_1, \dots, f_n$  are given and  $\varphi$  is unknown. When  $F$  is linear and the functions  $f_1, \dots, f_n$  form a group under composition on their domain, S. Presić [29, 36, 37] characterized all solutions of (3). The unique solution of a special case in (3) is determined by M. Bessenyei [10] under additional regularity assumptions. Further investigations have been carried out by M. Bessenyei and his collaborators [11, 12] for the unique differentiable solution of (3).

The equation (2) is another special case of (3). With the methods of linear algebra combined with a version of recurrent iteration, we present exact solutions of (2) and the formulas of solutions are different from those in [32]. We also present some examples and applications.

## 2 The main results

The equation (2) is a class of linear functional equations and the corresponding homogeneous equation is

$$\varphi = \pm \varphi \circ f. \quad (4)$$

Similar to [29, p. 101, Theorem 3.1.5], we have the superposition principle for the linear functional equation (2).

**Lemma 1.** *Let  $S$  be a set and  $(G, +)$  a group,  $f : S \rightarrow S$  and  $g : S \rightarrow G$  be two given mappings. Then the general solution  $\varphi : S \rightarrow G$  of equation (2) is given by  $\varphi = \varphi_1 + \varphi_2$ , where  $\varphi_1 : S \rightarrow G$  is a particular solution of (2), and  $\varphi_2 : S \rightarrow G$  is the general solution of equation (4).*

*Proof.* Let  $\varphi : S \rightarrow G$  be an arbitrary solution of (2) and  $\varphi_1 : S \rightarrow G$  a particular solution of (2). Then

$$\begin{aligned} \varphi &= \pm \varphi \circ f + g, \\ \varphi_1 &= \pm \varphi_1 \circ f + g. \end{aligned}$$

Thus  $(\varphi - \varphi_1) = \pm(\varphi - \varphi_1) \circ f$ . It follows that  $\varphi - \varphi_1$  is a solution of (4).

On the other hand, let  $\varphi_2 : S \rightarrow G$  be an arbitrary solution of (4) and  $\varphi_1 : S \rightarrow G$  a particular solution of (2). Then

$$\begin{aligned} \varphi_2 &= \pm \varphi_2 \circ f, \\ \varphi_1 &= \pm \varphi_1 \circ f + g. \end{aligned}$$

Thus  $(\varphi_1 + \varphi_2) = \pm(\varphi_1 + \varphi_2) \circ f + g$ . It follows that  $\varphi_1 + \varphi_2$  is a solution of (2).  $\square$

In what follows, it suffices to find the general solution of the homogeneous equation (4) and one particular solution of (2).

**Lemma 2.** *Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$  and the unknown  $\varphi$  maps the set  $S$  to a set  $G$ . Then the general solution of  $\varphi = \varphi \circ f$  is given by*

$$\varphi(x) = H \left( x, f(x), f^2(x), \dots, f^{n-1}(x) \right),$$

where  $H : S^n \rightarrow G$  is any function satisfying

$$H \left( x, f(x), \dots, f^{n-1}(x) \right) = H \left( f(x), f^2(x), \dots, f^{n-1}(x), x \right).$$

*Proof.* Let  $\varphi : S \rightarrow G$  be a solution of  $\varphi = \varphi \circ f$ . Then define  $H : S^n \rightarrow G$  in this way: take an arbitrary  $x_0 \in S$ ,

$$H \left( x_0, f(x_0), \dots, f^{n-1}(x_0) \right) := \varphi(x_0), \quad \forall x_0 \in S;$$

on other points  $(x_1, x_2, \dots, x_n) \in S^n$ , define  $H$  arbitrarily. We see that  $C_f(x_0) := \{x_0, f(x_0), \dots, f^{n-1}(x_0)\}$  is an orbit of  $x_0$ . It follows from [28, Theorem 1.6] that  $\varphi$  is

constant on  $C_f(x_0)$ . So

$$H(x_0, f(x_0), \dots, f^{n-1}(x_0)) = \varphi(x_0) = \varphi(f(x_0)) = H(f(x_0), \dots, f^{n-1}(x_0), x_0).$$

On the other hand, a simple calculation shows that  $\varphi := H$  satisfies  $\varphi = \varphi \circ f$ .  $\square$

Let  $n$  be an integer greater than or equal to 2. A *uniquely n-divisible Abelian group*  $(K, +)$  is an Abelian group having the property that for each  $x \in K$  there is a unique  $y \in K$  such that  $x = ny$ . So we can denote  $y$  by  $\frac{x}{n}$ .

**Lemma 3.** *Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ , and  $(G, +)$  is a uniquely  $n$ -divisible Abelian group. Then the general solution  $\varphi : S \rightarrow G$  of  $\varphi = \varphi \circ f$  is given by*

$$\varphi(x) = \sum_{i=0}^{n-1} h(f^i(x)), \quad (5)$$

where  $h : S \rightarrow G$  is an arbitrary function.

*Proof.* For an arbitrary function  $h : S \rightarrow G$ , the function

$$\varphi(x) := \sum_{i=0}^{n-1} h(f^i(x))$$

evidently satisfies  $\varphi = \varphi \circ f$ . On the other hand, if  $\varphi$  is a solution of the equation  $\varphi = \varphi \circ f$ , then  $\varphi = \varphi \circ f^i$  for every positive integer  $i$ . Since  $(G, +)$  is a uniquely  $n$ -divisible Abelian group, we have for any  $x \in S$

$$\begin{aligned} \varphi(x) &= \frac{\varphi(x)}{n} + \frac{\varphi(f(x))}{n} + \dots + \frac{\varphi(f^{n-1}(x))}{n} \\ &= \sum_{i=0}^{n-1} \frac{\varphi(f^i(x))}{n}. \end{aligned}$$

Set  $h(x) := \frac{\varphi(x)}{n}$ . Then (5) holds.  $\square$

**Lemma 4.** *Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ ,  $n$  is odd, and  $(G, +)$  is a group and for each  $y \in G$ ,  $2y = 0$  if and only if  $y = 0$ . Then  $\varphi = -\varphi \circ f$  has a unique solution from  $S$  to  $G$  given by  $\varphi(x) = 0$ .*

*Proof.* By successively substituting  $f^j(x)$  for  $x$  in  $\varphi(x) = -\varphi \circ f(x)$  for each  $j = 1, 2, \dots, n-1$ , we obtain a set of  $n$  equations in the  $n$  unknowns  $\varphi(f^j(x))$ :

$$\left\{ \begin{array}{l} \varphi(x) + \varphi(f(x)) = 0, \\ \varphi(f(x)) + \varphi(f^2(x)) = 0, \\ \vdots \\ \varphi(f^{n-2}(x)) + \varphi(f^{n-1}(x)) = 0, \\ \varphi(f^{n-1}(x)) + \varphi(x) = 0. \end{array} \right. \quad (6)$$

Since  $n$  is odd, we have

$$\varphi(x) = -\varphi(f(x)) = \varphi(f^2(x)) = \cdots = \varphi(f^{n-1}(x)) = -\varphi(x).$$

Thus  $\varphi(x) = 0$ .  $\square$

With similar arguments as in Lemmas 2, 3, proofs of the following two lemmas are easily supplied.

**Lemma 5.** Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ ,  $n$  is even, and  $(G, +)$  is a group. Then the general solution  $\varphi : S \rightarrow G$  of  $\varphi = -\varphi \circ f$  is given by

$$\varphi(x) = H(x, f(x), \dots, f^{n-1}(x)),$$

where  $H : S^n \rightarrow G$  is any function satisfying

$$H(x, f(x), \dots, f^{n-1}(x)) + H(f(x), f^2(x), \dots, f^{n-1}(x), x) = 0.$$

**Lemma 6.** Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ ,  $n$  is even, and  $(G, +)$  is a uniquely  $n$ -divisible Abelian group. Then the general solution  $\varphi : S \rightarrow G$  of  $\varphi = -\varphi \circ f$  is given by

$$\varphi(x) = \sum_{i=0}^{n-1} (-1)^i h(f^i(x)),$$

where  $h : S \rightarrow G$  is an arbitrary function.

Now we shall give exact solutions of (2).

**Theorem 1.** Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ , and  $(G, +)$  is a uniquely  $n$ -divisible Abelian group. Then there exists a solution  $\varphi : S \rightarrow G$  of  $\varphi = \varphi \circ f + g$  if and only if  $\sum_{i=0}^{n-1} g \circ f^i = 0$ . Further, the general solution  $\varphi : S \rightarrow G$  is given by

$$\varphi(x) = \sum_{i=0}^{n-1} h(f^i(x)) + \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g(f^i(x)), \quad (7)$$

where  $h : S \rightarrow G$  is an arbitrary function.

*Proof.* By the recurrent iteration to  $\varphi = \varphi \circ f + g$ , we have  $\sum_{i=0}^{n-1} g \circ f^i = 0$ . On the other hand, assume that  $\sum_{i=0}^{n-1} g \circ f^i = 0$ . Set

$$\varphi(x) := \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g(f^i(x)) \quad (8)$$

which yields that

$$\begin{aligned}
\varphi - \varphi \circ f &= \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g \circ f^i - \sum_{i=0}^{n-2} \frac{(n-1-i)}{n} g \circ f^{i+1} \\
&= \sum_{i=0}^{n-1} \frac{(n-1-i)}{n} g \circ f^i - \sum_{i=1}^{n-1} \frac{(n-i)}{n} g \circ f^i \\
&= \frac{(n-1)}{n} g - \sum_{i=1}^{n-1} \frac{1}{n} g \circ f^i \\
&= g.
\end{aligned}$$

So (8) is a particular solution of  $\varphi = \varphi \circ f + g$ . By Lemmas 1, 3, (7) is the general solution.  $\square$

**Theorem 2.** Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ ,  $n$  is odd,  $(G, +)$  is a uniquely 2-divisible Abelian group. Then  $\varphi = -\varphi \circ f + g$  has a unique solution from  $S$  to  $G$  given by

$$\varphi(x) = \sum_{i=0}^{n-1} \frac{(-1)^i g(f^i(x))}{2}. \quad (9)$$

*Proof.* By induction, we have

$$\varphi(f^j(x)) = (-1)^j \varphi(f^j(x)) + \sum_{i=0}^{j-1} (-1)^i g(f^i(x)), \quad j = 1, 2, \dots \quad (10)$$

Since  $n$  is odd, set  $j = n$ , then (10) becomes

$$\varphi(x) = -\varphi(x) + \sum_{i=0}^{n-1} (-1)^i g(f^i(x)).$$

Thus (9) follows. One can check that (9) is a particular solution of  $\varphi = -\varphi \circ f + g$ . By Lemmas 1, 4, (9) is a unique solution.  $\square$

**Theorem 3.** Suppose  $f$  is globally periodic with the prime period  $n$  on a set  $S$ ,  $n$  is even,  $(G, +)$  is a uniquely  $n$ -divisible Abelian group. Then there exists a solution  $\varphi : S \rightarrow G$  of  $\varphi = -\varphi \circ f + g$  if and only if  $\sum_{i=0}^{n-1} (-1)^i g(f^i(x)) = 0$ . Further, the general solution  $\varphi : S \rightarrow G$  is given by

$$\varphi(x) = \sum_{i=0}^{n-1} (-1)^i h(f^i(x)) + \sum_{i=0}^{n-2} \frac{(-1)^i (n-1-i) g(f^i(x))}{n}, \quad (11)$$

where  $h : S \rightarrow G$  is an arbitrary function.

*Proof.* Since  $n$  is even, set  $j = n$ , then (10) becomes

$$\varphi(x) = \varphi(x) + \sum_{i=0}^{n-1} (-1)^i g(f^i(x)),$$

which implies that  $\sum_{i=0}^{n-1} (-1)^i g(f^i(x)) = 0$ .

On the other hand, assume that  $\sum_{i=0}^{n-1} (-1)^i g(\varphi^i(x)) = 0$  holds. Set

$$\varphi(x) := \sum_{i=0}^{n-2} \frac{(-1)^i (n-1-i) g(f^i(x))}{n}. \quad (12)$$

Then one can check that (12) is a particular solution of  $\varphi = -\varphi \circ f + g$ . By Lemmas 1, 6, (11) is the general solution.  $\square$

Remark that the conditions of Theorems 1 and 3 respectively, have a close connection with the following two functional equations

$$\sum_{i=0}^{n-1} \varphi \circ f^i = 0, \quad n > 2, \quad (13)$$

$$\sum_{i=0}^{n-1} (-1)^i \varphi \circ f^i = 0, \quad n > 2 \text{ is even}, \quad (14)$$

where  $f$  is a given globally periodic map with the prime period  $n$ . The general solutions of these two equations are defined with the method of iterative construction in the paper [32]. However, for some applications, it remains interesting to give exact solutions, which are *not* of the form of a piecewise function.

### 3 Applications and examples

In this section, we conclude with some examples. The interested reader can find exact solutions for more functional equations on the website [35] with a nice classification.

**Example 4.1.** Find the function  $\varphi : (0, +\infty) \rightarrow \mathbb{R}$  satisfying  $\varphi(x) + \varphi(1/x) = 1$ .

Observe that  $1/2$  is a particular solution. By Theorem 3, the exact solution of this equation is

$$\varphi(x) = h(x) - h(1/x) + 1/2,$$

where  $h : (0, +\infty) \rightarrow \mathbb{R}$  is an arbitrary function. With the method of iterative construction in [28, Chp.1] or [32], the general solution with the form of piecewise function is given by

$$\varphi(x) = \begin{cases} \varphi_0(x), & \text{if } x \in (0, 1) \\ 1/2, & \text{if } x = 1 \\ 1 - \varphi_0(1/x), & \text{if } x \in (1, \infty) \end{cases}$$

where  $\varphi_0 : (0, 1) \rightarrow \mathbb{R}$  is an arbitrary function.

**Example 4.2.** Consider the Knuth mapping  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  in this form [14]

$$T(x, y) = (-y + |x|, x),$$

which is globally periodic with the prime period 9.

By Theorem 1, all first integrals of  $T$  are of the form  $F(x, y) = \sum_{j=0}^8 h(T^j(x, y))$ , where  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  is an arbitrary non-constant function. In particular, choosing  $h(x, y) = y$ , we get a first integral

$$F(x, y) = y + |y - |x|| + |x - |y - |x||| + |y - |x - |y||| + |x - |y| + |y - |x - |y||||.$$

**Example 4.3.** Find the function  $\varphi : \mathbb{R} \setminus \{-1, 2\} \rightarrow \mathbb{R}$  satisfying  $\varphi(x) - \varphi(f(x)) = g(x)$ , where  $f(x) = \frac{2x-7}{x+1}$  is globally periodic with the prime period 3.

One can examine

$$x \xrightarrow{f} \frac{2x-7}{x+1} \xrightarrow{f} -\frac{x+7}{x-2} \xrightarrow{f} x.$$

By Theorem 1, there exists a solution of this equation if and only if

$$g(x) + g\left(\frac{2x-7}{x+1}\right) + g\left(-\frac{x+7}{x-2}\right) = 0.$$

Further, the exact solution is given by

$$f(x) = \frac{2g(x) + g\left(\frac{2x-7}{x+1}\right)}{3} + h(x) + h\left(\frac{2x-7}{x+1}\right) + h\left(-\frac{x+7}{x-2}\right),$$

where  $h : \mathbb{R} \setminus \{-1, 2\} \rightarrow \mathbb{R}$  is an arbitrary function.

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Yong-Guo Shi and Xiao-Bing Gong  
 Key Laboratory of Numerical Simulation of Sichuan Province  
 College of Mathematics and Information Science  
 Neijiang Normal University  
 Neijiang, Sichuan 641112, P. R. China  
 e-mail: [scumat@163.com](mailto:scumat@163.com) (Y.-G. Shi)  
[gxb525@163.com](mailto:gxb525@163.com) (X.-B. Gong, corresponding author)