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**Autor:** Høibakk, Ralph / Lukkassen, Dag  
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## Power means with integer values

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Ralph Høibakk and Dag Lukkassen

Ralph Høibakk obtained his Master degree in physics from the Norwegian Institute of Technology in 1962. He is now a part-time professor at Narvik University College within Enterprise Development.

Dag Lukkassen obtained his Dr. Scient. degree in mathematics from Tromsø University in 1996. He is a full professor of mathematics at Narvik University College since 2000. His main field of interest is the theory of partial differential equations and their applications.

Throughout this paper  $a$  and  $b$  will denote positive real numbers. The  $k$ -th power mean  $P_k = P_k(a, b)$  (with equal weights) of  $a$  and  $b$  is defined by

$$P_k = \begin{cases} \left( \frac{a^k + b^k}{2} \right)^{1/k} & \text{if } k \neq 0, \\ \sqrt{ab} & \text{if } k = 0. \end{cases}$$

The most well-known power means are the arithmetic, the geometric and the harmonic mean given by the formulae

$$A = \frac{a+b}{2}, \quad G = \sqrt{ab}, \quad H = \frac{2ab}{a+b},$$

respectively. For more detailed information on power means, see e.g. the books [3] and [7].

Das  $k$ -te Potenzmittel  $P_k(a, b)$  zweier positiver reeller Zahlen  $a, b$  ist für  $k \neq 0$  durch  $((a^k + b^k)/2)^{1/k}$  bzw. im Fall  $k = 0$  durch  $\sqrt{ab}$  gegeben. In der vorliegenden Arbeit gehen die beiden Autoren der Frage nach, für welche Paare  $(a, b)$  natürlicher Zahlen das Mittel  $P_k(a, b)$  ebenfalls ganzzahlig ist. Unter Verwendung eines Ergebnisses von H. Darmon und L. Merel aus dem Jahr 1997, das seinerseits auf dem Beweis der Fermat-Vermutung durch A. Wiles beruht, beweisen sie, dass für  $|k| > 2$  keine solchen Zahlenpaare  $(a, b)$  existieren können. In den verbleibenden Fällen  $k = 0, \pm 1, \pm 2$  geben die Autoren eine vollständige Klassifikation der möglichen Zahlenpaare  $(a, b)$  mit der gewünschten Eigenschaft.

In this paper we consider the problem of determining integers  $a$  and  $b$  such that the corresponding power mean  $P_k$  becomes integer valued. For integer values of  $k$  satisfying  $|k| \geq 3$  the answer follows easily by the following variant of Fermat's Last Theorem. We recall that a solution  $(x, y, z)$ , where  $\gcd(x, y, z) = 1$ , is called trivial if  $xyz = 0$  or  $\pm 1$ , and is called non-trivial otherwise.

**Theorem 1.** *The equation  $x^n + y^n = 2z^n$  has no non-trivial solution for integer values of  $n \geq 3$ .*

This result was conjectured and even proved for  $2 < n < 31$  by Denés [6] in 1952. Motivated by Wiles' famous breakthrough in the early 90's, which led to the proof of Fermat's Last Theorem, Ribet [10] proved the theorem when  $n$  is divisible by a prime which is congruent to 1 mod 4. By retracing the steps in Ribet's argument, Darmon and Merel [4] were able to give the final proof of the theorem in 1997.

Thanks to this theorem we can easily verify the following result:

**Corollary 2.** *If  $a \neq b$  are integers and  $k$  is an integer satisfying  $|k| \geq 3$ , then  $P_k$  is not an integer.*

*Proof.* The case  $k \geq 3$  follows directly from Theorem 1 by putting  $n = k$ ,  $x = a$ ,  $y = b$  and  $z = P_k$ . For  $k \leq -3$  we put  $n = -k$ . Hence, by rearranging, we obtain that

$$a^n + b^n = 2 \left( \frac{ab}{P_k} \right)^n. \quad (1)$$

Therefore, if  $P_k$  is an integer, so is  $ab/P_k$ . But this contradicts Theorem 1 for  $x = a$ ,  $y = b$  and  $z = ab/P_k$ . Hence, the proof is complete.  $\square$

It is certainly easy to find all integers making the arithmetic mean or the geometric mean integer valued. Concerning the harmonic mean we are going to prove the following result:

**Theorem 3.** *The integers  $a$  and  $b$  making the harmonic mean integer valued are precisely those of the form*

$$a = tp(p + q), \quad b = tq(p + q) \quad (2)$$

*(in this case  $H = 2tpq$ ) and the form*

$$a = t(2p + 1)(p + q + 1), \quad b = t(2q + 1)(p + q + 1) \quad (3)$$

*(in this case  $H = t(2p + 1)(2q + 1)$ ), where  $p, q$  and  $t$  are positive integers.*

*Proof.* The fact that  $H$  is an integer when  $a$  and  $b$  are of the forms (2) and (3) is seen directly by inspection. It remains to prove that  $H$  is an integer only if  $a$  and  $b$  are of these forms. Assume that  $H$  is an integer and let  $r$  and  $s$  be the integers given by

$$r = 2a - H, \quad s = 2b - H. \quad (4)$$

From the identity  $H = 2ab/(a + b)$  we obtain the simple relation

$$H^2 = rs. \quad (5)$$

Next we put  $r = wr_1$ ,  $s = ws_1$ , where  $w = \gcd(r, s)$  (such that  $\gcd(r_1, s_1) = 1$ ). By (5) we see that  $r_1s_1$  is a perfect square, and since  $r_1$  and  $s_1$  do not have any prime factors in common,  $r_1$  and  $s_1$  have to be perfect squares, i.e. of the forms  $r_1 = k^2$ ,  $s_1 = l^2$ . Thus, by (4) and (5) we find that

$$2a = kw(k + l), \quad 2b = lw(k + l). \quad (6)$$

Investigating the only two possible combinations

1.  $(k, l) = (2p + 1, 2q + 1)$  and  $w = t$ ,
2.  $(k, l) = (p, q)$  and  $w = 2t$ ,

where  $p$ ,  $q$  and  $t$  are integers, we arrive at the two possible representations (2) and (3). This completes the proof.  $\square$

**Remark 4.** Let  $t = 1$  and put  $p = 1$  and  $p = 0$  in (2) and (3), respectively. Then the above result shows that

$$H = \begin{cases} 2q & \text{if } a = 1 + q, b = q(1 + q), \\ 2q + 1 & \text{if } a = 1 + q, b = (2q + 1)(1 + q). \end{cases}$$

This formula shows that  $H$  takes all possible integer values, and that all these values, except  $H = 1$  and  $H = 2$ , may be attained in a nontrivial manner, that is such that  $a \neq b$ .

**Corollary 5.** *The integers  $a$  and  $b$  making the harmonic mean, arithmetic mean and the geometric mean integer valued are precisely those of the form*

$$a = 2t\alpha^2(\alpha^2 + \beta^2), \quad b = 2t\beta^2(\alpha^2 + \beta^2),$$

and the form

$$\begin{aligned} a &= t(2\alpha + 1)^2(2\alpha + 2\alpha^2 + 2\beta + 2\beta^2 + 1), \\ b &= t(2\beta + 1)^2(2\alpha + 2\alpha^2 + 2\beta + 2\beta^2 + 1), \end{aligned}$$

where  $\alpha$ ,  $\beta$  and  $t$  are positive integers.

*Proof.* By Theorem 3,  $(a, b)$  must belong to the class (2) or (3). By inspection, the latter is directly seen to generate integer values of  $A$ . But (2) gives integer values of  $A$  only if both  $p$  and  $q$  are odd (for which (2) may be written in the form (3)) or both even, or  $t$  is even. In any case, if both  $H$  and  $A$  are integers, we end up with the form

$$a = 2tp(p + q), \quad b = 2tq(p + q), \quad (7)$$

and the form

$$a = t(2p + 1)(p + q + 1), \quad b = t(2q + 1)(p + q + 1). \quad (8)$$

Using that the product  $G^2 = ab$  is a perfect square if and only if the integers  $a$  and  $b$  are of the form  $a = \tau r_1^2$ ,  $b = \tau r_2^2$ , we finally end up with the above representation. This completes the proof.  $\square$

Concerning the remaining two power means for which  $k$  is an integer we have the following result:

**Theorem 6.** *Let  $t \geq 0$  be integer valued and let  $x_1$  and  $y_1$  denote integers of the form*

$$x_1 = |p^2 - 2pq - q^2|, \quad y_1 = |p^2 + 2pq - q^2|,$$

*for some integers  $p$  and  $q$ . The integers  $a$  and  $b$  making  $P_2$  integer valued are precisely those of the form*

$$a = tx_1, \quad b = ty_1 \quad (9)$$

*(in this case  $P_2 = t(p^2 + q^2)$ ). The integers  $a$  and  $b$  making  $P_{-2}$  integer valued are precisely those of the form*

$$a = t(p^2 + q^2)x_1, \quad b = t(p^2 + q^2)y_1 \quad (10)$$

*(in this case  $P_{-2} = tx_1y_1$ ). Moreover, the integers  $a$  and  $b$  making all the power means  $P_{-2}$ ,  $H$ ,  $A$  and  $P_2$  integer valued are precisely those of the form*

$$a = t(p^2 + q^2)x_1 \frac{x_1 + y_1}{2}, \quad (11)$$

$$b = t(p^2 + q^2)y_1 \frac{x_1 + y_1}{2}. \quad (12)$$

*In the representations above it is enough to use numbers  $p, q$  which are coprime and have opposite parity (one odd, the other even).*

*Proof.* If  $y^2, z^2, x^2$  is a primitive 3-element arithmetic progression of squares, that is  $\gcd(y^2, z^2, x^2) = 1$  and  $z^2 - y^2 = x^2 - z^2$ , i.e.

$$x^2 + y^2 = 2z^2, \quad (13)$$

then it is always possible to find two numbers  $p, q$  of opposite parity (one odd, the other even) and coprime such that

$$x^2 = (p^2 - 2pq - q^2)^2, \quad y^2 = (p^2 + 2pq - q^2)^2, \quad z^2 = (p^2 + q^2)^2, \quad (14)$$

(see e.g. [5, pp. 437–438]). Conversely, by inspection we see that  $x^2 + y^2 = 2z^2$  for all squares of the form (14) and for any integers  $p, q$ . Thus, (9) follows by noting that  $a^2 + b^2 = 2P_2^2$ .

In the rest of the proof we assume that  $P_{-2}$  is an integer. Since

$$a^2 + b^2 = 2 \left( \frac{ab}{P_{-2}} \right)^2,$$

we see that  $ab/P_{-2}$  is an integer. Hence,  $x = a$ ,  $y = b$  and  $z = ab/P_{-2}$  satisfy (13). Putting  $x = dx_1$ ,  $y = dy_1$  and  $z = dz_1$ , where  $x_1, y_1, z_1$  are positive integers and  $d = \gcd(x, y, z)$  (note that by (13) we also have that  $d = \gcd(x, z) = \gcd(y, z)$ ), we obtain that

$dx_1y_1 = P_{-2}z_1$ , and since  $\gcd(x_1y_1, z_1) = 1$ , this shows that  $d = tz_1$  for some integer  $t$ . Using that  $x_1, y_1, z_1$  can be parametrized as in (14), i.e.

$$x_1 = |p^2 - 2pq - q^2|, \quad y_1 = |p^2 + 2pq - q^2|, \quad z_1 = p^2 + q^2, \quad (15)$$

we finally obtain the representation (10) (by inspection we also see that (10) makes  $P_{-2}$  integer valued for any integers  $p, q$ ). Since

$$(a, b) = d(x_1, y_1) = tz_1(x_1, y_1) \quad (16)$$

also satisfies (9), we know that  $P_2$  also is an integer. It is easy to see that  $x_1, y_1$  and  $z_1$  are odd numbers. Hence,  $A = d(x_1 + y_1)/2$  is also an integer. If, in addition,  $H$  is an integer, then

$$a = T(2P + 1)(P + Q + 1), \quad b = T(2Q + 1)(P + Q + 1), \quad (17)$$

for some integers  $T, P$  and  $Q$  by (3). Moreover, according to the proof of Theorem 3 we can combine these integers in such a way that  $2P + 1$  and  $2Q + 1$  become relatively prime, i.e. such that

$$x_1 = |p^2 - 2pq - q^2| = 2P + 1, \quad y_1 = |p^2 + 2pq - q^2| = 2Q + 1 \quad (18)$$

(note that (2) is of the same form as (17) due to the fact that both  $x_1$  and  $y_1$  are odd numbers). Thus,

$$P + Q + 1 = \frac{x_1 + y_1}{2}.$$

Hence, by (17) we obtain that

$$a = Tx_1 \frac{x_1 + y_1}{2}, \quad b = Ty_1 \frac{x_1 + y_1}{2},$$

so according to (16),

$$T \frac{x_1 + y_1}{2} = t(p^2 + q^2). \quad (19)$$

Since  $p$  and  $q$  are coprime, so are  $u = (x_1 + y_1)/2$  and  $v = p^2 + q^2$ . To see this, first observe that according to (18)  $u$  takes either the forms  $u = \pm(p^2 - q^2)$  or  $u = 2pq$ . By adding and subtracting  $u = \pm(p^2 - q^2) = kr$  and  $v = ks$ , we obtain that  $2p^2 = k(s \pm r)$ ,  $2q^2 = k(s \mp r)$ , which shows that  $k = 1$  or  $2$ , since  $p^2$  and  $q^2$  are coprime. But the latter possibility is excluded since  $z_1 = v = ks$  is odd.

$$\begin{aligned} ks + kr &= p^2 + q^2 \pm 2pq = (p \pm q)^2, \\ ks - kr &= p^2 + q^2 \mp 2pq = (p \mp q)^2. \end{aligned}$$

Similarly, by adding and subtracting  $u = 2pq = kr$  and  $v = ks$ , we obtain that  $k(s + r) = (p + q)^2$  and  $k(s - r) = (p - q)^2$ , which give the solutions  $2p = \sqrt{k}(\sqrt{s + r} \pm \sqrt{s - r})$ ,  $2q = \sqrt{k}(\sqrt{s + r} \mp \sqrt{s - r})$ , i.e.  $4p^2 = k(\sqrt{s + r} \pm \sqrt{s - r})^2$ ,  $4q^2 = k(\sqrt{s + r} \mp \sqrt{s - r})^2$ , and, as above, we conclude that  $k = 1$ . Since  $(x_1 + y_1)/2$  and  $p^2 + q^2$  are coprime, (12) gives that  $T = \tau(p^2 + q^2)$  for some integer  $\tau$ , and (11) and (12) follows. Moreover, by inspection we also see that this representation yields integer values of these four means for any integers  $p, q$ . This completes the proof.  $\square$

**Remark 7.** By Corollary 5 and Theorem 6 we have a representation of all pairs  $(a, b)$  making  $H, G, A$  and  $P_{-2}, H, A$  and  $P_2$  integer valued, respectively. It is therefore natural to ask the question: What integers  $a \neq b$  make all of the power means  $P_{-2}, H, G, A$  and  $P_2$  integer valued? The answer is: None! In fact, there are no integers  $a \neq b$  making both  $G$  and  $P_2$  (or  $P_{-2}$ ) integer valued. Indeed, if  $a \neq b$  is of the form (9) then  $a^2 + b^2 = 2w^2$ , where  $w = P_2$ . Without loss of generality we may assume that  $\gcd(a, b) = 1$ . If  $G$  also is an integer, then  $a$  and  $b$  have to be perfect squares  $a = u^2$  and  $b = v^2$ . Hence,

$$u^4 + v^4 = 2w^2. \quad (20)$$

But this equation has no nontrivial integer solution.

**Remark 8.** The fact that (20) has no nontrivial integer solution has been known for centuries (see [5] and [9]). Concerning the solvability of the more general equation  $u^p + v^p = 2w^2$  it was recently proved in [8] that there are no nontrivial integer solution such that  $\gcd(u, v, w) = 1$  if  $p \geq 7$  is a prime number. In the cases  $p = 2$  and  $p = 3$  there are infinitely many solutions. For both cases we are in the so called "spherical case" of the generalized Fermat equation  $Ax^p + By^q = Cz^r$ , i.e. when  $1/p + 1/q + 1/r > 1$  (see [1]). The case  $p = 5$  was treated in [2] where it was shown that  $(3, -1, 11), (-1, 3, 11)$  are the only nontrivial solutions such that  $\gcd(u, v, w) = 1$ .

**Remark 9.** In this paper we are mainly interested in cases when the order  $k$  of the power mean  $P_k$  itself is an integer. However, an interesting problem is to find all real values of  $k$  making  $P_k = P_k(a, b)$  integer valued for some integers  $a \neq b$ . Even though we do not have the solution to this problem, it is easy to see that the number of such values is infinite, in contrast to the case when  $k$  is integer valued. Indeed, let  $m$  be a positive integer. Then, for any odd numbers  $a$  and  $b$  we have that

$$(a^m)^{1/m} + (b^m)^{1/m} = 2(c^m)^{1/m},$$

where  $2c = a + b$ . Hence,  $P_{1/m}(a^m, b^m) = c^m$ . Similarly, by putting  $a = p(p + q)$ ,  $b = q(p + q)$ , where  $p$  and  $q$  are any positive integers, we obtain that

$$(a^m)^{1/m} + (b^m)^{1/m} = 2 \left( \frac{a^m b^m}{r^m} \right)^{1/m},$$

where  $r = 2pq$ . Accordingly, by (1), we obtain that  $P_{-1/m}(a^m, b^m) = r^m$ .

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## References

- [1] Beukers, F.: The Diophantine equation  $Ax^p + By^q = Cz^r$ . *Duke Math. J.* 91 (1998)1, 61–88.
- [2] Bruin, N.: *Chabauty methods and covering techniques applied to generalized Fermat equations*. PhD-thesis, University of Leiden, 1999.
- [3] Bullen, P.S.; Mitrinovic, D.S.; Vasic, P.M.: *Means and their inequalities*. D. Reidel Publishing Company Dordrecht, 1988.
- [4] Darmon, H.; Merel, L.: Winding quotients and some variants of Fermat’s Last Theorem. *J. Reine Angew. Math.* 490 (1997), 81–100.
- [5] Dickson, L.E.: *History of the Theory of Numbers, Vol. 2: Diophantine Analysis*. New York, Dover 2005.
- [6] Denés, P.: Über die Diophantische Gleichung  $x^l + y^l = cz^l$ . *Acta Math.* 88 (1952), 241–251.
- [7] Hardy, G.H.; Littlewood, J.E.; Pólya, G.: *Inequalities*. Cambridge University Press, 1934.
- [8] Ivorra, W.: Sur les équations  $x^p + 2^\beta y^p = z^2$  et  $x^p + 2^\beta y^p = 2z^2$ . *Acta Arith.* 108 (2003), 327–338.
- [9] Ribenboim, P.: *Catalan’s conjecture. Are 8 and 9 the only consecutive powers?* Academic Press Inc., Boston MA 1994.
- [10] Ribet, K.: On the equation  $a^p + 2^\alpha b^p + c^p = 0$ . *Acta Arith.* 79 (1997)1, 7–16.
- [11] Ribet, K.: *Private communication*, 2007.

Ralph Høibakk  
 Narvik University College  
 N–8505 Narvik, Norway

Dag Lukkassen  
 Narvik University College  
 P.O. Box 385  
 N–8505 Narvik, Norway

and

Norut Narvik  
 P.O. Box 250  
 N–8504 Narvik, Norway  
 e-mail: Dag.Lukkassen@hin.no