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## A new class of permutation polynomials of $\mathbb{F}_q$

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Let  $\mathbb{F}_q$  be the finite field of characteristic  $p$  containing  $q = p^r$  elements. A polynomial  $f(x) \in \mathbb{F}_q[x]$  is called a permutation polynomial of  $\mathbb{F}_q$  if the induced map  $f : \mathbb{F}_q \rightarrow \mathbb{F}_q$  is one to one. The study of permutation polynomials goes back to Hermite [2] for  $\mathbb{F}_p$  and to Dickson [1] for  $\mathbb{F}_q$ . One of the open problems proposed by Lidl and Mullen [3], is to find new classes of permutation polynomials of  $\mathbb{F}_q$ . We refer to [4] or [5] for the basic results on permutation polynomials. Wan and Lidl [7], gave conditions on a polynomial of the form  $x^r f(x^{(q-1)/d})$  to be a permutation polynomial. The conditions are not explicitly given in terms of  $q$  and  $r$ , and may be difficult to verify in general. In the present note, without using the characterization of Wan and Lidl [7], but using only an elementary method, we exhibit a new class of permutation polynomials. We prove the following:

**Theorem 1** *Let  $q = p^r$ , where  $p$  is a prime number and  $r$  is a positive integer. Let  $u$  be a positive integer and let*

$$f(x) = x^u \left( x^{\frac{q-1}{2}} + x^{\frac{q-1}{4}} + 1 \right). \quad (1)$$

Unter einem Permutationspolynom des kommutativen Ringes  $R$  mit Einselement versteht man ein Polynom  $p \in R[x]$ , für welches die durch  $\pi\alpha := p(\alpha)$  definierte Abbildung von  $R$  nach  $R$  eine Permutation der Ringelemente ist. Permutationspolynome sind beliebte Studienobjekte der Zahlentheorie, der Algebra und der Kombinatorik. Am besten untersucht ist wohl der Fall, wenn  $R$  ein endlicher Körper ist. Die Autoren der vorliegenden Arbeit steuern zu dieser Theorie eine neue, einfache Klasse von Permutationspolynomen bei.

Assume that the following conditions hold:

- (i)  $\gcd(u, q-1) = 1$ .
- (ii)  $q \equiv 1 \pmod{8}$ .
- (iii)  $3^{\frac{q-1}{4}} \equiv 1 \pmod{p}$ .

Then  $f(x)$  is a permutation polynomial of  $\mathbb{F}_q$ .

*Proof.* We will prove that under the above conditions, the polynomial  $f$  induces a one-to-one application on  $\mathbb{F}_q$ . Suppose that  $f(a) = f(b)$  for some elements  $a$  and  $b$  of  $\mathbb{F}_q$ . If one of them, say  $a$ , is 0, then  $b^u(b^{\frac{q-1}{2}} + b^{\frac{q-1}{4}} + 1) = 0$ . Suppose that  $b \neq 0$ , then  $b^{\frac{q-1}{2}} + b^{\frac{q-1}{4}} + 1 = 0$ . Set  $c = b^{\frac{q-1}{4}}$ , then  $c^2 + c + 1 = 0$  and  $c$  is a cubic root of unity. Condition (iii) implies  $c \neq 1$ . We have  $c = c^4 = b^{q-1} = 1$ , which is a contradiction. It follows that  $b = 0 = a$ .

From now on we may suppose that  $ab \neq 0$ . It is clear that  $a^{\frac{q-1}{2}} = \pm 1$  and  $b^{\frac{q-1}{2}} = \pm 1$ . By symmetry, we have to consider only the following three cases:

*Case 1:* If  $a^{\frac{q-1}{2}} = b^{\frac{q-1}{2}} = 1$ . If  $a^{\frac{q-1}{4}} = b^{\frac{q-1}{4}} = 1$ , then  $a^u = b^u$ , hence  $(\frac{a}{b})^u = 1$ . Therefore  $a = b$  by (i). To complete Case 1, we may suppose that  $a^{\frac{q-1}{4}} = 1$  and  $b^{\frac{q-1}{4}} = -1$ . From equation (1) we have  $3a^u = b^u$  or  $(\frac{b}{a})^u = 3$ . We deduce that  $(\frac{b^{\frac{q-1}{4}}}{a^{\frac{q-1}{4}}}) = 3^{\frac{q-1}{4}}$ , hence  $(-1)^u = 1$  by (iii). By (i),  $u$  is odd and we reached a contradiction.

*Case 2:* If  $a^{\frac{q-1}{2}} = b^{\frac{q-1}{2}} = -1$ . From equation (1) we get:  $a^{u+\frac{q-1}{4}} = b^{u+\frac{q-1}{4}}$ , hence  $(b/a)^{u+\frac{q-1}{4}} = 1$ . The order  $\delta$  of  $b/a$  in  $\mathbb{F}_q$  divides  $q-1$  and  $u + \frac{q-1}{4}$ . Let  $l$  be a prime factor of  $\delta$ . Because  $u$  is odd and by (ii), we may exclude the case  $l = 2$ . It follows that  $l$  is odd and  $l \mid \frac{q-1}{4}$ , therefore  $l \mid u$ , which contradicts (i).

*Case 3:* If  $a^{\frac{q-1}{2}} = -b^{\frac{q-1}{2}} = 1$ . Here we have  $a^{\frac{q-1}{4}} = \pm 1$  and  $b^{\frac{q-1}{4}} = \zeta$ , where  $\zeta$  is a primitive quartic root of unity.

- If  $a^{\frac{q-1}{4}} = -1$  and  $b^{\frac{q-1}{4}} = \zeta$ , then by equation (1), we have  $a^u = \zeta b^u$ . We deduce that  $(a/b)^u = \zeta$ , therefore  $(a/b)^{4u} = 1$ . Using (i), we conclude that  $(a/b)^4 = 1$ . If  $a/b = -1$ , then  $a^{\frac{q-1}{2}} = (-1)^{\frac{q-1}{2}} b^{\frac{q-1}{2}} = b^{\frac{q-1}{2}}$ , which is a contradiction. Suppose next that  $a/b = \pm \zeta$ , then  $a^{\frac{q-1}{2}} = (\pm \zeta)^{\frac{q-1}{2}} b^{\frac{q-1}{2}}$ . Hence  $1 = (\zeta^4)^{\frac{q-1}{8}} (-1) = -1$ , which is a contradiction. We conclude that  $a = b$ .
- Suppose now that  $a^{\frac{q-1}{4}} = 1$  and  $b^{\frac{q-1}{4}} = \zeta$ , then by equation (1) we have  $3a^u = \zeta b^u$ . By condition (iii), the characteristic of the field is  $\neq 3$ , hence we may write this equation in the form:  $(a/b)^u = \zeta/3$ . It follows that  $(\frac{a^{\frac{q-1}{4}}}{b^{\frac{q-1}{4}}})^u = \frac{\zeta^{\frac{q-1}{4}}}{3^{\frac{q-1}{4}}}$ , hence  $3^{\frac{q-1}{4}} = -1$ , contradicting (iii).  $\square$

**Remark 1** The minimal example for Theorem 1 is when  $p = 7$  and  $q = 7^2$ .

**Example 1** Let  $p$  be a prime number such that  $p \equiv 1 \pmod{8}$  and  $p \equiv 1 \pmod{3}$  and let  $q = p^r$  where  $r$  is positive and even. It is clear that condition (ii) of Theorem 1 is satisfied. Euler criteria gives that  $3^{\frac{p-1}{2}} = \left(\frac{3}{p}\right) = 1$  (see [6]). It follows that  $3^{\frac{p-1}{4}} = \pm 1$ , hence  $3^{\frac{q-1}{4}} = (3^{\frac{p-1}{4}})^{(1+p+\dots+p^{r-1})} = 1$  and condition (iii) of Theorem 1 is fulfilled. By Dirichlet's theorem (see [6]), there exist infinitely many prime numbers  $p \equiv 1 \pmod{8}$  and  $p \equiv 1 \pmod{3}$ . The smallest such prime is 73. Any polynomial  $f(x)$  of the form (1) such that  $u$  satisfies condition (i) of Theorem 1 induces a permutation of  $\mathbb{F}_q$ . We may put  $u = 1$  for example.

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### References

- [1] Dickson, L.E.: The analytic representation of substitutions on a power of a prime number of letters with a discussion of the linear group. *Ann. of Math.* 11 (1896/97) 16, 161–183.
- [2] Hermite, C.: Sur les fonctions de sept lettres. *C.R. Acad. Sci. Paris* 57 (1863) 750–757.
- [3] Lidl, R.; Mullen, G.L.: When does a polynomial over a finite field permute the elements of the field? *Amer. Math. Monthly* 95 (1988) 243–246.
- [4] Lidl, R.; Niederreiter, H.: *Finite fields (Encyclopedia of Mathematics and its Applications)*. Cambridge Univ. Press, 2008.
- [5] Small, C.: *Arithmetic of finite fields*. Marcel Dekker, Inc., 1991.
- [6] Strayer, J.K.: *Elementary Number Theory*. PWS Publishing Company, Boston 1994.
- [7] Wan, D.Q.; Lidl, R.: Permutation polynomials of the form  $x^r f(x^{(q-1)/d})$  and their group structure. *Monatsh. Math.* 112 (1991) 2, 149–163.

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