

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 68 (2013)

Artikel: On the diagonal weights of inscribed polytopes
Autor: Akiyama, Jin / Sato, Ikuro
DOI: <https://doi.org/10.5169/seals-515899>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 10.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On the diagonal weights of inscribed polytopes

Jin Akiyama and Ikuro Sato

Jin Akiyama received a D.Sc. from Tokyo University of Science for his work in graph theory and combinatorics. He is now the director of the Research Center for Science and Math Education at Tokyo University of Science and also serves as the founding editor of the journal of Graphs and Combinatorics. He is interested in graph theory, discrete and computational geometry, and also in mathematics education.

Ikuro Sato graduated from Tohoku University School of Medicine in 1981, and received a M.D. in pathology. He is working at the Research Institute of Miyagi Cancer Center as the director and clinical professor of pathology. He has also published many mathematical papers on polytopes in higher dimensions.

1 The diagonal weight

For a given polytope Π with v vertices P_1, P_2, \dots, P_v , we define the *diagonal weight* of Π as the sum of the squares of the lengths of all diagonals and sides of Π , and denote it by $\alpha(\Pi)$. That is, the diagonal weight $\alpha(\Pi)$ of Π is defined as follows:

$$\alpha(\Pi) = \sum_{i,j} |P_i P_j|^2$$

where $|P_i P_j|$ is the distance between P_i and P_j , and the sum is taken over all possible pairs of P_i and P_j .

We first consider a few regular polygons on the plane, namely, the equilateral triangle, the square, the regular pentagon and the regular hexagon inscribed in unit circles as illustrated in Figure 1.

Berechnet man von einem dem Einheitskreis einbeschriebenen regulären n -Eck die Summe der Quadrate der Seitenlängen und Diagonalen, so erhält man n^2 . Hätten Sie's gewusst? Die Autoren der vorliegenden Arbeit haben genauer hingeschaut und bemerkt, dass diese Eigenschaft in beliebigen Dimensionen $m \geq 2$ für alle der Einheits-sphäre einbeschriebenen Polytope mit n Ecken gilt, wenn der Eckenschwerpunkt mit dem Zentrum der Umkugel zusammenfällt. Die Autoren untersuchen auch die Frage, welche dreidimensionalen Polyeder, deren Seiten reguläre Vielecke sind, diese Schwerpunktsbedingung erfüllen.

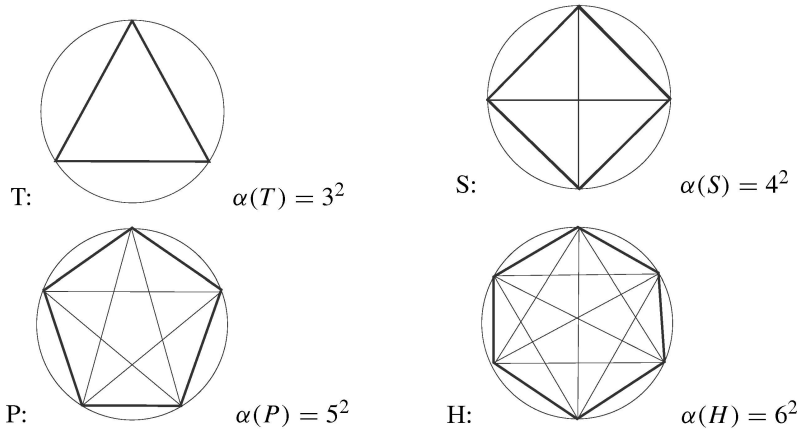


Fig. 1

Then it follows at once that the diagonal weights of these regular polygons are 9, 16, 25 and 36, respectively. This simple observation suggests to us that the diagonal weight $\alpha(\Pi) = v^2$ holds for every regular polygon Π with v vertices.

Figure 2 illustrates two of the polyhedra in 3-dimensional space: the cube and the regular octahedron. The following theorem shows that the result holds not only on the plane but also in any higher dimension n ($n \geq 2$).

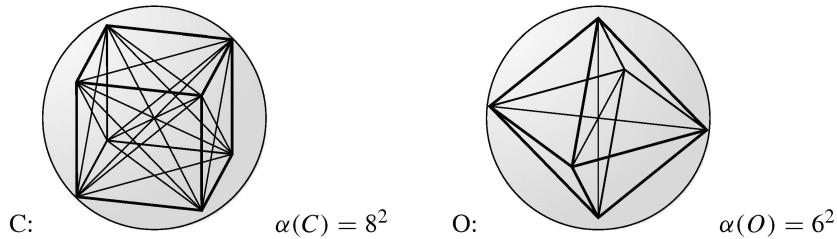


Fig. 2

Theorem 1. *Let R be a regular n -dimensional polytope with v vertices P_1, P_2, \dots, P_v which is inscribed in a unit n -sphere. Then the diagonal weight $\alpha(R)$ is v^2 for every dimension $n \geq 2$.*

Proof. Let

$$\begin{aligned}
 Q &= \sum_{j=1}^v |P_1 P_j|^2 \\
 &= (\mathbf{p}_1 - \mathbf{p}_1) \cdot (\mathbf{p}_1 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) + \cdots + (\mathbf{p}_v - \mathbf{p}_1) \cdot (\mathbf{p}_v - \mathbf{p}_1),
 \end{aligned}$$

where each \mathbf{p}_j is a position vector of P_j in relation to the center of R . Since

$$(\mathbf{p}_j - \mathbf{p}_1) \cdot (\mathbf{p}_j - \mathbf{p}_1) = \mathbf{p}_1 \cdot \mathbf{p}_1 - 2 \mathbf{p}_1 \cdot \mathbf{p}_j + \mathbf{p}_j \cdot \mathbf{p}_j, \quad \mathbf{p}_1 \cdot \mathbf{p}_1 = 1 \quad \text{and} \quad \mathbf{p}_j \cdot \mathbf{p}_j = 1,$$

we have

$$(\mathbf{p}_j - \mathbf{p}_1) \cdot (\mathbf{p}_j - \mathbf{p}_1) = 2 - 2 \mathbf{p}_1 \cdot \mathbf{p}_j.$$

Thus we obtain

$$Q = 2v - 2 \mathbf{p}_1 \cdot (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_v).$$

Since the centroid of the vertices of R is the origin,

$$\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_v = \mathbf{0},$$

and hence

$$Q = 2v.$$

Since the length of each diagonal occurs twice in $v \cdot Q$,

$$\alpha(R) = \frac{v}{2} \cdot Q = v^2.$$

The argument above holds for every n -dimensional polytope whose centroid is the origin. \square

There are many applications of this theorem. For instance, the diagonal weight of a regular 4-dimensional 120-cell is 600^2 , since it has 600 vertices. Likewise, it is straight forward to find the length d_n of a side of an n -simplex inscribed in a unit n -sphere, since $\binom{n+1}{2} d_n^2 = (n+1)^2$.

Corollary 1. *Let P be an n -polytope inscribed in a unit n -sphere and having the property that $\sum \mathbf{p}_j = \mathbf{0}$. Then the diagonal weight $\alpha(P) = v^2$ holds for P .*

2 Well-balanced polyhedra

We next generalize Theorem 1 slightly for a polytope whose centroid does not coincide with its circumcenter.

Theorem 2. *Let Π be an n -polytope with v vertices P_1, P_2, \dots, P_v inscribed in a unit n -sphere and c be the distance between the centroid and the circumcenter of Π . Then we have*

$$\alpha(\Pi) = v^2(1 - c^2), \quad \text{where} \quad 0 \leq c \leq 1.$$

Proof. Let \mathbf{c} be the vector defined as $\mathbf{c} = (\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_v)/v$, and c be its norm, i.e., $c = |\mathbf{c}|$.

As $Q_j = 2v(1 - \mathbf{p}_j \cdot \mathbf{c})$, we have $\alpha(\Pi) = 1/2 \sum Q_j = v^2(1 - c^2)$, $0 \leq c \leq 1$. \square

Theorem 2 implies that $\alpha(\Pi) \leq v^2$ for every inscribed n -polytope with v vertices. An inscribed n -polytope P with v vertices is said to be *well-balanced* if P satisfies $\alpha(P) = v^2$. That is, the diagonal weight of a balanced polytope attains the upper bound.

In the 3-dimensional case, there are infinitely many convex well-balanced polyhedra other than the Platonic solids. Our next purpose is to determine all well-balanced convex polyhedra with regular polygons as faces.

All convex polyhedra with regular polygons as their faces are known. There are five Platonic solids, thirteen Archimedean solids, the two infinite sequences of prisms and antiprisms, and 92 other polyhedra discovered by N.W. Johnson. We will denote Johnson's polyhedra by J1 to J92, following Johnson's numbering in his original paper [2]. Drawings and nets of other solids may be viewed on Wikipedia. We checked all such polyhedra with the property that their circumcenters coincide with the centroids on the basis of their coordinates.

Corollary 2. *The following are all the well-balanced convex polyhedra with regular polygons as faces:*

*all Platonic solids, all Archimedean solids, all Archimedean prisms,
all Archimedean antiprisms, J27, J34, J37, J72, J73, J74, J75 and J80.*

Acknowledgement

The authors thank Dr. Nobuaki Muto for computing the centroid of every Johnson polyhedron by computer search.

References

- [1] Maehara, H.: Geometry on Circles and Spheres. Asakura Shoten (1998), in Japanese.
- [2] Johnson, N.W.: Convex solids with regular faces. *Canad. J. Math.* 18 (1966), 169–200.

Jin Akiyama
Research Center for Science
and Math Education
Tokyo University of Science
e-mail: ja@jin-akiyama.com

Ikuro Sato
Department of Pathology
Research Institute
Miyagi Cancer Center, Japan
e-mail: sato-ik510@miyagi-pho.jp