Zeitschrift: Elemente der Mathematik

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 65 (2010)

Artikel: A straightened proof for the uncountability of R

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DOI: https://doi.org/10.5169/seals-130686

Elem. Math. 65 (2010) 26 – 28
0013-6018/10/010026-3
DOI 10.4171/EM/135

Elemente der Mathematik

A straightened proof for the uncountability of $\mathbb{R}$

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In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.
G. Cantor, 1867 [8, p. 31]

In 1873, Georg Cantor (1845–1918) presented the first proof of the uncountability of the set of real numbers by establishing essentially the following statement.

Theorem. *For every sequence of real numbers* $(x_n)_{n\in\mathbb{N}}$*, there is an* $x_0 \in\,]0,1[$ *with* $x_0 \neq x_n$ *for all* $n \in \mathbb{N}$. □

Cantor's proof of this result [2, §2] makes use of nested intervals, but today a proof based on another ingenious idea of Cantor is more popular, namely the *diagonal method*, which he introduced in 1891 to prove the uncountability of $2^{\mathbb{N}}$ [3]. However, Cantor himself did not employ diagonalization directly in the proof of uncountability of $\mathbb{R}$, but gave a rather intricate derivation of $|2^{\mathbb{N}}| = |\mathbb{R}|$ in [4, §4]. The origin of the now standard argument for

Der heute gebräuchliche Beweis für die Überabzählbarkeit der Menge der reellen Zahlen stützt sich auf das sogenannte *zweite Diagonalisierungsverfahren* von Cantor, das dieser jedoch zum Nachweis der Überabzählbarkeit der Potenzmenge der natürlichen Zahlen verwandt hatte. Wegen der Nicht-Eindeutigkeit der üblicherweise eingesetzten Dezimaldarstellung reeller Zahlen müssen aber künstliche Zusatzbedingungen gestellt werden, die im Dualsystem versagen. Dies wurde von Fraenkel beobachtet, dessen Ausweg in der Betrachtung einer „flacheren“ Diagonalen bestand. Doch auch sein Argument enthält eine Bedingung an die Dualdarstellung der Diagonalenelemente. Im vorliegenden Beitrag wird nun ein von Zahlensystem *und* Darstellung unabhängiger Beweis vorgestellt.

the theorem is unknown (to me). It starts off from decimal representations

$$|x_n| = \sum_{k \in \mathbb{Z}} x_{n,k} \cdot 10^{-k}, \quad x_{n,k} \in \{0, \ldots, 9\},$$

and assigns a digit different from $x_{n,n}$ to $x_{0,n}$. To avoid the possibility that x_0 coincides with some x_n which has two decimal representations, one might either assume that for these the (unique) finite representation had been chosen and exclude 9 as a value of $x_{0,n}$ (w.l., 0 is a member of $(x_n)_{n \in \mathbb{N}}$, such that $x_0 \neq 0$), or one might restrict the values for $x_{0,n}$ to $\{1, \ldots, 8\}$ in the first place.

Presumably, it is this "classical procedure" which David Hilbert (1862–1943) considered worth explicating to Albert Einstein (1879–1955), and "Einstein, who seized everything immediately, was totally overwhelmed by the splendor of these thoughts..." (translated from a letter (1918) of Hilbert to Cantor's daughter Else, as quoted in [7, p. 176]). On the other hand, the same proof stirred a lot of controversy among lesser minds, even leading to a court case (cf. [6]).

Although ennobled by its inclusion in *The Book* [1, p. 92f] and therefore regarded as perfect [1, p. V], this proof has the disadvantage of being rather artificial in the construction of the wanted number, depending on the base p of the number system employed and, more seriously, it does not work at all for the dual system! (The second variant even has problems in base $p = 3$, which can, however, be overcome by putting $x_{0,n} = 1$, if $x_{n,n} \neq 1$, and otherwise $x_{0,n} = 0$, if n is odd, and $x_{0,n} = 2$, if n is even.) This was noted by Abraham Fraenkel (1891–1965) [5, p. 66f], whose way out was to insert bits 0 into the binary expansion of x_0 between any two switched entries from the sequence of reals, thereby using a less inclined diagonal. Again, this only works if a representation not ending in 1s is assumed for all numbers. Therefore we propose the following most straightforward proof of the theorem for base $p = 2$.

Proof. Let

$$\forall n \in \mathbb{N} : \ |x_n| = \sum_{k \in \mathbb{Z}} x_{n,k} \cdot 2^{-k}, \quad x_{n,k} \in \{0, 1\},$$

and define

$$\forall n \in \mathbb{N} : \ x_{0,2n-1} = x_{n,2n}, \quad x_{0,2n} = 1 - x_{n,2n},$$

and

$$x_0 = \sum_{k \in \mathbb{N}} x_{0,k} \cdot 2^{-k}.$$

Then $(x_{0,k})_{k \in \mathbb{N}}$ is not eventually constant since $x_{0,2n-1} \neq x_{0,2n}$ and therefore $x_0 \in]0, 1[$ and $\forall n \in \mathbb{N} : \ x_0 \neq x_n$ because $x_{0,2n} \neq x_{n,2n}$. □

This proof does not depend on the base p of the number system, because we may put $x_{0,2n} = (x_{n,2n} + 1) \bmod p$. It is constructive in the sense of Cantor, who asserts that one can determine ("bestimmen") x_0 in the theorem. In fact, since he had shown in [2, §1] that the set of algebraic numbers can be arranged in a sequence $(x_n)_{n \in \mathbb{N}}$, the corresponding x_0 from our proof of the theorem is an explicit transcendental number.

Acknowledgement. I am grateful to Günter M. Ziegler for a valuable discussion.

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