Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 67 (2012)

Artikel: Iterative geometric triangle transformations

Autor: Vartziotis, Dimitris / Huggenberger, Simon

DOI: https://doi.org/10.5169/seals-283512

Elem. Math. 67 (2012) 68 – 83
0013-6018/12/020068-16
DOI 10.4171/EM/196

Elemente der Mathematik

Iterative geometric triangle transformations

Dimitris Vartziotis and Simon Huggenberger

Dimitris Vartziotis studied aerospace engineering and civil engineering at the University of Stuttgart, Germany. He is head of the Mathematics Research Department at TWT GmbH Science & Innovation, Stuttgart. Furthermore, he is currently researcher at the Institute of Structural Analysis & Antiseismic Research at the National Technical University of Athens, Greece.

Simon Huggenberger studied mathematics at the University of Tübingen, Germany, and did his Ph.D. at the University of Ghent, Belgium, in the field of discrete geometry. Currently, he is a member of the Mathematics Research Department at TWT GmbH Science & Innovation, Stuttgart.

0 Introduction and notations

A well-known theorem in geometry of triangles is the following:

> *If equilateral triangles are erected externally on the sides of any triangle, their centres form an equilateral triangle.*

This theorem can be found in [3] and is often attributed to Napoleon Bonaparte, although it is questionable whether he knew enough geometry for this achievement, see [1].

There are several generalisations of this theorem. One is the following, which can be found in [7, 4.] or [1, Theorem 3.36]:

Seit über einem Jahrhundert sind geometrische Transformationen von Dreiecken Gegenstand mathematischer Untersuchungen. In dem vorliegenden Artikel wird ein Ausgangsdreieck in ein neues Dreieck transformiert, indem auf jeder der Dreiecksseiten ein gleichschenkliges, nach außen gerichtetes Aufsatzdreieck errichtet wird. Die Scheitelpunkte der Aufsatzdreiecke bilden die Eckpunkte des neuen Dreiecks. Durch wiederholtes Anwenden dieser Transformation erhält man eine Folge von Dreiecken. Die Form der Dreiecke dieser Folge konvergiert gegen eine charakteristische Dreiecksform, von der nachgewiesen wird, dass sie mit jedem Transformationsschritt echt angenähert wird.

If similar triangles PCB, CQA, and BAR are erected externally on the sides of any triangle ABC, their circumcentres form a triangle similar to the three triangles.

I.M. Yaglom proves in [8, I.2, 22] the following generalisation of the theorem above:

On the sides of an arbitrary triangle ABC, exterior to it, isosceles triangles CBA_1, ACB_1, BAC_1 are erected with angles at the vertices A_1, B_1, and C_1, respectively equal to α, β, and γ. If $\alpha + \beta + \gamma = 2\pi$, then the angles of the triangle $A_1B_1C_1$ are equal to $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, and $\frac{1}{2}\gamma$.

Note that the case $\alpha = \beta = \gamma = \frac{2}{3}\pi$ is just the same as taking the centres of equilateral triangles. It is easy to check that these two generalisations are equivalent by decomposing each external triangle into three isosceles triangles that have a common vertex in the circumcentre (note that the angles at the circumcentre are just the double of the angles of the triangle).

Considering the formulation of Yaglom, we drop the condition $\alpha + \beta + \gamma = 2\pi$ and repeat the transformation to obtain an infinite sequence of triangles. In doing so, the angles α, β, and γ stay fixed. We analyse two cases. In the first case, all three angles are the same. In the second case, two angles coincide and the third equals π (hence, the corresponding erected triangle is degenerate). Equivalently to the second case, we may erect only two similar isosceles triangles and take the centre of the remaining side as the third vertex of the new triangle. We prove that in both cases, the shape of the triangles converge to the shape of the triangle one would get if the condition $\alpha + \beta + \gamma = 2\pi$ were satisfied. That is an equilateral triangle in the first case and a rectangular isosceles triangle in the second case.

In this article Δ_0 always denotes the initial triangle with vertices A_0, B_0, and C_0 (ordered counterclockwise). For $n \in \mathbb{N}$ the points A_{n+1}, B_{n+1}, and C_{n+1} are defined recursively such that $A_nC_nB_{n+1}$, $B_nA_nC_{n+1}$, $C_nB_nA_{n+1}$ are isosceles triangles. The triangle with vertices A_n, B_n, and C_n is denoted by Δ_n. The side-lengths of Δ_n are denoted by $x_n := \overline{B_nC_n}$, $y_n := \overline{C_nA_n}$, and $z_n := \overline{A_nB_n}$ and the angles are denoted by $\alpha_n := \angle B_nA_nC_n$, $\beta_n := \angle C_nB_nA_n$, and $\gamma_n := \angle A_nC_nB_n$.

We do not demand the triangle to be non-degenerate. The degenerate case where all three points are pairwise distinct but on a common line will be used to show that some given bounds are sharp. In the other degenerate cases there have to be vertices of the triangle that coincide and hence there has to be a side of length 0. Although such a side has no direction, this does not cause any problems for the iteration since in this case, the erected triangle always degenerates to a single point. Thus, the new vertex coincides with the two old ones. As a direct consequence, the degenerate case where all three points coincide is without any interest since such a triangle is invariant under all transformations introduced above. Therefore, we exclude the case $A_0 = B_0 = C_0$ for all of this article.

1 Equilateral case

In this section, the triangles erected externally on the sides of Δ_n are similar to each other. More precisely, there is an angle $0 < \theta < \pi/2$ such that $\angle A_{n+1}B_nC_n = \angle B_nC_nA_{n+1} =$

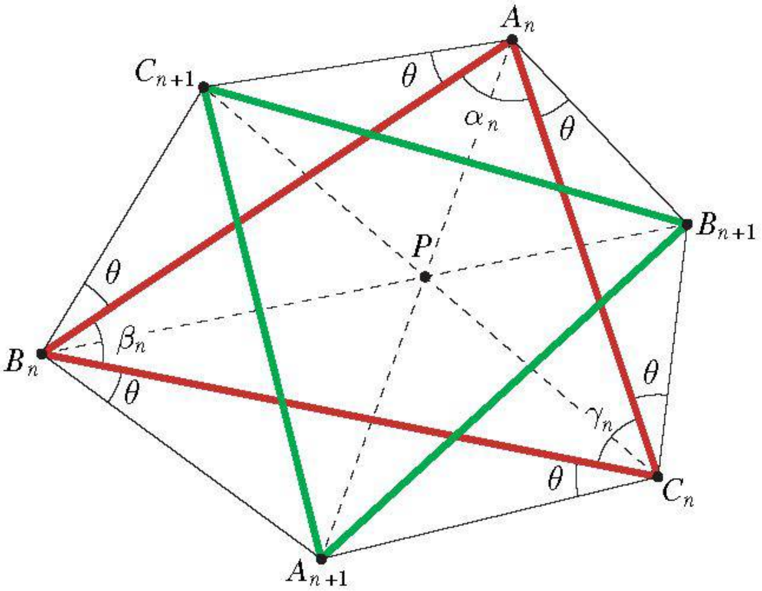

Fig. 1 Transformation with similar isosceles externally erected triangles

$\angle B_{n+1}C_nA_n = \angle C_nA_nB_{n+1} = \angle C_{n+1}A_nB_n = \angle A_nB_nC_{n+1} = \theta$. The so erected triangles are also known as Kiepert triangles, see [2]. The main result of this section can also be found in [5] where the convergence of this transformation is studied by considering the dominating eigenvalue. In this paper, we give a different proof of this result, which allows us a more detailed view at the process of convergence.

Remark 1.1. The lines A_nA_{n+1}, B_nB_{n+1}, and C_nC_{n+1} meet in a common point P, see [2] for a proof. Note that this point is inside the triangle Δ_n if and only if all angles of Δ_n are $< \pi - \theta$.

We first show how the side-lengths of the triangle Δ_{n+1} can be expressed in terms of the preceding triangle. Regarding the transformation there is no distinction between x_n, y_n, and z_n. Therefore we state the claims usually for only one instance, but we will use in the following the analogue statements as well.

Lemma 1.2. *Let E_n denote the area of Δ_n. Then the following identities hold:*

$$x_{n+1}^2 = \frac{1}{2}\tan^2\theta \cdot (y_n^2 + z_n^2) + \frac{1}{4}(1-\tan^2\theta)\cdot x_n^2 + 2\tan\theta \cdot E_n, \tag{1.1}$$

$$x_{n+1}^2 - y_{n+1}^2 = \frac{1}{4}(1-3\tan^2\theta)(x_n^2 - y_n^2). \tag{1.2}$$

Proof. Note that $\overline{A_nC_{n+1}} = 1/(2\cos\theta)\cdot z_n$ and $\overline{A_nB_{n+1}} = 1/(2\cos\theta)\cdot y_n$. Hence, applying the law of cosines to the triangle $A_nC_{n+1}B_{n+1}$ yields

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4\cos^2\theta}\cdot y_n^2 + \frac{1}{4\cos^2\theta}\cdot z_n^2 - 2\cdot\frac{1}{4\cos^2\theta}\cdot y_nz_n\cos(2\theta+\alpha_n) \\ &= \frac{1}{4\cos^2\theta}(y_n^2+z_n^2) - \frac{\cos(2\theta+\alpha_n)}{2\cos^2\theta}\cdot y_nz_n\,. \end{aligned}$$

Using the addition theorems for the cosine and the sine we obtain $\cos(2\theta+\alpha_n) = (\cos^2\theta - \sin^2\theta)\cos\alpha_n - 2\sin\theta\cos\theta\sin\alpha_n$. Applying this and $E_n = (\sin\alpha_n/2)y_n z_n$ to the identity above leads to

$$\begin{aligned} x_{n+1}^2 &= \frac{1}{4\cos^2\theta}(y_n^2+z_n^2) - \frac{(1-\tan^2\theta)}{2}\cos\alpha_n \cdot y_n z_n + \tan\theta\sin\alpha_n \cdot y_n z_n \\ &= \frac{1}{4\cos^2\theta}(y_n^2+z_n^2) - \frac{1-\tan^2\theta}{2}\cdot\frac{y_n^2+z_n^2-x_n^2}{2} + 2\tan\theta\cdot E_n \\ &= \left(\frac{1-\cos^2\theta}{4\cos^2\theta} + \frac{1}{4}\tan^2\theta\right)\cdot(y_n^2+z_n^2) + \frac{1}{4}(1-\tan^2\theta)\cdot x_n^2 + 2\tan\theta\cdot E_n \\ &= \frac{1}{2}\tan^2\theta\cdot(y_n^2+z_n^2) + \frac{1}{4}(1-\tan^2\theta)\cdot x_n^2 + 2\tan\theta\cdot E_n. \end{aligned}$$

The second equation is a direct consequence of the first one together with its analogue for y_{n+1}. □

Corollary 1.3. *Let $x_n \geq y_n$. Then*

$$\begin{aligned} x_{n+1} &\geq y_{n+1} \qquad \text{if} \quad \theta \leq \frac{\pi}{6} \quad \text{and} \\ x_{n+1} &\leq y_{n+1} \qquad \text{if} \quad \theta \geq \frac{\pi}{6}, \end{aligned}$$

where equality on the left-hand sides holds if and only if $\theta = \pi/6$ or $x_n = y_n$.

Proof. This is a direct consequence of (1.2) since $\tan^2(\pi/6) = 1/3$. □

Remark 1.4. We may assume that in the initial triangle x_0 is the greatest side and z_0 is the smallest. If $\theta < \pi/6$, the corollary above implies that x_n is the greatest side of Δ_n and z_n is the smallest one for every n. If $\theta > \pi/6$, things are different. For even n, we still have $x_n \geq y_n \geq z_n$, whereas $x_n \leq y_n \leq z_n$ holds for odd n. In the special case $\theta = \pi/6$, the triangle Δ_n is equilateral for every $n > 0$. This observation is a motivation to consider the two subsequences $(\Delta_{2n})_{n\in\mathbb{N}}$ and $(\Delta_{2n+1})_{n\in\mathbb{N}}$ sometimes separately.

To study the behaviour of corresponding side-lengths during the iteration, we state some estimates. First, note the following two simple inequalities, which will be used several times.

Lemma 1.5. *Let r and s be two positive real numbers. Then $r^2+s^2 \geq \frac{1}{2}(r+s)^2$ and $r^2+s^2 \geq 2rs$.*

Proof. Since $(r-s)^2$ is positive, we obtain $4r^2+4s^2 \geq (4r^2-(r-s)^2)+(4s^2-(s-r)^2) = (3r-s)(r+s)+(3s-r)(s+r) = (2r+2s)(r+s)$. Subtracting $2(r^2+s^2)$ on both sides implies the second claim. □

Lemma 1.6. *For the corresponding side-lengths of subsequent triangles, the following lower bounds hold:*

$$x_{n+1} \geq \frac{1}{2} x_n, \tag{1.3}$$

$$x_{n+2}^2 \geq \frac{1}{16}(9\tan^4\theta + 1) \cdot x_n^2. \tag{1.4}$$

Moreover, if Δ_n is non-degenerate, both bounds are strict.

Proof. By Lemma 1.5 we obtain $y_n^2 + z_n^2 \geq \frac{1}{2}(y_n + z_n)^2 \geq \frac{1}{2}x_n^2$. Applying this to (1.1) yields

$$x_{n+1}^2 \geq \frac{1}{4}\tan^2\theta \cdot x_n^2 + \frac{1}{4}(1-\tan^2\theta)\cdot x_n^2 + 2\tan\theta \cdot E_n \geq \frac{1}{4}x_n^2.$$

Now the first inequality follows directly. In the last step we subtracted $2\tan\theta \cdot E_n$. Since $\tan\theta > 0$, equality cannot occur if Δ_n is non-degenerate. Using the analogues of (1.1) provides the following identity:

$$\begin{aligned} y_{n+1}^2 + z_{n+1}^2 &= \frac{\tan^2\theta}{2}(2x_n^2 + y_n^2 + z_n^2) + \frac{1-\tan^2\theta}{4}(y_n^2+z_n^2) + 4\tan\theta \cdot E_n \\ &= \tan^2\theta \cdot x_n^2 + \frac{1+\tan^2\theta}{4}(y_n^2+z_n^2) + 4\tan\theta\cdot E_n. \end{aligned}$$

We apply this to the analogue of (1.1) for x_{n+2}^2:

$$\begin{aligned} x_{n+2}^2 \geq{}& \frac{1}{2}\tan^2\theta\left(\tan^2\theta \cdot x_n^2 + \frac{1+\tan^2\theta}{4}(y_n^2+z_n^2) + 4\tan\theta\cdot E_n\right) \\ &+ \frac{1-\tan^2\theta}{4}\left(\frac{\tan^2\theta}{2}(y_n^2+z_n^2) + \frac{1-\tan^2\theta}{4}\cdot x_n^2 + 2\tan\theta\cdot E_n\right) \\ ={}& \frac{\tan^2\theta}{4}(y_n^2+z_n^2) + \frac{8\tan^4\theta + (1-\tan^2\theta)^2}{16}\cdot x_n^2 + \frac{3\tan^3\theta+\tan\theta}{2}\cdot E_n \\ \geq{}& \frac{\tan^2\theta}{8}\cdot x_n^2 + \frac{9\tan^4\theta - 2\tan^2\theta + 1}{16}\cdot x_n^2 \\ ={}& \frac{9}{16}\tan^4\theta\cdot x_n^2 + \frac{1}{16}x_n^2. \end{aligned}$$

In the last but one step we subtracted $(3\tan^3\theta + \tan\theta)E_n/2$. Since $3\tan^3\theta + \tan\theta > 0$, equality cannot occur if Δ_n is non-degenerate. □

The first lower bound given in the previous lemma is sharp as one can check by considering the degenerate case $\alpha_n = \pi$ and $y_n = z_n$. The second lower bound is sharp, too. This can be seen by considering again the degenerate case $\alpha_n = \pi$ and $y_n = z_n$ while θ tends to 0. As a consequence of these two estimates, we state the following theorems.

Theorem 1.7. *For $n \geq 1$, the side-lengths of Δ_n are all > 0.*

Proof. We may assume that Δ_{n-1} is degenerate since otherwise the claim follows directly from (1.3).

Since Δ_0 has at least one side of length > 0, (1.3) implies that every triangle has at least one side of length > 0. Suppose $x_n = 0$. Then (1.3) yields $x_{n-1} = 0$. Thus, $E_{n-1} = 0$ and consequently, $x_n^2 = \tan^2\theta \cdot (y_{n-1}^2 + z_{n-1}^2)/2$ by (1.1). We know $y_{n-1}^2 + z_{n-1}^2 > 0$ since otherwise all vertices of Δ_{n-1} would coincide. With $\tan^2\theta > 0$ we obtain $x_n > 0$, a contradiction. □

Theorem 1.8. *We have:*

$$\begin{array}{lll} \textit{If} \quad 0 < \theta < \frac{\pi}{6}, & \textit{then} & \lim_{n\to\infty} x_n = 0. \\ \textit{If} \quad \frac{\pi}{6} < \theta < \frac{\pi}{2}, & \textit{then} & \lim_{n\to\infty} x_n = \infty. \\ \textit{If} \quad \theta = \frac{\pi}{6}, & \textit{then} & x_n = x_1 \qquad \forall n > 0. \end{array}$$

Proof. First let $0 < \theta < \pi/6$. Assume $x_0 \geq y_0 \geq z_0$. Then $x_n \geq y_n \geq z_n$ for every n by Remark 1.4. Hence, $\lim_{n\to\infty} x_n = 0$ implies $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$. Thus, it suffices to prove the claim for the case $x_0 \geq y_0 \geq z_0$. Since $x_n \geq y_n \geq z_n$, we obtain $\alpha_n \geq \beta_n \geq \gamma_n$ by the law of sines. This implies $\gamma_n \leq \pi/6$ and hence, $\sin\gamma_n \leq \sqrt{3}/2$. With $E_n = (\sin\gamma_n/2)x_n y_n$, we obtain by (1.1)

$$\begin{aligned} x_{n+1}^2 &\leq \frac{1}{2}\tan^2\theta \cdot (y_n^2 + z_n^2) + \frac{1}{4}(1 - \tan^2\theta)\cdot x_n^2 + \frac{\sqrt{3}}{2}\tan\theta \cdot x_n y_n \\ &\leq \left(\frac{1}{4} + \frac{3}{4}\tan^2\theta + \frac{\sqrt{3}}{2}\tan\theta\right)\cdot x_n^2 . \end{aligned}$$

Now $\theta < \pi/6$ implies $\tan\theta < \sqrt{3}/3$ and consequently $\frac{1}{4} + \frac{3}{4}\tan^2\theta + \frac{\sqrt{3}}{2}\tan\theta < 1$. The claim follows.

Now let $\pi/6 < \theta < \pi/2$. Assume $x_0 \leq y_0 \leq z_0$. Then $x_{2n} \leq y_{2n} \leq z_{2n}$ for every n by Remark 1.4. Hence, $\lim_{n\to\infty} x_{2n} = \infty$ implies $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} z_{2n} = \infty$. Thus, to prove $\lim_{n\to\infty} x_{2n} = \infty$ it suffices to consider the case $x_0 \leq y_0 \leq z_0$. Let n be even. Since $x_n \leq y_n \leq z_n$, we obtain $\alpha_n \leq \beta_n \leq \gamma_n$. This implies $\gamma_n \geq \pi/6$ and hence, $\sin\gamma_n \geq \sqrt{3}/2$. With $E_n = (\sin\gamma_n/2)x_n y_n$, we obtain by (1.1)

$$\begin{aligned} z_{n+1}^2 &\geq \frac{1}{2}\tan^2\theta \cdot (x_n^2 + y_n^2) + \frac{1}{4}(1 - \tan^2\theta)\cdot z_n^2 + \frac{\sqrt{3}}{2}\tan\theta \cdot x_n y_n \\ &\geq \left(\frac{1}{4} + \frac{3}{4}\tan^2\theta + \frac{\sqrt{3}}{2}\tan\theta\right)\cdot x_n^2 . \end{aligned}$$

Now $\theta < \pi/6$ implies $\tan\theta < \sqrt{3}/3$ and consequently $\kappa := \frac{1}{4} + \frac{3}{4}\tan^2\theta + \frac{\sqrt{3}}{2}\tan\theta > 1$. Since $z_{n+1} \leq y_{n+1} \leq x_{n+1}$ by Remark 1.4, we conclude analogously $x_{n+2}^2 \geq \kappa z_{n+1}^2$. Thus, $x_{n+2}^2 \geq \kappa^2 x_n^2$ whenever n is even and consequently $\lim_{n\to\infty} x_{2n} = \infty$. By analogous reasons we obtain $\lim_{n\to\infty} x_{2n+1} = \infty$ and the claim follows.

In the last case, we conclude by Corollary 1.3 that for every $n > 0$ the triangle Δ_n is equilateral. Furthermore, $\tan\theta = \sqrt{3}/3$. Thus, for $n > 0$, (1.1) yields $x_{n+1}^2 = x_n^2$. $\square$

Due to the unbounded growth of the triangles for $\theta > \pi/6$ and the fact that for $\theta < \pi/6$, the triangle sequence collapses to a single point, the only reasonable case to study seems to be the case where θ equals $\pi/6$. However, since we are interested in the shape of the triangle, the size of the triangle does not matter.

We state another two simple inequalities for our further estimates.

Lemma 1.9. *Let r, s, and t be positive real numbers all smaller than 1. Then $1 - s^2 < r(1 - t^2)$ implies $(1 - s) < r(1 - t)$.*

Proof. We obtain $1 - s^2 < 1 - t^2$ and therefore $s > t$. Thus, $(1 - s) = (1 - s^2)/(1 + s) < r(1 - t^2)/(1 + s) < r(1 - t^2)/(1 + t) < r(1 - t)$. $\square$

The following general proposition applies separately to the two triangle sequences, i.e. the one with the even and the one with the odd indices. As in Remark 1.4 we assume $x_0 \geq y_0$ for the following proposition.

Proposition 1.10. *Let $x_0 > y_0$. Then for every angle θ, there is a constant $0 \leq \kappa < 1$ such that*

$$0 \leq \left|1 - \frac{y_{n+2}}{x_{n+2}}\right| \leq \kappa \left|1 - \frac{y_n}{x_n}\right|,$$

where equality holds if and only if $\theta = \pi/6$ or $y_n = x_n$.

Proof. First note that for $n \geq 1$, Theorem 1.7 states $x_n > 0$. Furthermore, since $x_0 \geq y_0$ and the case $A_0 = B_0 = C_0$ is excluded, we obtain $x_0 > 0$. Now, (1.2) provides

$$x_{n+2}^2 - y_{n+2}^2 = \left(\frac{1}{4} - \frac{3}{4}\tan^2\theta\right)^2 \cdot (x_n^2 - y_n^2)$$

and hence,

$$1 - \frac{y_{n+2}^2}{x_{n+2}^2} = \frac{(1 - 3\tan^2\theta)^2}{16} \cdot \frac{x_n^2}{x_{n+2}^2}\left(1 - \frac{y_n^2}{x_n^2}\right).$$

Thus, we may assume $x_n \neq y_n$ since otherwise we are done. Applying (1.4) yields

$$\left|1 - \frac{y_{n+2}^2}{x_{n+2}^2}\right| \leq \frac{(1 - 3\tan^2\theta)^2}{1 + 9\tan^4\theta}\left|1 - \frac{y_n^2}{x_n^2}\right|.$$

Since $0 \leq (1 - 3\tan^2\theta)^2 < 1 + 9\tan^4\theta$ the claim follows for $\kappa := (1 - 3\tan^2\theta)^2/(1 + 9\tan^4\theta)$ by Lemma 1.9. Note that $\kappa = 0$ if and only if $\theta = \pi/6$. $\square$

For $\theta \leq \pi/4$, the ratio of side-lengths tends to 1 in every step of the iteration. In other terms, the function $n \mapsto 1 - \min\{x_n, y_n\}/\max\{x_n, y_n\}$ is strictly decreasing as long as the values differ from 0. Note that for $\theta > \pi/6$, the role of the smaller side-length alternates.

Proposition 1.11. *Let $x_0 > y_0$ and $\theta \neq \pi/6$. Then there is a positive constant $\kappa < 1$ that only depends on θ such that the following holds:*

$$0 < 1 - \frac{y_{n+1}}{x_{n+1}} \leq \kappa \left(1 - \frac{y_n}{x_n}\right) \qquad \text{if} \quad \theta < \frac{\pi}{6},$$

$$0 < 1 - \frac{x_{n+1}}{y_{n+1}} \leq \kappa \left(1 - \frac{y_n}{x_n}\right) \qquad \text{if} \quad \frac{\pi}{6} < \theta \leq \frac{\pi}{4} \text{ and } n \text{ even},$$

$$0 < 1 - \frac{y_{n+1}}{x_{n+1}} \leq \kappa \left(1 - \frac{x_n}{y_n}\right) \qquad \text{if} \quad \frac{\pi}{6} < \theta \leq \frac{\pi}{4} \text{ and } n \text{ odd}.$$

Proof. First note that $x_n > 0$ and $y_n > 0$ for $n \geq 1$ by Theorem 1.7. Moreover, $x_0 > 0$ since $x_0 \geq y_0$ and the case $A_0 = B_0 = C_0$ is excluded.

We set $\omega := 1 - 3\tan^2\theta$. First assume $\theta < \pi/6$. Then $x_n > y_n$ by Corollary 1.3. Furthermore, $0 < \tan^2\theta < 1/3$ and therefore $0 < \omega < 1$. By (1.3) we know $x_n \leq 2x_{n+1}$. Hence, dividing both sides of (1.2) by x_{n+1}^2 provides

$$1 - \frac{y_{n+1}^2}{x_{n+1}^2} = \frac{\omega}{4} \cdot \frac{x_n^2}{x_{n+1}^2}\left(1 - \frac{y_n^2}{x_n^2}\right) \leq \omega \left(1 - \frac{y_n^2}{x_n^2}\right).$$

Now the claim follows from Lemma 1.9 by setting $\kappa := \omega$.

For $\theta > \pi/6$, we know by Corollary 1.3 that $x_n > y_n$ if n is even and $y_n > x_n$ otherwise. We restrict ourselves to the case where n is even. The other case can be obtained by exchanging x_n with y_n, x_{n+1} with y_{n+1}, and z_0 with z_1. Note that $\omega < 0$ for $\theta > \pi/6$.

We assume $\pi/6 < \theta < \pi/4$. Then $\tan^2\theta \leq 1$. Hence, (1.1) yields $y_{n+1}^2 \geq \frac{1}{2}(x_n^2 + z_n^2)\tan^2\theta$. Furthermore, $\tan^2\theta \leq 1$ implies $-\omega \leq 2\tan^2\theta$. We divide both sides of (1.2) by $-y_{n+1}^2$ and obtain

$$\begin{aligned} 1 - \frac{x_{n+1}^2}{y_{n+1}^2} &= \frac{-\omega}{4} \cdot \frac{x_n^2}{y_{n+1}^2}\left(1 - \frac{y_n^2}{x_n^2}\right) \\ &\leq \frac{\tan^2\theta}{2} \cdot \frac{2x_n^2}{(x_n^2 + z_n^2)\tan^2\theta}\left(1 - \frac{y_n^2}{x_n^2}\right) \\ &= \frac{x_n^2}{x_n^2 + z_n^2}\left(1 - \frac{y_n^2}{x_n^2}\right). \end{aligned}$$

Now set $\varepsilon := \min\{1, (z_0/x_0)^2\}$. Then Proposition 1.10 together with induction implies $z_n^2 \geq \varepsilon x_n^2$. The claim follows for $\kappa := 1/(1+\varepsilon)$ by using Lemma 1.9. □

Remark 1.12. For $\theta > \pi/4$ and $x_n > y_n$, it is possible that $1 - x_{n+1}/y_{n+1}$ exceeds $1 - y_n/x_n$, especially if θ is close to $\pi/2$. However, in this situation there is another observation one can make: While θ tends to $\pi/2$, the angle α_{2n} tends to α_0 for every $n \in \mathbb{N}$. Analogously, $\lim_{\theta\to\pi/2} \beta_{2n} = \beta_0$ and $\lim_{\theta\to\pi/2} \gamma_{2n} = \gamma_0$. Hence, the shape of Δ_2 tends to the shape of Δ_0. On the other hand $\lim_{\theta\to\pi/2} x_1 = \lim_{\theta\to\pi/2} y_1 = \lim_{\theta\to\pi/2} z_1 = \infty$ as long as Δ_0 is non-degenerate. Thus, one cannot speak of a limit triangle.

To avoid the enormous growth of the triangles, one can dilate each transformed triangle after the iteration with the reciprocal of the largest side-length. Equivalently, one can apply this dilation before the step of iteration. By doing so, Δ_1 converges pointwise while θ tends to $\pi/2$ as long as we take a fixed centre for the dilations. Thus, we obtain a limit triangle which we call Δ_1'. Repeating this process leads to two sequences of pairwise similar triangles Δ_{2n}' and Δ_{2n+1}'. The triangles Δ_0 and Δ_1' do not have to be similiar.

Clearly, while θ tends to $\pi/2$, the factor of the dilation we apply to Δ_0 tends to 0. Thus, the vertices A_1', B_1', and C_1' of Δ_1' lie on the lines through the dilation centre that are perpendicular to one of the sides of Δ_0. Moreover, the proportions of the distances from the dilation centre to A_1', B_1', C_1' match the proportions of z_1, y_1, x_1. Hence, a triangle similar to Δ_1' can be obtained by taking three concurrent rays r_0, r_1, r_2 such that r_0 and r_1 span the angle $2\alpha_0$, r_0 and r_2 span the angle $2\beta_0$, and r_1 and r_2 span the angle $2\gamma_0$. Taking the points on r_0, r_1, r_2 at distance z_0, y_0, x_0, respectively, to the intersection of the three rays provides a triangle similar to Δ_1'.

Since Δ_2' is similar to Δ_0 again, the shape of Δ_1' can be seen as some kind of dual shape to the shape of Δ_0.

The following theorem is our main result, namely, regarding only the shape of the triangles, Δ_n tends to an equilateral triangle for $n \to \infty$.

Theorem 1.13. *For every initial triangle Δ_0 and every angle $0 < \theta < \pi/2$, the following two limits hold:*

$$\lim_{n\to\infty} \frac{x_n}{y_n} = 1, \qquad \lim_{n\to\infty} \alpha_n = \frac{\pi}{3}.$$

Proof. The first limit is a direct consequence of Proposition 1.10. The second limit follows by the first together with the law of sines. □

We conclude this section by stating two theorems concerning the position and the orientation of the triangles.

Theorem 1.14. *For every $n > 0$, the centroid of Δ_n coincides with the centroid of Δ_0.*

Proof. We consider the Euclidean plane as vector space. For $n \in \mathbb{N}$, let a_n, b_n, and c_n be the vectors representing the points A_n, B_n, and C_n, respectively. Let δ be the linear transformation that rotates the Euclidean plane by $\pi/2$. Then $a_{n+1} = \frac{1}{2}(b_n + c_n) + (\frac{1}{2}\tan\theta(b_n - c_n))^\delta$. Since δ is linear, this implies $a_{n+1} + b_{n+1} + c_{n+1} = a_n + b_n + c_n$. Thus, the centroid of Δ_n, defined as $\frac{1}{3}(a_n + b_n + c_n)$ coincides with the one of Δ_{n+1}. The claim follows by induction. □

Theorem 1.15. *For every $n > 0$, the triangle Δ_n is non-degenerate and counterclockwise oriented.*

Proof. Assume that Δ_n is non-degenerate and counterclockwise oriented. By symmetric reasons we may assume $x_n \geq y_n \geq z_n$. Hence, $\alpha_n \geq \beta_n \geq \gamma_n$ by the law of sines and therefore $\beta_n \leq \pi/2$ and $\gamma_n \leq \pi/2$.

Let A'_n, B'_n, and C'_n denote the centres of B_nC_n, C_nA_n, and A_nB_n, respectively. Since the triangle $A'_nC_nB'_n$ is similar to Δ_n, we obtain $\angle C_nA'_nB'_n = \beta_n$. On the other hand, let l be the perpendicular bisector of the side C_nA_n, i. e. the line through B'_n and B_{n+1}. Since $x_n \geq z_n$, the line l intersects x_n in a point S. We obtain $\angle C_nSB_{n+1} = \pi/2 - \gamma_n$. Thus, $\min\{\beta_n, \pi/2 - \gamma_n\} \leq \angle C_nA'_nB_{n+1} \leq \max\{\beta_n, \pi/2 - \gamma_n\}$ and therefore $0 < \angle C_nA'_nB_{n+1} < \pi/2$. Analogously, $0 < \angle C_{n+1}A'_nB_n < \pi/2$. This implies that the angles $\angle C_{n+1}A'_nA_{n+1}$ and $\angle A_{n+1}A'_nB_{n+1}$ are greater than $\pi/2$ and smaller than π and consequently, $\angle B_{n+1}A'_nC_{n+1} < \pi$. We conclude that A'_n is inside the triangle Δ_{n+1} and Δ_{n+1} is counterclockwise oriented since $\angle A_{n+1}A'_nB_{n+1} < \pi$. Now the claim follows by induction. □

2 Rectangular isosceles case

As in the previous section, the points B_{n+1} and C_{n+1} are the apices of similar isosceles triangles erected to the outside of Δ_n over the edges y_n and z_n, respectively. More precisely, there is an angle $0 < \theta < \pi/2$ such that $\angle B_{n+1}C_nA_n = \angle C_nA_nB_{n+1} = \angle C_{n+1}A_nB_n = \angle A_nB_nC_{n+1} = \theta$. In contrast to the previous section, the point A_{n+1} is the centre of B_nC_n (or, equivalently, the apex of a degenerate isosceles triangle with angle π).

Again, we first give equations for the side-lengths of the triangle Δ_{n+1} in terms of Δ_n. Regarding the transformation there is no distinction between y_n and z_n except for the orientation. Therefore we state the claims usually for only one instance, but we will use the analogue statements as well in the following.

Lemma 2.1. *Let E_n be the area of Δ_n. Then the following identities hold:*

$$x_{n+1}^2 = \frac{1}{2}\tan^2\theta \cdot (y_n^2 + z_n^2) + \frac{1}{4}(1 - \tan^2\theta) \cdot x_n^2 + 2\tan\theta \cdot E_n, \tag{1.1}$$

$$y_{n+1}^2 = \frac{1}{4} \cdot y_n^2 + \frac{1}{4}\tan^2\theta \cdot z_n^2 + \tan\theta \cdot E_n. \tag{2.1}$$

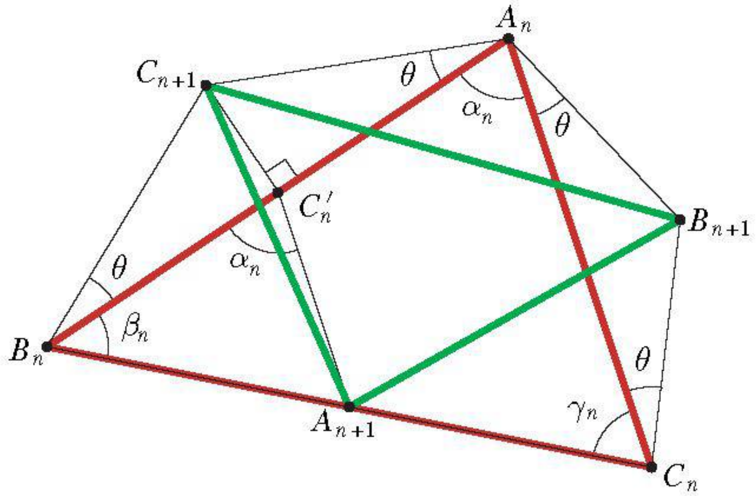

Fig. 2 Transformation with two similar isosceles externally erected triangles and one midpoint of a side

Proof. The first equation is obtained in precisely the same way as in the proof of Lemma 1.2.

Let C_n' be the centre of A_nB_n. Then $\overline{C_{n+1}C_n'} = \frac{1}{2}\tan\theta \cdot z_n$ and $\overline{A_{n+1}C_n'} = \frac{1}{2} \cdot y_n$. Applying the law of cosines to the triangle $A_{n+1}C_n'C_{n+1}$ (possibly degenerate for $\alpha_n = \pi/2$ and oriented clockwise for $\alpha_n > \pi/2$) yields

$$y_{n+1}^2 = \frac{1}{4} \cdot y_n^2 + \frac{1}{4}\tan^2\theta \cdot z_n^2 - 2 \cdot \frac{1}{4}\tan\theta \cdot y_n z_n \cos\left(\frac{\pi}{2} + \beta_n + \gamma_n\right).$$

With $\cos(\pi/2 + \beta_n + \gamma_n) = \sin(-\beta_n - \gamma_n) = \sin(\alpha_n - \pi/2) = -\sin(\alpha_n)$ and $E_n = \frac{1}{2}\sin\alpha_n \cdot y_n z_n$, the second identity follows. □

The following identities are immediate consequences of the previous lemma.

$$y_{n+1}^2 - z_{n+1}^2 = \frac{1}{4}(1 - \tan^2\theta) \cdot (y_n^2 - z_n^2), \tag{2.2}$$

$$y_{n+1}^2 + z_{n+1}^2 = \frac{1}{4}(1 + \tan^2\theta) \cdot (y_n^2 + z_n^2) + 2\tan\theta \cdot E_n, \tag{2.3}$$

$$x_{n+1}^2 - y_{n+1}^2 - z_{n+1}^2 = \frac{1}{4}(1 - \tan^2\theta) \cdot (x_n^2 - y_n^2 + z_n^2). \tag{2.4}$$

Corollary 2.2. *Let $y_n \geq z_n$. Then*

$$y_{n+1} \geq z_{n+1} \quad \textit{if} \quad \theta \leq \frac{\pi}{4} \quad \textit{and}$$
$$y_{n+1} \leq z_{n+1} \quad \textit{if} \quad \theta \geq \frac{\pi}{4},$$

where equality on the left-hand side holds if and only if $\theta = \pi/4$ or $y_n = z_n$.

Proof. This is a direct consequence of (2.2) since $\tan(\pi/4) = 1$. □

Since A_{n+1} is obtained in a different way than B_{n+1} and C_{n+1}, there is no corresponding condition that involves x_n and x_{n+1}. Our next step is to give lower bounds for the side-lengths after two steps of iteration.

Lemma 2.3. *For the corresponding side-lengths of subsequent triangles, the following lower bounds hold:*

$$x_{n+2}^2 \geq \frac{1}{16}(1 + \tan^4\theta)x_n^2, \tag{2.5}$$

$$y_{n+2}^2 \geq \frac{1}{16}(1 + \tan^4\theta)y_n^2. \tag{2.6}$$

Proof. We apply (2.3) and (1.1) to the analogue of (1.1) for x_{n+2}^2:

$$\begin{aligned} x_{n+2}^2 \geq{}& \frac{\tan^2\theta}{2}\left(\frac{1+\tan^2\theta}{4}\cdot(y_n^2+z_n^2)+2\tan\theta\cdot E_n\right) \\ &+\frac{1-\tan^2\theta}{4}\left(\frac{\tan^2\theta}{2}\cdot(y_n^2+z_n^2)+\frac{1-\tan^2\theta}{4}\cdot x_n^2+2\tan\theta\cdot E_n\right) \\ ={}& \frac{\tan^2\theta}{4}\cdot(y_n^2+z_n^2)+\frac{(1-\tan^2\theta)^2}{16}\cdot x_n^2+\frac{\tan\theta+\tan^3\theta}{2}\cdot E_n. \end{aligned}$$

Now Lemma 1.5 implies $y_n^2+z_n^2 \geq (y_n+z_n)^2/2 \geq x_n^2/2$ and thus,

$$\begin{aligned} x_{n+2}^2 &\geq \frac{\tan^2\theta}{8}\cdot x_n^2+\frac{1-2\tan^2\theta+\tan^4\theta}{16}\cdot x_n^2 \\ &= \frac{1}{16}x_n^2+\frac{1}{16}\tan^4\theta\cdot x_n^2. \end{aligned}$$

Using (2.1) repeatedly yields

$$\begin{aligned} y_{n+2}^2 &\geq \frac{1}{4}y_{n+1}^2+\frac{1}{4}\tan^2\theta\cdot z_{n+1}^2 \\ &\geq \frac{1}{16}y_n^2+\frac{1}{8}\tan^2\theta\cdot z_{n+1}^2+\frac{1}{16}\tan^4\theta\cdot y_n^2 \\ &\geq \frac{1}{16}y_n^2+\frac{1}{16}\tan^4\theta\cdot y_n^2. \end{aligned}$$ □

As in the previous section, the lemma above motivates us to consider the sequence of triangles as two separated sequences.

Theorem 2.4. *For $n \geq 1$, the side-lengths of Δ_n are all > 0.*

Proof. The side-length x_n does not depend on A_n and hence, for a given triangle Δ_{n-1} and a given angle θ, the side-length x_n is just the same as in the previous section. Thus, $x_n > 0$ by Theorem 1.7.

For y_n and z_n, we may assume that Δ_{n-1} is degenerate since otherwise the claim follows directly from (2.1).

Since Δ_0 has at least two sides of length > 0, (2.1) implies $y_1 > 0$ and analogously, $z_1 > 0$. Now the claim follows by induction using (2.1). □

We proceed by studying the ratio of corresponding side-lengths.

Proposition 2.5. *Let $y_0 \geq z_0$. Then for every angle $0 < \theta < \pi/2$, there is a constant $0 \leq \kappa < 1$ such that*

$$0 \leq \left|1-\frac{z_{n+2}}{y_{n+2}}\right| \leq \kappa\left|1-\frac{z_n}{y_n}\right|,$$

where equality holds if and only $\theta = \pi/4$ or $y_n = z_n$.

Proof. Note that $y_n > 0$ for $n \geq 1$ by Theorem 2.4. Moreover, $y_0 > 0$ since otherwise $z_0 = y_0 = 0$ and hence, $A_0 = B_0 = C_0$. Now, (2.2) provides

$$y_{n+2}^2 - z_{n+2}^2 = \left(\frac{1}{4} - \frac{1}{4}\tan^2\theta\right)^2 \cdot (y_n^2 - z_n^2)$$

and hence,

$$1 - \frac{z_{n+2}^2}{y_{n+2}^2} = \frac{(1-\tan^2\theta)^2}{16} \cdot \frac{y_n^2}{y_{n+2}^2}\left(1 - \frac{z_n^2}{y_n^2}\right).$$

We may assume $y_n \neq z_n$ since otherwise we are done. Applying (2.6) yields

$$\left|1 - \frac{z_{n+2}^2}{y_{n+2}^2}\right| \leq \frac{(1-\tan^2\theta)^2}{1+\tan^4\theta}\left|1 - \frac{z_n^2}{y_n^2}\right|.$$

Since $0 \leq (1-\tan^2\theta)^2 < 1+\tan^4\theta$ the claim follows for $\kappa := (1-\tan^2\theta)^2/(1+\tan^4\theta)$ by Lemma 1.9. Note that $\kappa = 0$ if and only if $\theta = \pi/4$. □

Proposition 2.6. *For every angle θ, there is a constant $0 \leq \kappa < 1$ such that*

$$0 \leq \left|1 - \frac{y_{n+2}^2 + z_{n+2}^2}{x_{n+2}^2}\right| \leq \kappa \cdot \left|1 - \frac{y_n^2 + z_n^2}{x_n^2}\right|,$$

where equality holds if and only $\theta = \pi/4$ or $x_n^2 = y_n^2 + z_n^2$.

Proof. By Theorem 2.4, the only possibility where one of the fractions is not defined is the case $n = 0$ and $x_0 = 0$. In this case, the term on the right-hand side can be understood as a term of inifinite value, which makes the claim obviously true for this case. Formula (2.4) provides

$$x_{n+2}^2 - y_{n+2}^2 - z_{n+2}^2 = \left(\frac{1}{4} - \frac{1}{4}\tan^2\theta\right)^2 \cdot (x_n^2 - y_n^2 - z_n^2)$$

and hence,

$$1 - \frac{y_{n+2}^2 + z_{n+2}^2}{x_{n+2}^2} = \frac{\left(1-\tan^2\theta\right)^2}{16} \cdot \frac{x_n^2}{x_{n+2}^2}\left(1 - \frac{y_n^2 + z_n^2}{x_n^2}\right).$$

We may assume $x_n^2 \neq y_n^2 + z_n^2$ since otherwise we are done. Applying (2.5) yields

$$\left|1 - \frac{y_{n+2}^2 + z_{n+2}^2}{x_{n+2}^2}\right| \leq \frac{(1-\tan^2\theta)^2}{1+\tan^4\theta} \cdot \left|1 - \frac{y_n^2 + z_n^2}{x_n^2}\right|.$$

For $\kappa := (1-\tan^2\theta)^2/(1+\tan^4\theta)$, the claim follows since $0 \leq (1-\tan^2\theta)^2 < 1+\tan^4\theta$. Note that $\kappa = 0$ if and only if $\theta = \pi/4$. □

We are now ready to state our main result. Regarding only the shape of the triangles, Δ_n tends to a rectangular isosceles triangle for $n \to \infty$.

Theorem 2.7. *For every initial triangle* Δ_0 *and every angle* $0 < \theta < \pi/2$*, the following limits hold:*

$$\lim_{n\to\infty} \frac{y_n^2 + z_n^2}{x_n^2} = 1, \qquad \lim_{n\to\infty} \alpha_n = \frac{\pi}{2},$$
$$\lim_{n\to\infty} \frac{y_n}{z_n} = 1, \qquad \lim_{n\to\infty} \beta_n = \frac{\pi}{4}.$$

Proof. The limits on the left-hand side are immediate consequences of Propositions 2.5 and 2.6. By the law of cosines we know $\cos\alpha_n = \frac{1}{2}(y_n^2 + z_n^2 - x_n^2)/(y_n z_n)$. Now the limits on the left-hand side imply $\lim_{n\to\infty} \cos\alpha_n = 0$ and hence, $\lim_{n\to\infty} \alpha_n = \pi/2$. The last limit follows from $\lim_{n\to\infty} y_n/z_n = 1$ together with the law of sines. □

As in the previous section, the size of the triangles becomes stable for only one specific choice of θ. For every greater angle, the triangles grow unboundedly and for every smaller angle, the triangles collapse to a single point.

Theorem 2.8.

If $0 < \theta < \frac{\pi}{4}$, *then* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0.$

If $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, *then* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \infty.$

If $\theta = \frac{\pi}{4}$, *then* $x_n = x_1 = \sqrt{2}\cdot y_1 = \sqrt{2}\cdot y_n \qquad \forall n > 0.$

Proof. First let $0 < \theta < \pi/4$. Then $\tan\theta < 1$ and hence, (2.3) implies

$$\begin{aligned} y_{n+1}^2 + z_{n+1}^2 &< \frac{1}{2}(y_n^2 + z_n^2) + \tan\theta \sin\alpha_n \cdot y_n z_n \\ &\le \frac{1}{2}(y_n^2 + z_n^2) + \tan\theta \cdot y_n z_n\,. \end{aligned}$$

With Lemma 1.5 we conclude $y_{n+1}^2 + z_{n+1}^2 < \frac{1}{2}(1+\tan\theta)(y_n^2+z_n^2)$. Since $\frac{1}{2}(1+\tan\theta) < 1$, this implies $\lim_{n\to\infty}(y_n^2 + z_n^2) = 0$ and hence, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$. The claim follows.

For $\pi/4 < \theta$, we obtain $\tan\theta > 1$. Let $\varepsilon > 0$ such that $\tan\theta > (1+\varepsilon)^2$. By Theorem 2.7 there are natural numbers n_z and n_α such that $(1+\varepsilon/2)z_n > y_n$ for every $n > n_z$ and $(1+\varepsilon/2)\sin\alpha_n > 1$ for every $n > n_\alpha$. Set $n_0 := \max\{n_z, n_\alpha\}$. Then for every $n > n_0$, (2.1) implies

$$\begin{aligned} y_{n+1}^2 &> \frac{1}{4}\cdot y_n^2 + \frac{1}{4}(1+\varepsilon)^4 \cdot z_n^2 + (1+\varepsilon)^2 \cdot \frac{1}{2}\sin\alpha_n \cdot y_n z_n \\ &> \frac{1}{4}\cdot y_n^2 + \frac{1}{4}(1+\varepsilon)^2 \cdot y_n^2 + \frac{1}{2}\cdot y_n^2 \\ &> \left(1+\frac{\varepsilon}{2}\right) y_n^2. \end{aligned}$$

Thus, $\lim_{n\to\infty} y_n = \infty$. Now $\lim_{n\to\infty} x_n = \infty$ follows from Theorem 2.7.

For the last case, Corollary 2.2 implies $y_n = z_n$ for every $n > 0$. The rest follows from (2.4). □

We conclude this section with a statement concerning the orientation of the triangles Δ_n.

Theorem 2.9. *For every $n > 0$, the triangle Δ_n is non-degenerate and counterclockwise oriented.*

Proof. Assume that Δ_n is non-degenerate and counterclockwise oriented. By symmetric reasons we may assume $y_n \geq z_n$. Let B'_n and C'_n denote the centres of C_nA_n and A_nB_n, respectively.

First we consider the case $x_n \geq y_n$. Then $\alpha_n \geq \beta_n$ and $\alpha_n \geq \gamma_n$ by the law of sines and therefore $\beta_n \leq \pi/2$ and $\gamma_n \leq \pi/2$. Since the triangle $A_{n+1}C_nB'_n$ is similar to Δ_n, we obtain $\angle C_nA_{n+1}B'_n = \beta_n$. On the other hand, let l be the perpendicular bisector of the side y_n, i. e. the line through B'_n and B_{n+1}. Since $x_n \geq z_n$, the line l intersects B_nC_n in a point S. We obtain $\angle C_nSB_{n+1} = \pi/2 - \gamma_n$. Thus, $\min\{\beta_n, \pi/2-\gamma_n\} \leq \angle C_nA_{n+1}B_{n+1} \leq \max\{\beta_n, \pi/2-\gamma_n\}$ and therefore $0 < \angle C_nA_{n+1}B_{n+1} < \pi/2$. Analogously, $0 < \angle C_{n+1}A_{n+1}B_n < \pi/2$. This implies $\angle C_{n+1}A_{n+1}B_n < \pi$ and the claim holds for Δ_{n+1}.

Now assume $x_n < y_n$. Then we obtain analogously to the above $0 < \angle A_nB'_nC_{n+1} < \pi/2$ and $\angle A_{n+1}B'_nC_n = \alpha_n < \pi/2$. Thus, both angles $\angle A_{n+1}B'_nB_{n+1}$ and $\angle B_{n+1}B'_nC_{n+1}$ are greater than $\pi/2$ and smaller than π and consequently, $\angle C_{n+1}B'_nA_{n+1} < \pi$. We conclude that B'_n is inside the triangle Δ_{n+1} and that the claim holds for Δ_{n+1}.

We complete the proof by applying induction. □

3 Further remarks and outlook

The "missing" case where only on one side, say B_nC_n, an isosceles triangle is erected and for the other two sides the centre is taken is not very interesting. Following I.M. Yaglom [8, I.2, 22], the distinguished angle for the isosceles triangle would be $\angle C_nA_{n+1}B_n = 0$ and therefore $\theta = \angle A_{n+1}B_nC_n = \angle B_nC_nA_{n+1} = \pi$, which is not possible. One gains the idea that the shape to which the triangles converge should be degenerate. Moreover, since every possible choice θ is smaller than π, the triangles should collapse to a single point. These claims are easy to prove: One can see immediately $x_{n+1} = x_n/2$. Furthermore, $y_{n+1} < y_n/2+\tan\theta\cdot x_n/2$ and $z_{n+1} < z_n/2+\tan\theta\cdot x_n/2$. Thus, the ratios x_n/y_n and y_n/z_n tend to 0 while y_n/z_n converges to 1. After reaching the point $\tan\theta\cdot x_n < y_n$, we obtain additionally $y_{n+1} < y_n$. The analogue holds for z_n. Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0$.

Consequently, for triangles, the next task would be to consider three different angles θ_x, θ_y, and θ_z for the isosceles triangles that are erected on the sides of Δ_n. Furthermore, instead of isosceles triangles one can erect arbitrary triangles on the sides of Δ_n. One possibility to determine the triangles uniquely is to demand besides the angle θ (which we now ask for only one of the two possibilities) also the ratio λ in which the side adjacent to the old triangle is subdivided by the orthocentre. This is in the spirit of [6] where the convergence of this transformation is studied by using its eigenvalues.

Another possible generalisation would be to consider instead of the triangle Δ_0 an arbitrary polygon. Of course, the transformation still uses isosceles triangles erected on the sides of the polygon. The convergence of these transformations has been studied in [5] together with the first case of this article. A possible application of this transformation is an element oriented mesh smoothing method based on successively applying this transformation to the polygonal bounded elements of triangular element surface meshes [4].

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Dimitris Vartziotis
Institute of Structural Analysis & Antiseismic Research
National Technical University Athens (NTUA)
GR-15780 Athens, Greece

and

TWT GmbH Science & Innovation, Research Department
Bernhäuser Straße 40–42
D-73765 Neuhausen, Germany
e-mail: dimitris.vartziotis@nikitec.gr

Simon Huggenberger
TWT GmbH Science & Innovation, Research Department
Bernhäuser Straße 40–42
D-73765 Neuhausen, Germany
e-mail: simon.huggenberger@twt-gmbh.de