❙ Elemente der Mathematik

On the Miquel point of simplices

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1 Introduction and preliminaries

If points are marked on each side of a planar triangle, one on each side (or on a side's extension), then the three circles (each passing through a vertex and the marked points on the adjacent sides) are concurrent at a point M. This interesting fact was first proved and published by Augueste Miquel [3] in 1838, see also Weisstein [4] for further details and extensions. This result is well-known in planar geometry as *Miquel's theorem*, and M is called the *Miquel point*. However, much less known (even amongst geometers) is the following multidimensional generalization of Miquel's theorem: If one point is marked on each of the $d(d+1)/2$ edges of a d-simplex

$$S(\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_d}) = \left\{ \mathbf{x_0} + \sum_{i=1}^{d} \mu_i (\mathbf{x_i} - \mathbf{x_0}) : \sum_{i=1}^{d} \mu_i \leq 1, \mu_i \geq 0, i = 1, \ldots, d \right\} \quad (1)$$

Der Satz von Miquel ist ein klassisches Resultat der Elementargeometrie: Wird auf jeder Seite eines gegebenen Dreiecks oder deren Verlängerung ein Punkt beliebig festgelegt, und wird durch jeweils eine Ecke und die beiden markierten Punkte auf den Nachbarseiten ein Kreis gezeichnet, so schneiden sich diese drei Kreise in einem Punkt. In der vorliegenden Arbeit wird mit einfachen Hilfsmitteln der linearen Algebra erstmals ein vollständiger Beweis der Verallgemeinerung des Miquelschen Satzes auf d-dimensionale Simplizes angegeben: Wird auf den $d(d+1)/2$ Kanten je ein Punkt beliebig markiert, dann haben die $d+1$ Kugeln, wovon jede durch je einen Eckpunkt und die markierten Punkte auf den d in diesen Eckpunkt einlaufenden Kanten festgelegt wird, genau einen Punkt gemeinsam.

with positive volume and a sphere S_i is drawn through each vertex $\mathbf{x}_i$ and the points marked on the d edges which meet in $\mathbf{x}_i$, then these $d + 1$ spheres S_i, $i = 0, 1, \ldots, d$, all meet in a point M which will also be called Miquel's point in this note. By the best of the author's knowledge the only known proof of this result seems to be that given by Konnully [2]. Konnully's proof is based on the fact that there exists a common orthogonal sphere with respect to the so-called *Miquel spheres* $S_0, S_1, \ldots, S_d$ and it is shown that the radius of this sphere equals zero. However, such a sphere does not always exist even in the planar case, namely if the unique point which has the same circle power w.r.t. three pairwise non-concentric circles lies in the interior of each of these circles.

The aim of the present note is to provide a rigorous analytical proof which requires only simple facts from analytic geometry and linear algebra. As a by-product we obtain a family of upper bounds of Gram's determinant (including Hadamard's inequality) which seems to be of interest in its own right. This auxiliary result in Section 3 is valid in all Euclidean vector spaces.

It should be mentioned that an analogous construction with points marked on the $(d - 1)$-faces (instead of on the edges) of the simplex does in general not yield a common point that belongs to all spheres.

Let the points of the Euclidean space $\mathbb{R}^d$ be represented by column vectors $\mathbf{x} = (x_1, \ldots, x_d)'$ having the Euclidean norm $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$, where the scalar product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \mathbf{y}, \mathbf{z} \rangle = \mathbf{y}'\mathbf{z} = y_1 z_1 + \ldots + y_d z_d$$

for

$$\mathbf{y} = (y_1, \ldots, y_d)' \quad \text{and} \quad \mathbf{z} = (z_1, \ldots, z_d)'.$$

Furthermore, we recall the well-known fact from analytic geometry that the circumsphere of the d-simplex (1) consists of all points $\mathbf{x} \in \mathbb{R}^d$ satisfying the equation

$$\begin{vmatrix} \|\mathbf{x}\|^2 & \|\mathbf{x_0}\|^2 & \|\mathbf{x_1}\|^2 & \cdots & \|\mathbf{x_d}\|^2 \\ \mathbf{x} & \mathbf{x_0} & \mathbf{x_1} & \cdots & \mathbf{x_d} \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = 0 \,, \tag{2}$$

where the left-hand side is for a $(d + 2) \times (d + 2)$ determinant.

Without loss of generality, we shall assume the vertex $\mathbf{x_0}$ of the d-simplex (1) to coincide with the origin $\mathbf{o} = (0, \ldots, 0)'$ and the vertices $\mathbf{x_i} = (x_{1i}, \ldots, x_{di})'$, $i = 1, \ldots, d$, to be linearly independent vectors, i.e.

$$\Delta := \det \mathbf{X} \neq 0 \,, \tag{3}$$

where $\mathbf{X} = (\mathbf{x_1}, \ldots, \mathbf{x_d})$ denotes the quadratic matrix with columns $\mathbf{x_1}, \ldots, \mathbf{x_d}$.

For $0 \leq i < j \leq d$, let $\mathbf{x_{ij}}$ be a fixed point on the edge joining the vertices $\mathbf{x_i}$ and $\mathbf{x_j}$ being distinct from its end-points, i.e. $\mathbf{x_{ij}} = \mathbf{x_i} + \lambda_{ij} (\mathbf{x_j} - \mathbf{x_i})$ for some $\lambda_{ij} \notin \{0, 1\}$. For notational ease put $\lambda_{ji} = 1 - \lambda_{ij}$ for $i < j$ and $\lambda_{ii} = 1/2$ for $i = 0, 1, \ldots, d$.

In accordance with (2), the unique d-sphere S_0 passing through the origin $\mathbf{o}$ and the points $\mathbf{x_{01}}, \ldots, \mathbf{x_{0d}}$ consists of all points $\mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d$ satisfying

$$\Delta_0(\mathbf{x}) := \begin{vmatrix} \|\mathbf{x}\|^2 & 0 & \lambda_{01}^2 \|\mathbf{x_1}\|^2 & \cdots & \lambda_{0d}^2 \|\mathbf{x_d}\|^2 \\ x_1 & 0 & \lambda_{01} x_{11} & \cdots & \lambda_{0d} x_{1d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_d & 0 & \lambda_{01} x_{d1} & \cdots & \lambda_{0d} x_{dd} \\ 1 & 1 & 1 & \cdots & 1 \end{vmatrix} = 0 \, .$$

Since $\prod_{i=0}^{d} \lambda_{0i} \neq 0$, the latter equation is equivalent to

$$\begin{vmatrix} \|\mathbf{x}\|^2 & \lambda_{01} \|\mathbf{x_1}\|^2 & \cdots & \lambda_{0d} \|\mathbf{x_d}\|^2 \\ x_1 & x_{11} & \cdots & x_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ x_d & x_{d1} & \cdots & x_{dd} \end{vmatrix} = 0 \, . \tag{4}$$

Likewise, a point $\mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d$ belongs to the unique d-sphere S_i containing the vertex $\mathbf{x_i}$ and the marked points $\mathbf{x_{0i}}, \ldots, \mathbf{x_{i-1,i}}, \mathbf{x_{i,i+1}}, \ldots, \mathbf{x_{id}}$ on the adjacent edges if and only if

$$\Delta_i(\mathbf{x}) := \begin{vmatrix} \|\mathbf{x}\|^2 & \|\mathbf{x_i}\|^2 & \lambda_{0i}^2 \|\mathbf{x_i}\|^2 & \|\mathbf{x_i} + \lambda_{ij}(\mathbf{x_j} - \mathbf{x_i})\|^2 \\ x_1 & x_{1i} & \lambda_{0i} x_{1i} & x_{1i} + \lambda_{ij}(x_{1j} - x_{1i}) \\ \vdots & \vdots & \vdots & \vdots \\ x_d & x_{di} & \lambda_{0i} x_{di} & x_{di} + \lambda_{ij}(x_{dj} - x_{di}) \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$\underbrace{\hphantom{x_{di} + \lambda_{ij}(x_{dj} - x_{di})}}_{j \in \{1, \ldots, d\} \setminus \{i\}}$$

for $i = 1, \ldots, d$.

By appealing to the well-known transformation rules for determinants we obtain

$$\Delta_i(\mathbf{x}) = (\lambda_{0i} - 1) \begin{vmatrix} \|\mathbf{x}\|^2 - \lambda_{0i}^2 \|\mathbf{x_i}\|^2 & (1 + \lambda_{0i}) \|\mathbf{x_i}\|^2 & \lambda_{0i} \|\mathbf{x_i}\|^2 & \|\mathbf{x_i} + \lambda_{ij}(\mathbf{x_j} - \mathbf{x_i})\|^2 \\ x_1 - \lambda_{0i} x_{1i} & x_{1i} & 0 & x_{1i} + \lambda_{ij}(x_{1j} - x_{1i}) \\ \vdots & \vdots & \vdots & \vdots \\ x_d - \lambda_{0i} x_{di} & x_{di} & 0 & x_{di} + \lambda_{ij}(x_{dj} - x_{di}) \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

$$\underbrace{\hphantom{x_{di} + \lambda_{ij}(x_{dj} - x_{di})}}_{j \in \{1, \ldots, d\} \setminus \{i\}}$$

$$= (\lambda_{0i} - 1) \prod_{j \neq i} \lambda_{ij} \begin{vmatrix} \|\mathbf{x}\|^2 - \lambda_{0i}^2 \|\mathbf{x_i}\|^2 & c_{ii} & \lambda_{0i} \|\mathbf{x_i}\|^2 & c_{ij} \\ x_1 - \lambda_{0i} x_{1i} & x_{1i} & 0 & x_{1j} \\ \vdots & \vdots & \vdots & \vdots \\ x_d - \lambda_{0i} x_{di} & x_{di} & 0 & x_{dj} \\ 0 & 0 & -1 & 0 \end{vmatrix} \, ,$$

$$\underbrace{\hphantom{c_{ij}}}_{j \in \{1, \ldots, d\} \setminus \{i\}}$$

where

$$c_{ij} := \frac{\|\mathbf{x_i} + \lambda_{ij}(\mathbf{x_j} - \mathbf{x_i})\|^2 - \|\mathbf{x_i}\|^2}{\lambda_{ij}} + (1 + \lambda_{0i})\|\mathbf{x_i}\|^2$$

$$= \lambda_{0i}\|\mathbf{x_i}\|^2 + \|\mathbf{x_j}\|^2 - \lambda_{ji}\|\mathbf{x_i} - \mathbf{x_j}\|^2 \quad \text{for} \quad i, j = 1, \ldots, d.$$

The latter equality follows by using the identity $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle$. Since $c_{ii} = (1 + \lambda_{0i})\|\mathbf{x_i}\|^2$ we may simplify the previous determinant by multiplying the second column by λ_{0i} and adding it to the first column.

Consequently, in view of $\lambda_{ij} \notin \{0, 1\}$ for $i \neq j$, the equation $\Delta_i(\mathbf{x}) = 0$ can be expressed as

$$\begin{vmatrix} \|\mathbf{x}\|^2 + \lambda_{0i}\|\mathbf{x_i}\|^2 & c_{i1} & \cdots & c_{id} \\ x_1 & x_{11} & \cdots & x_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ x_d & x_{d1} & \cdots & x_{dd} \end{vmatrix} = 0 \quad \text{for} \quad i = 1, \ldots, d. \tag{5}$$

Obviously, the set of points $\mathbf{x} \in \mathbb{R}^d$ satisfying both equations (4) and (5) coincides with the $(d-1)$-dimensional sphere $S_0 \cap S_i$. By subtracting equation (4) from equation (5) and applying the summation law for determinants differing in only one row we obtain the linear equation

$$D_i(\mathbf{x}) := \begin{vmatrix} \lambda_{0i}\|\mathbf{x_i}\|^2 & c_{i1} - \lambda_{01}\|\mathbf{x_1}\|^2 & \cdots & c_{id} - \lambda_{0d}\|\mathbf{x_d}\|^2 \\ x_1 & x_{11} & \cdots & x_{1d} \\ \vdots & \vdots & \ddots & \vdots \\ x_d & x_{d1} & \cdots & x_{dd} \end{vmatrix} = 0$$

that holds for all $\mathbf{x} = (x_1, \ldots, x_d)' \in \mathbb{R}^d$ belonging to the (uniquely determined) hyperplane H_i containing the $(d-1)$-sphere $S_0 \cap S_i$. Combining the equations $D_i(\mathbf{x}) = 0$, $i = 1, \ldots, d$, yields a system of d linear equations whose solution (if it exists!) coincides with the point of intersection of the hyperplanes $H_1, \ldots, H_d$.

To find this point we introduce the matrix $\mathbf{A}_\Lambda = (a_{ij})_{i,j=1}^d$ with entries

$$a_{ij} := c_{ij} - \lambda_{0j}\|\mathbf{x_j}\|^2 = \lambda_{0i}\|\mathbf{x_i}\|^2 + (1 - \lambda_{0j})\|\mathbf{x_j}\|^2 - \lambda_{ji}\|\mathbf{x_i} - \mathbf{x_j}\|^2. \tag{6}$$

Let us first note the remarkable fact that, in view of $\lambda_{ij} + \lambda_{ji} = 1$ for $i \neq j$ and $\lambda_{ii} = 1/2$,

$$a_{ij} + a_{ji} = \|\mathbf{x_i}\|^2 + \|\mathbf{x_j}\|^2 - (\lambda_{ij} + \lambda_{ji})\|\mathbf{x_i} - \mathbf{x_j}\|^2 = 2\langle \mathbf{x_i}, \mathbf{x_j} \rangle \quad \text{for} \quad i, j = 1, \ldots, d,$$

which can be expressed concisely by

$$\frac{1}{2}(\mathbf{A}_\Lambda + \mathbf{A}'_\Lambda) = G(\mathbf{X}) := (\langle \mathbf{x_i}, \mathbf{x_j} \rangle)_{i,j=1}^d, \tag{7}$$

where $G(\mathbf{X})$ and $\det G(\mathbf{X})$ are called Gram's matrix and Gram's determinant, respectively, of the vectors $\mathbf{x_1}, \ldots, \mathbf{x_d}$. In $\mathbb{R}^d$ we have $G(\mathbf{X}) = \mathbf{X}'\mathbf{X}$ and thus $\det G(\mathbf{X}) = \Delta^2$. On the

other hand, the relation (7) is meaningful in any real vector space V on which a symmetric, positive definite, bilinear form $\langle \cdot, \cdot \rangle$ – briefly called scalar product – is defined. Such vector spaces are usually called *Euclidean vector spaces*.

Proposition. *For arbitrary linearly independent vectors* $\mathbf{x_1}, \ldots, \mathbf{x_d} \in \mathbb{R}^d$ *and real numbers* λ_{ij}, $i, j = 0, 1, \ldots, d$, *satisfying* $\lambda_{ij} + \lambda_{ji} = 1$ *and* $\lambda_{ij} \notin \{0, 1\}$ *for* $i, j = 0, 1, \ldots, d$, *there exists a unique point of intersection* $H_1 \cap \ldots \cap H_d = \mathbf{x}^* = (x_1^*, \ldots, x_d^*)'$, *which is given by*

$$\mathbf{x}^* = \mathbf{X} \mathbf{A}_\Lambda^{-1} \mathbf{x}_\lambda \quad \textit{with} \quad \mathbf{x}_\lambda = (\lambda_{01} \|\mathbf{x_1}\|^2, \ldots, \lambda_{0d} \|\mathbf{x_d}\|^2)' . \tag{8}$$

The first step in proving this result is to show that $\mathbf{x} = (x_1, \ldots, x_d)'$ obeys the linear equations $D_i(\mathbf{x}) = 0$ for $i = 1, \ldots, d$ if and only if it satisfies the equation $\mathbf{A}_\Lambda \mathbf{X}^{-1} \mathbf{x} = \mathbf{x}_\lambda$. This is left to the reader as an exercise. To see the invertibility of the matrix $\mathbf{A}_\Lambda$ we use (7) and decompose $\mathbf{A}_\Lambda$ as follows:

$$\mathbf{A}_\Lambda = \mathbf{B}_\Lambda + G(\mathbf{X}) \quad \text{with} \quad \mathbf{B}_\Lambda = \frac{1}{2}\left(\mathbf{A}_\Lambda - \mathbf{A}_\Lambda'\right) . \tag{9}$$

Here the matrix $\mathbf{B}_\Lambda$ is skew-symmetric, i.e. $\mathbf{B}_\Lambda' = -\mathbf{B}_\Lambda$. This skew-symmetry and the positive definiteness of the Gram matrix $G(\mathbf{X})$ show the quadratic form $\mathbf{x}' \mathbf{A}_\Lambda \mathbf{x} = \mathbf{x}' G(\mathbf{X}) \mathbf{x}$ to be strictly positive for any $\mathbf{x} \neq \mathbf{o}$. This in turn implies $\mathbf{A}_\Lambda \mathbf{x} \neq \mathbf{o}$ for any $\mathbf{x} \neq \mathbf{o}$, which is equivalent to $\det \mathbf{A}_\Lambda \neq 0$ for all λ_{ij} satisfying $\lambda_{ij} + \lambda_{ji} = 1$. This combined with a simple continuity argument tells us that $\det \mathbf{A}_\Lambda < 0$ contradicts $\det G(\mathbf{X}) > 0$, leaving as the only possibility $\det \mathbf{A}_\Lambda > 0$. By the same argument we get $\det(\mathbf{B}_\Lambda + \alpha G(\mathbf{X})) > 0$ for any $\alpha > 0$, entailing $\det \mathbf{B}_\Lambda \geq 0$ by letting $\alpha \downarrow 0$. However $\det \mathbf{B}_\Lambda$ may take positive values only for even $d \geq 2$ due to the very definition of skew-symmetry. Moreover, we shall bound $\det \mathbf{A}_\Lambda$ uniformly from below by Gram's determinant $\det G(\mathbf{X})$ in Section 3.

Finally, note that the proposition does not answer the question whether $\mathbf{x}^* \in S_i$ holds for some or even all $i = 0, 1, \ldots, d$. This will be the subject of the next section.

2   The main result and its proof

Theorem. *Under the conditions of the above proposition the point* $\mathbf{x}^*$ *given in* (8) *belongs to each of the spheres* S_i, $i = 0, 1, \ldots, d$, *that is,* $\mathbf{x}^*$ *is the unique point of intersection* $S_0 \cap S_1 \cap \ldots \cap S_d$. *In other words,* $\mathbf{x}^*$ *coincides with Miquel's point* M *of the simplex* $S(\mathbf{o}, \mathbf{x_1}, \ldots, \mathbf{x_d})$ *w.r.t. the marked points* $\mathbf{x_{ij}} = \mathbf{x_i} + \lambda_{ij}(\mathbf{x_j} - \mathbf{x_i})$ *on its edges.*

Proof. After transposing and expanding the determinant on the left-hand side of (4) along the first row we recognize that $\Delta_0(\mathbf{x}^*) = 0$ is equivalent to

$$\|\mathbf{x}^*\|^2 \Delta - \sum_{j=1}^{d} x_j^* \begin{vmatrix} x_{11} & \cdots & \lambda_{01}\|\mathbf{x_1}\|^2 & \cdots & x_{d1} \\ x_{12} & \cdots & \lambda_{02}\|\mathbf{x_2}\|^2 & \cdots & x_{d2} \\ \vdots & & \vdots & & \vdots \\ x_{1d} & \cdots & \lambda_{0d}\|\mathbf{x_d}\|^2 & \cdots & x_{dd} \end{vmatrix} = 0 . \tag{10}$$
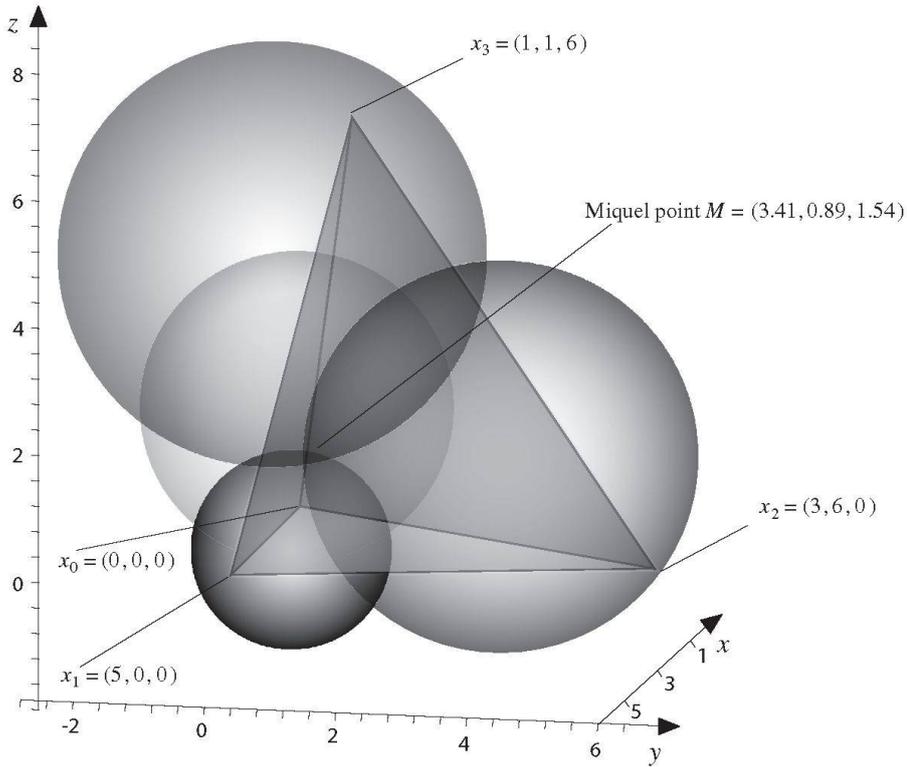
Fig. 1: Tetrahedron $S(\mathbf{x_0}, \mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ and the four Miquel spheres with common Miquel point M for $\lambda_{01} = 1/2$, $\lambda_{02} = 1/4$, $\lambda_{03} = 2/3$, $\lambda_{12} = 1/3$, $\lambda_{13} = 1/4$, $\lambda_{23} = 2/3$

Dividing by $\Delta \neq 0$ the latter equation takes the form $\langle \mathbf{x}^*, \mathbf{x}^* - \mathbf{z_0} \rangle = 0$, where the components of $\mathbf{z_0} = (z_1, \ldots, z_d)'$ are given by

$$z_j = \frac{1}{\Delta} \begin{vmatrix} x_{11} & \cdots & \lambda_{01} \|\mathbf{x_1}\|^2 & \cdots & x_{d1} \\ x_{12} & \cdots & \lambda_{02} \|\mathbf{x_2}\|^2 & \cdots & x_{d2} \\ \vdots & & \vdots & & \vdots \\ x_{1d} & \cdots & \lambda_{0d} \|\mathbf{x_d}\|^2 & \cdots & x_{dd} \end{vmatrix} \quad \text{for} \quad j = 1, \ldots, d \,.$$

Applying Cramer's rule we see that $\mathbf{z_0}$ satisfies the linear equation $\mathbf{X}' \mathbf{z_0} = \mathbf{x}_\lambda$, i.e. $\langle \mathbf{x_i} / \|\mathbf{x_i}\|, \mathbf{z_0} \rangle = \lambda_{0i} \|\mathbf{x_i}\|$ for $i = 1, \ldots, d$, so that $\mathbf{z_0} = (\mathbf{X}')^{-1} \mathbf{x}_\lambda$. The geometric interpretation of the above relations reveals that the orthogonal projection of $\mathbf{z_0}$ onto the edge joining $\mathbf{o}$ and $\mathbf{x_i}$ equals $\mathbf{x_{0i}} = \lambda_{0i} \mathbf{x_i}$ for $i = 1, \ldots, d$. Furthermore, $\mathbf{z_0}$ lies on the sphere S_0 and, provided that $\mathbf{x}^* \in S_0$, the point $\mathbf{z_0}/2$ coincides with the centre of S_0 by appealing to the converse of Thales' theorem.

Hence, $\Delta_0(\mathbf{x}^*) = 0$ holds if and only if

$$\langle \mathbf{x}^*, \mathbf{x}^* - (\mathbf{X}')^{-1} \mathbf{x}_\lambda \rangle = \mathbf{x}_\lambda' (\mathbf{A}_\Lambda^{-1})' \mathbf{X}' \left(\mathbf{X} \mathbf{A}_\Lambda^{-1} - (\mathbf{X}')^{-1} \right) \mathbf{x}_\lambda = \mathbf{x}_\lambda' \mathbf{C}_\Lambda \mathbf{x}_\lambda = 0 \,,$$

where $\mathbf{C}_\Lambda := (\mathbf{A}_\Lambda^{-1})' \mathbf{X}' \mathbf{X} \mathbf{A}_\Lambda^{-1} - (\mathbf{A}_\Lambda^{-1})'$.
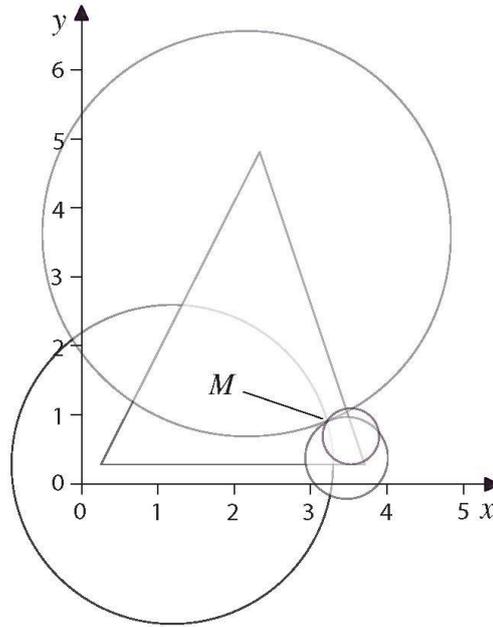
Fig. 2: Intersection of the tetrahedron and the Miquel spheres in Fig. 1 with the plane $z = 1.54$ parallel to the xy-plane

Using (7) we find that

$$\mathbf{C}_\Lambda = \frac{1}{2}\,(\mathbf{A}_\Lambda^{-1})'\,\left(\mathbf{A}_\Lambda + \mathbf{A}_\Lambda'\right)\,\mathbf{A}_\Lambda^{-1} - (\mathbf{A}_\Lambda^{-1})' = \frac{1}{2}\left(\mathbf{A}_\Lambda^{-1} - (\mathbf{A}_\Lambda^{-1})'\right) = -\mathbf{C}_\Lambda'\,,$$

thus proving the skew-symmetry of the matrix $\mathbf{C}_\Lambda$ which is necessary and sufficient for the quadratic form $\mathbf{x}_\lambda'\,\mathbf{C}_\Lambda\,\mathbf{x}_\lambda$ to disappear for any real $\lambda_{01}, \ldots, \lambda_{0d}$ and any $\mathbf{x_1}, \ldots, \mathbf{x_d} \in \mathbb{R}^d$, as we wished to prove.

It remains to show that $\Delta_i(\mathbf{x}^*) = 0$ for $i = 1, \ldots, d$. For this we transpose the determinant on the left-hand side of (5) and expand it along the first row leading to the system of equations

$$(\|\mathbf{x}^*\|^2 + \lambda_{0i}\|\mathbf{x_i}\|^2)\,\Delta - \sum_{j=1}^{d} x_j^* \begin{vmatrix} x_{11} & \cdots & c_{i1} & \cdots & x_{d1} \\ \vdots & & \vdots & & \vdots \\ x_{1d} & \cdots & \underbrace{c_{id}}_{j\text{-th column}} & \cdots & x_{dd} \end{vmatrix} = 0\,, \quad i = 1, \ldots, d\,.$$

In the next step we subtract equation (9) from the latter equation and divide the difference by $\Delta \neq 0$. Finally, using the abbreviation (6) we arrive at the equation

$$\lambda_{0i}\|\mathbf{x_i}\|^2 = \sum_{j=1}^{d} x_j^*\,w_{ij} \quad \text{with} \quad w_{ij} = \frac{1}{\Delta}\begin{vmatrix} x_{11} & \cdots & a_{i1} & \cdots & x_{d1} \\ \vdots & & \vdots & & \vdots \\ x_{1d} & \cdots & \underbrace{a_{id}}_{j\text{-th column}} & \cdots & x_{dd} \end{vmatrix}\,, \quad j = 1, \ldots, d\,,$$

which is equivalent to $\Delta_i(\mathbf{x}^*) = 0$ for each $i = 1, \ldots, d$. Again, according to Cramer's rule the vector $\mathbf{w_i} = (w_{i1}, \ldots, w_{id})'$ satisfies the equation $\mathbf{X}' \mathbf{w_i} = (a_{i1}, \ldots, a_{id})'$, that is, we may write $\mathbf{w_i} = (\mathbf{X}')^{-1} \mathbf{A}'_\Lambda \mathbf{e_i}$ with $\mathbf{e_i} = (0, \ldots, 1, \ldots, 0)'$ denoting the i-th column vector of the $d \times d$ identity matrix. Hence, the above equations can be rewritten as

$$\lambda_{0i} \|\mathbf{x_i}\|^2 = \mathbf{w_i}' \mathbf{x}^* = \mathbf{e_i}' \mathbf{A}_\Lambda \mathbf{X}^{-1} \mathbf{x}^* \quad \text{for} \quad i = 1, \ldots, d.$$

However, these equations follow directly from (8) and vice versa. Thus the proof of the Theorem is complete. $\qquad\Box$

Note that the point $\mathbf{z_i} = \mathbf{w_i} + \mathbf{z_0} - \mathbf{x_i}$ which can be shown to belong to the sphere S_i satisfies the orthogonality relation

$$\langle \mathbf{x}^* - \mathbf{z_i}, \mathbf{x}^* - \mathbf{x_i} \rangle = 0 \quad \text{for} \quad i = 1, \ldots, d.$$

This is easily verified by straightforward computations using the expressions of $\mathbf{x}^*$, $\mathbf{w_i}$, and $\mathbf{z_0}$ given in the above proof. As a result, the sphere S_i has the centre $(\mathbf{z_i} + \mathbf{x_i})/2 = (\mathbf{w_i} + \mathbf{z_0})/2$.

3 Bounds for determinants

In the subsequent lemma we establish a lower bound for the determinant of $\mathbf{A}_\Lambda$ that is uniform in all the varying parameters λ_{ij}, $0 \le i < j \le d$, and even positive provided the vectors $\mathbf{x_1}, \ldots, \mathbf{x_d}$ are linearly independent. Note that the below inequality (11) is valid for any d (≥ 2) elements of an arbitrary Euclidean vector space.

Lemma. *Let V be a Euclidean vector space equipped with scalar product $\langle \cdot, \cdot \rangle$. For any $\mathbf{x_1}, \ldots, \mathbf{x_d} \in V$ and any real numbers λ_{ij}, $i, j = 0, 1, \ldots, d$, satisfying $\lambda_{ij} + \lambda_{ji} = 1$ for $i, j = 0, 1, \ldots, d$, the inequality*

$$\det \mathbf{A}_\Lambda \ge \det \mathbf{B}_\Lambda + \det G(\mathbf{X}) \ge \det G(\mathbf{X}) \tag{11}$$

holds, where $\mathbf{A}_\Lambda$, $\mathbf{B}_\Lambda$, and $G(\mathbf{X})$ are defined by (6), (9), and (7), respectively. Equality is attained in (11) if

$$\lambda_{ij} \|\mathbf{x_i} - \mathbf{x_j}\|^2 = (1 - \lambda_{0i}) \|\mathbf{x_i}\|^2 + \lambda_{0j} \|\mathbf{x_j}\|^2 - \langle \mathbf{x_i}, \mathbf{x_j} \rangle \quad \text{for} \quad 1 \le i < j \le d.$$

Proof. As already pointed out at the end of Section 1 the determinant of any skew-symmetric $d \times d$ matrix $\mathbf{B}$ is non-negative and equals zero if d is odd. Thus, the second part of (11) is trivial and instead of the first part we prove the slightly more general inequality

$$\det(\mathbf{B} + G(\mathbf{X})) \ge \det \mathbf{B} + \det G(\mathbf{X}). \tag{12}$$

By applying the well-known principal axis theorem to the non-negative definite Gram matrix $G(\mathbf{X})$ we find an orthogonal $d \times d$ matrix $\mathbf{O}$ (with $\det \mathbf{O} = 1$) such that $\mathbf{D} = \mathbf{O}' G(\mathbf{X}) \mathbf{O}$ is a diagonal matrix with non-negative diagonal elements. The multiplication rule for determinants enables us to replace $G(\mathbf{X})$ by $\mathbf{D}$ and $\mathbf{B}$ by the skew-symmetric matrix

O' B O without changing the inequality (12). Therefore it suffices to verify $\det(B + D) \geq \det B + \det D$ only for diagonal matrices D with non-negative diagonal elements $y_1, \ldots, y_d$ and any skew-symmetric $d \times d$ matrix B. The Taylor expansion of the function

$$f(y_1, \ldots, y_d) = \det(B + D) = \begin{vmatrix} y_1 & b_{12} & \cdots & b_{1d} \\ -b_{12} & y_2 & \cdots & b_{2d} \\ \vdots & \vdots & & \vdots \\ -b_{1d} & -b_{2d} & \cdots & y_d \end{vmatrix}$$

leads to

$$f(y_1, \ldots, y_d) = \det B + \sum_{k=1}^{d-2} \sum_{1 \leq i_1 < \cdots < i_k \leq d} y_{i_1} \cdots y_{i_k} \det B_{i_1, \ldots, i_k} + y_1 \cdots y_d \, ,$$

where the $(d - k) \times (d - k)$ matrix $B_{i_1, \ldots, i_k}$ emerges from B by deleting the rows $i_1, \ldots, i_k$ and columns $i_1, \ldots, i_k$. Obviously, all these matrices are skew-symmetric which, together with $y_1, \ldots, y_d \geq 0$, implies

$$f(y_1, \ldots, y_d) \geq \det B + y_1 \cdots y_d = \det B + \det D \, .$$

Thus the inequality (11) is proved. The proof of the lemma is complete by noting that $A_\Lambda = A'_\Lambda$ and (7) imply equality in (11). $\square$

Remark. The above proof turns out that the inequality (12) remains valid if $G(X)$ is replaced by any other non-negative definite $d \times d$ matrix.

Corollary. *If $x_1, \ldots, x_d \in V$ are linearly independent, then $\det A_\Lambda$ is positive which in turn implies the existence of the inverse A_Λ^{-1} for all real λ_{ij} satisfying $\lambda_{ij} + \lambda_{ji} = 1$ for $i, j = 0, 1, \ldots, d$. As a special case, (11) includes the well-known Hadamard inequality*

$$\det G(X) \leq \|x_1\|^2 \cdot \ldots \cdot \|x_d\|^2 \quad \text{for any} \quad x_1, \ldots, x_d \in V \, ,$$

which reads $|\Delta| \leq \|x_1\| \cdot \ldots \cdot \|x_d\|$ in $\mathbb{R}^d$.

The first part of the corollary follows from the well-known fact that $\det G(X) > 0$ characterizes the linear independence of $x_1, \ldots, x_d \in V$. The second part follows by setting $\lambda_{01} = \ldots = \lambda_{0d} = 0$ and $\lambda_{ij} = \|x_i\|^2 / \|x_i - x_j\|^2$ so that by (6) $a_{ji} = 0$ for $1 \leq i < j \leq d$. In other words, A_Λ is an upper triangular matrix entailing $\det A_\Lambda = a_{11} \cdot \ldots \cdot a_{dd}$.

4   Concluding remarks

1. Formula (8) and the above theorem reveal that the Miquel point M of the simplex $S(o, x_1, \ldots, x_d)$ w.r.t. the marked points $x_{ij} = x_i + \lambda_{ij}(x_j - x_i)$ on the simplex' edges can be represented as a linear combination of the edges $x_1, \ldots, x_d$,

$$x^* = \sum_{i=1}^{d} u_i x_i = X u \quad \text{with weight vector} \quad u = (u_1, \ldots, u_d)' = A_\Lambda^{-1} x_\lambda \, . \tag{13}$$

From (6) and (8) it is seen that the weights $u_1, \ldots, u_d$ depend only on the parameters λ_{ij} and the squared edge lengths $\|x_i - x_j\|^2$ for $0 \le i < j \le d$ (with $x_0 = o$). For this reason the Miquel point can be defined in a meaningful way for any finite family of linearly independent elements of a Euclidean vector space V.

By (13) the Miquel points x_i^*, $i = 0, 1, \ldots, d$, of those $(d - 1)$-simplices having the d vertices x_j, $j \ne i$, can be easily determined. Geometrically spoken, x_i^* is the point of intersection of the d Miquel spheres S_j, $j \ne i$, with the hyperplane containing the $(d - 1)$-face of the original simplex which x_i does not belong to. In this way a new d-simplex $S(x_0^*, x_1^*, \ldots, x_d^*)$ arises and relations between it and the original one for given λ_{ij}'s could be of interest.

2. Since the matrix A_Λ is invertible for any λ_{ij} the point x^* is also well-defined for $\lambda_{ij} \in \{0, 1\}$. For special choices of the λ_{ij}'s we have interesting geometric intepretations, e.g. for $d = 2$, letting $\lambda_{01} \to 1$, $\lambda_{02} \to 0$, and $\lambda_{12} \to 1$ entails that each of the limiting Miquel circles touches one side of the triangle $\triangle\, x_0 x_1 x_2$ at a vertex and passes through the opposite vertex; the corresponding Miquel point turns into the *Brocard point*, see [1]. By means of the Theorem in Section 2 several generalizations of this point to higher dimensions are possible. In particular, for a tetrahedron $S(o, x_1, x_2, x_3)$ in $\mathbb{R}^3$ the choice $\lambda_{01} = \lambda_{02} = \lambda_{03} = \lambda_{13} = 1$ and $\lambda_{12} = \lambda_{23} = 0$ (in the above setting) yields x^* as common point of the circumsphere S_0 and $S_1 \cap S_2 \cap S_3$, where e.g. S_1 is the unique sphere through x_3 touching the face triangle $\triangle\, o x_1 x_2$ at x_1, and S_2, S_3 are defined analogously.

3. The vertices of a further d-simplex associated with $S(o, x_1, \ldots, x_d)$ and the given λ_{ij}'s coincide with the midpoints x_i^c of the Miquel spheres S_i, $i = 0, 1, \ldots, d$. In Section 2 we have derived the following formulas:

$$x_0^c = \frac{1}{2}(X')^{-1} x_\lambda \quad \text{and} \quad x_i^c = \frac{1}{2}(X')^{-1} A_\Lambda e_i + x_0^c, \quad i = 1, \ldots, d.$$

In the planar case simple geometric arguments show that the triangles $\triangle\, x_0 x_1 x_2$ and $\triangle\, x_0^c x_1^c x_2^c$ are similar, cf. e.g. [1]. It is natural to ask whether the simplex $S(x_0^c, x_1^c, \ldots, x_d^c)$ and the original simplex are similar for any $d \ge 2$. This fact can be expressed analytically by an orthogonal matrix O and some scaling factor $\gamma > 0$ by requiring

$$x_i^c - x_0^c = \frac{1}{2}(X')^{-1} A_\Lambda e_i = \gamma\, O x_i \quad \text{for} \quad i = 1, \ldots, d, \quad \text{where} \quad \gamma = \frac{1}{2}\left(\frac{\det A_\Lambda}{\Delta^2}\right)^{1/d}.$$

In view of $O' = O^{-1}$ and (7) these relations are equivalent to $G^{-1}(X) = 2\gamma^2\left(A_\Lambda^{-1} + (A_\Lambda')^{-1}\right)$. This equality holds actually only for $d = 2$ without additional restrictions on the λ_{ij}'s.

4. On the other hand, for any $d \ge 2$, the particular choice $\lambda_{ij} = 1/2$, $0 \le i < j \le d$, yields the Miquel point $x^* = \frac{1}{2}(X')^{-1}(\|x_1\|^2, \ldots, \|x_d\|^2)'$, which coincides with the circumcentre of the simplex $S(o, x_1, \ldots, x_d)$.

Acknowledgements

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