

Zeitschrift: Elemente der Mathematik
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 63 (2008)

Artikel: The conics of Lucas' configuration
Autor: Herrera Gómez, Blas
DOI: <https://doi.org/10.5169/seals-99057>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 17.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

The conics of Lucas' configuration

Blas Herrera Gómez

Blas Herrera Gómez obtained his Ph.D. in mathematics at the University Autònoma of Barcelona in 1994. Presently, he is professor of applied mathematics at the University Rovira i Virgili of Tarragona. His main fields of interest are: classical and differential geometry, and the application of geometry to fluid mechanics.

1 Introduction

Let us consider a figure formed by a triangle ABC and its three inscribed squares $X_1X_2Y_3Z_4$, $Y_1Y_2Z_3X_4$, $Z_1Z_2X_3Y_4$, where the sides X_1X_2 , Y_1Y_2 , Z_1Z_2 are on the sides AB , BC , CA of the triangle, and these three squares are homothetic to the external squares $BAB'A'$, $CBC'B'$, $ACA'C'$, respectively, from the vertices of CAB ; see Fig. 1. We will call this figure "Lucas' configuration".

In fact, there are another three squares inscribed in the triangle ABC . These are the three squares $X'_1X'_2Z'_3Y'_4$, $Y'_1Y'_2X'_3Z'_4$, $Z'_1Z'_2Y'_3X'_4$, where the sides $X'_1X'_2$, $Y'_1Y'_2$, $Z'_1Z'_2$ are on the sides $A'B'$, $B'C'$, $C'A'$ of the triangle, and these three squares are homothetic to the internal squares $ABA''B''$, $BCB''C''$, $CAC''A''$, respectively, from the vertices of CAB . We will call this figure "Lucas' internal configuration"; but the results and conditions are similar to Lucas' configuration.

In [3], I. Panakis shows the relations found by Édouard Lucas between the circumcircles of the triangles AX_4Z_3 , BY_4X_3 , CZ_4Y_3 and the length of the sides of the triangle ABC . In [1], A.P. Hatzipolakis and P. Yiu show that these three circumcircles are mutually tangent to each other, and tangent to the circumcircle of ABC ; see Fig. 1.

In this note we show that Lucas' configuration has more geometric peculiarities. We find the following result:

Der vorliegende Beitrag ist eine Variation zur sogenannten Lucas-Konfiguration. Diese ist beschrieben durch ein Dreieck und die ihm einbeschriebenen drei Quadrate, deren eine Seite jeweils auf einer der Dreiecksseiten liegt. Der Autor beweist nun das bemerkenswerte Resultat, dass die zwölf Eckpunkte der drei Quadrate in zwei Klassen mit je sechs Punkten zerfallen, so dass die Punkte beider Klassen jeweils einen Kegelschnitt beschreiben. Eine Klasse beschreibt dabei sogar eine Ellipse.

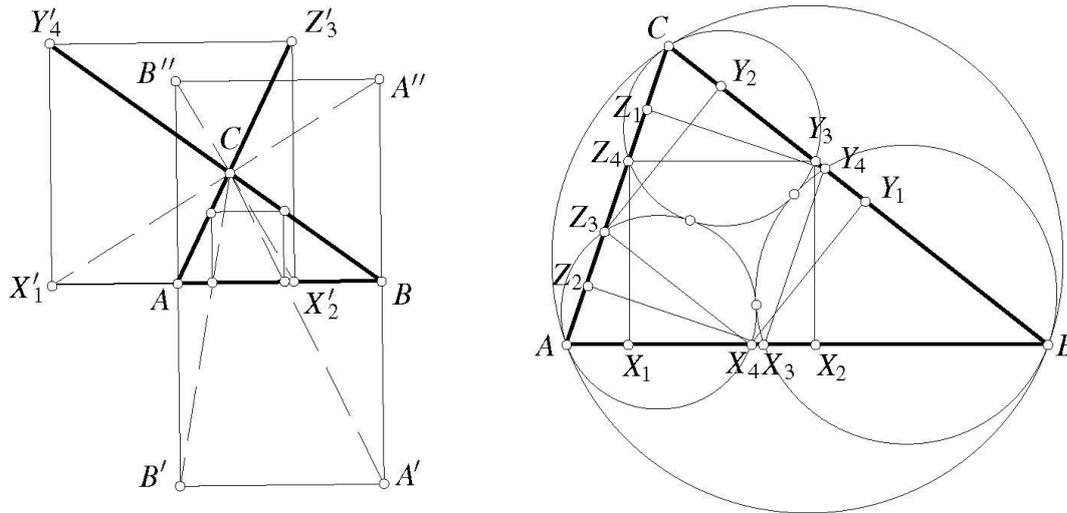


Fig. 1

2 Result

Theorem. Let ABC be a triangle and let $X_1X_2Y_3Z_4$, $Y_1Y_2Z_3X_4$, $Z_1Z_2X_3Y_4$ be its three inscribed squares forming Lucas' configuration. Then:

- a) The vertices $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ are on a conic.
- b) The vertices $Y_3, Z_4, Z_3, X_4, X_3, Y_4$ are on an ellipse.

See Figs. 2, 3 and 4.

To prove the result we will concentrate our efforts on finding the equations of the conic.

We point out that, in the case of the three squares $X'_1X'_2Z'_3Y'_4$, $Y'_1Y'_2X'_3Z'_4$, $Z'_1Z'_2Y'_3X'_4$, which form the Lucas's internal configuration, the result is the same, but the vertices $Z'_3, Y'_4, X'_3, Z'_4, Y'_3, X'_4$ are on a conic which is not necessarily an ellipse.

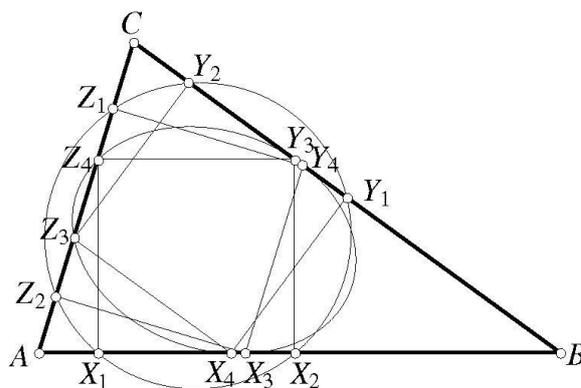


Fig. 2

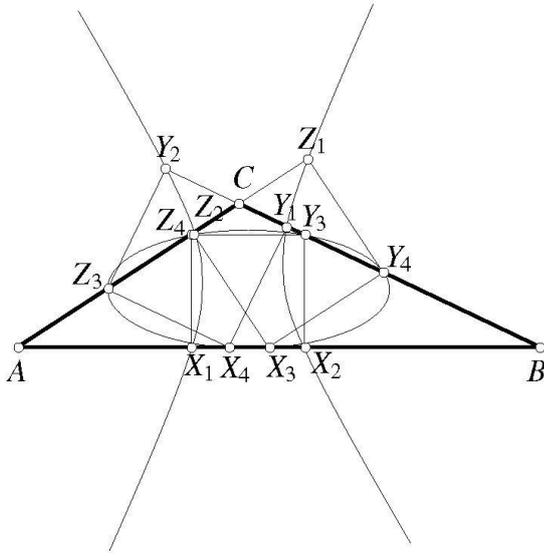


Fig. 3

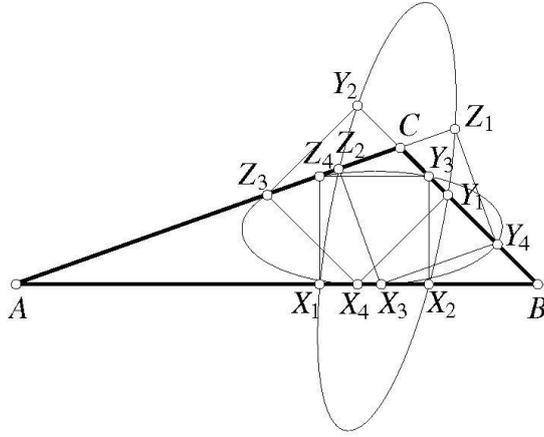


Fig. 4

Proof. To prove the result, let ABC be the triangle; we may assume that AB is the longest side, and we can consider a Cartesian system of coordinates such that

$$A = (0, 0), \quad B = (1, 0), \quad C = (a, b) \quad \text{with } a \in (0, 1], \quad b \in (0, 1].$$

In this system, after a calculation we have:

$$\begin{aligned} X_1 &= \Gamma(a, 0), \quad X_2 = \Gamma(a + b, 0), \quad Y_3 = \Gamma(a + b, b), \quad Z_4 = \Gamma(a, b), \\ Y_1 &= \Delta(b + 1, -a + 1), \quad Y_2 = \Delta(a + b, -a + b + 1), \quad Z_3 = \Delta(a, b), \quad X_4 = \Delta(1, 0), \\ Z_1 &= \Lambda(a^2 + ab, ab + b^2), \quad Z_2 = \Lambda(a^2, ab), \quad X_3 = \Lambda(a^2 + b^2, 0), \\ Y_4 &= \Lambda(a^2 + b^2 + ab, b^2) \end{aligned}$$

where Γ , Δ and Λ have positive values:

$$\Gamma = \frac{1}{b + 1}, \quad \Delta = \frac{b}{a^2 + b^2 - 2a + b + 1}, \quad \Lambda = \frac{1}{a^2 + b^2 + b}.$$

Then, with a long but straightforward calculation, we find that the points $X_1, X_2, Y_1, Y_2, Z_1, Z_2$ verify the following equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

with

$$\begin{aligned} A &= b^2(b + 1)^2, \\ B &= -3a^4 - a^2b^2 + b^4 + 6a^3 + ab^2 + 2b^3 - 3a^2 + b^2, \\ C &= b(2a - 1)(2a^2 + b^2 - 2a), \\ D &= -b^2(b + 1)(2a + b), \\ E &= -b(2a + b)(a^2 + b^2 - ab - a + b), \\ F &= ab^2(a + b). \end{aligned}$$

Also, with another long but straightforward calculation, we find that the points $Y_3, Z_4, Z_3, X_4, X_3, Y_4$ verify the following equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

with

$$A = b^2(a^2 + b^2 + b)(a^2 + b^2 - 2a + b + 1),$$

$$B = a^6 + 2a^4b^2 + a^2b^4 - 3a^5 + b^5 + 3a^4b - 4a^3b^2 + 4a^2b^3 - ab^4 + 4a^4 + 2b^4 - 6a^3b + 5a^2b^2 - 4ab^3 - 3a^3 + 2b^3 + 3a^2b - 3ab^2 + a^2 + b^2,$$

$$C = -b(2a - 1)(a^2 + b^2 - a + b)^2,$$

$$D = -b^2(a^4 + b^4 + 2a^2b^2 - 2a^3 + 2a^2b - 2ab^2 + 2b^3 + a^2 + 2b^2),$$

$$E = b(a^5 + 2a^3b^2 + ab^4 - 3a^4 - 2b^4 + 2a^3b - 5a^2b^2 + 2ab^3 + 3a^3 - 2a^2b + 4ab^2 - 2b^3 - a^2 - b^2),$$

$$F = b^3(a^2 + b^2).$$

Now that we have the previous equations, we can easily check that the first one corresponds to a conic which is not necessarily an ellipse, whereas the second one necessarily corresponds to an ellipse. \square

Remark. If instead of considering the three inscribed squares we consider the three inscribed equilateral triangles each with a side parallel to a side of ABC , then we can find similar results; see some of them in [2].

Acknowledgement. This work was supported by the “Dirección General de Investigación Científica y Técnica” (Spain), project no. CTQ2005-09182-C02-02.

References

- [1] Hatzipolakis, A.P.; Yiu, P.: The Lucas circles of a triangle. *Amer. Math. Monthly* 108 (2001), 444–446.
- [2] Herrera, B.: The notable configuration of the inscribed equilateral triangles in a triangle. *Elem. Math.* 57 (2002), 1–6.
- [3] Panakis, I.: Plane Trigonometry, Vol II. *Private publication*, Athens (1973).

Blas Herrera Gómez
 Departament d'Enginyeria Informàtica i Matemàtiques
 Universitat Rovira i Virgili
 Avinguda Països Catalans, 26
 43007 Tarragona, Spain
 e-mail: blas.herrera@urv.net