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The conics of Lucas' configuration

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1 Introduction

Let us consider a figure formed by a triangle ABC and its three inscribed squares $X_1X_2Y_3Z_4$, $Y_1Y_2Z_3X_4$, $Z_1Z_2X_3Y_4$, where the sides X_1X_2, Y_1Y_2, Z_1Z_2 are on the sides AB, BC, CA of the triangle, and these three squares are homothetic to the external squares $BAB'A'$, $CBC'B'$, $ACA'C'$, respectively, from the vertices of CAB; see Fig. 1. We will call this figure "Lucas' configuration".

In fact, there are another three squares inscribed in the triangle ABC. These are the three squares $X'_1X'_2Z'_3Y'_4$, $Y'_1Y'_2X'_3Z'_4$, $Z'_1Z'_2Y'_3X'_4$, where the sides $X'_1X'_2$, $Y'_1Y'_2$, $Z'_1Z'_2$ are on the sides $A'B'$, $B'C'$, $C'A'$ of the triangle, and these three squares are homothetic to the internal squares $ABA''B''$, $BCB''C''$, $CAC''A''$, respectively, from the vertices of CAB. We will call this figure "Lucas' internal configuration"; but the results and conditions are similar to Lucas' configuration.

In [3], I. Panakis shows the relations found by Édouard Lucas between the circumcircles of the triangles AX_4Z_3, BY_4X_3, CZ_4Y_3 and the length of the sides of the triangle ABC. In [1], A.P. Hatzipolakis and P. Yiu show that these three circumcircles are mutually tangent to each other, and tangent to the circumcircle of ABC; see Fig. 1.

In this note we show that Lucas' configuration has more geometric peculiarities. We find the following result:

Der vorliegende Beitrag ist eine Variation zur sogenannten Lucas-Konfiguration. Diese ist beschrieben durch ein Dreieck und die ihm einbeschriebenen drei Quadrate, deren eine Seite jeweils auf einer der Dreiecksseiten liegt. Der Autor beweist nun das bemerkenswerte Resultat, dass die zwölf Eckpunkte der drei Quadrate in zwei Klassen mit je sechs Punkten zerfallen, so dass die Punkte beider Klassen jeweils einen Kegelschnitt beschreiben. Eine Klasse beschreibt dabei sogar eine Ellipse.

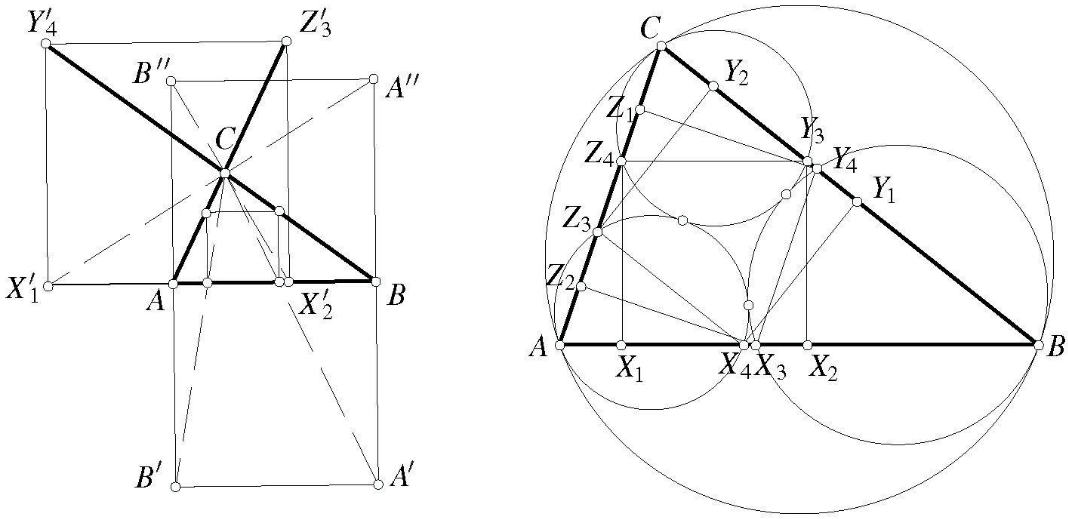

Fig. 1

2 Result

Theorem. *Let ABC be a triangle and let $X_1X_2Y_3Z_4$, $Y_1Y_2Z_3X_4$, $Z_1Z_2X_3Y_4$ be its three inscribed squares forming Lucas' configuration. Then:*

a) *The vertices X_1, X_2, Y_1, Y_2, Z_1, Z_2 are on a conic.*

b) *The vertices Y_3, Z_4, Z_3, X_4, X_3, Y_4 are on an ellipse.*

See Figs. 2, 3 and 4.

To prove the result we will concentrate our efforts on finding the equations of the conic.

We point out that, in the case of the three squares $X'_1X'_2Z'_3Y'_4$, $Y'_1Y'_2X'_3Z'_4$, $Z'_1Z'_2Y'_3X'_4$, which form the Lucas's internal configuration, the result is the same, but the vertices Z'_3, Y'_4, X'_3, Z'_4, Y'_3, X'_4 are on a conic which is not necessarily an ellipse.

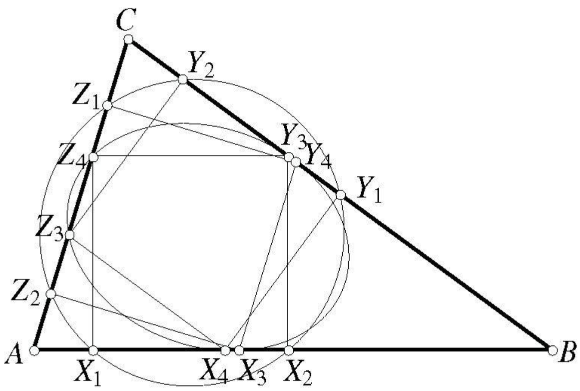

Fig. 2

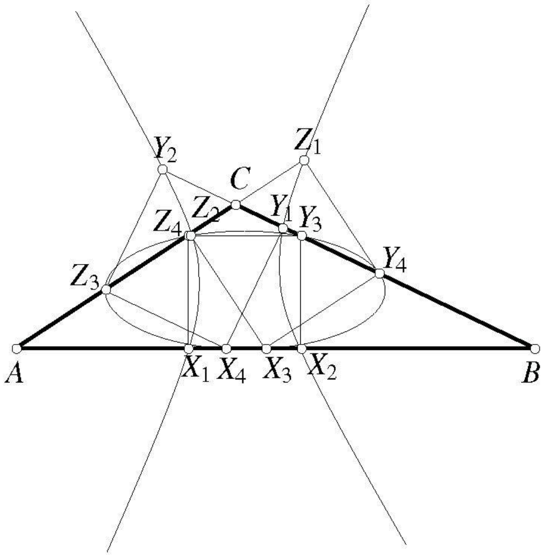

Fig. 3

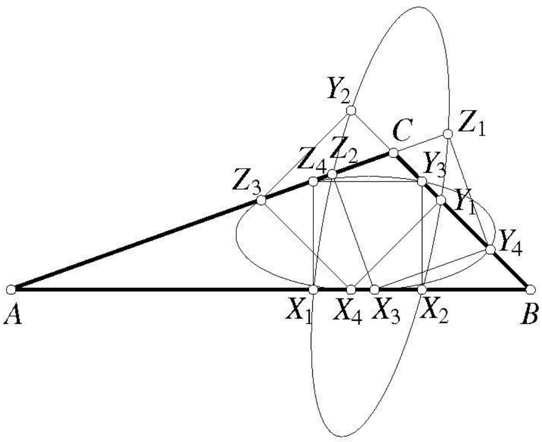

Fig. 4

Proof. To prove the result, let ABC be the triangle; we may assume that AB is the longest side, and we can consider a Cartesian system of coordinates such that

$$A = (0, 0),\ B = (1, 0),\ C = (a, b) \text{ with } a \in (0, 1],\ b \in (0, 1].$$

In this system, after a calculation we have:

$$X_1 = \Gamma(a, 0),\ X_2 = \Gamma(a + b, 0),\ Y_3 = \Gamma(a + b, b),\ Z_4 = \Gamma(a, b),$$
$$Y_1 = \Delta(b + 1, -a + 1),\ Y_2 = \Delta(a + b, -a + b + 1),\ Z_3 = \Delta(a, b),\ X_4 = \Delta(1, 0),$$
$$Z_1 = \Lambda(a^2 + ab, ab + b^2),\ Z_2 = \Lambda(a^2, ab),\ X_3 = \Lambda(a^2 + b^2, 0),$$
$$Y_4 = \Lambda(a^2 + b^2 + ab, b^2)$$

where Γ, Δ and Λ have positive values:

$$\Gamma = \frac{1}{b+1},\ \Delta = \frac{b}{a^2 + b^2 - 2a + b + 1},\ \Lambda = \frac{1}{a^2 + b^2 + b}.$$

Then, with a long but straightforward calculation, we find that the points X_1, X_2, Y_1, Y_2, Z_1, Z_2 verify the following equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

with

$$A = b^2(b + 1)^2,$$
$$B = -3a^4 - a^2b^2 + b^4 + 6a^3 + ab^2 + 2b^3 - 3a^2 + b^2,$$
$$C = b(2a - 1)(2a^2 + b^2 - 2a),$$
$$D = -b^2(b + 1)(2a + b),$$
$$E = -b(2a + b)(a^2 + b^2 - ab - a + b),$$
$$F = ab^2(a + b).$$

Also, with another long but straightforward calculation, we find that the points Y_3, Z_4, Z_3, X_4, X_3, Y_4 verify the following equation

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

with

$$A = b^2(a^2 + b^2 + b)(a^2 + b^2 - 2a + b + 1),$$
$$B = a^6 + 2a^4b^2 + a^2b^4 - 3a^5 + b^5 + 3a^4b - 4a^3b^2 + 4a^2b^3 - ab^4 + 4a^4 + 2b^4 - 6a^3b + 5a^2b^2 - 4ab^3 - 3a^3 + 2b^3 + 3a^2b - 3ab^2 + a^2 + b^2,$$
$$C = -b(2a - 1)(a^2 + b^2 - a + b)^2,$$
$$D = -b^2(a^4 + b^4 + 2a^2b^2 - 2a^3 + 2a^2b - 2ab^2 + 2b^3 + a^2 + 2b^2),$$
$$E = b(a^5 + 2a^3b^2 + ab^4 - 3a^4 - 2b^4 + 2a^3b - 5a^2b^2 + 2ab^3 + 3a^3 - 2a^2b + 4ab^2 - 2b^3 - a^2 - b^2),$$
$$F = b^3(a^2 + b^2).$$

Now that we have the previous equations, we can easily check that the first one corresponds to a conic which is not necessarily an ellipse, whereas the second one necessarily corresponds to an ellipse. □

Remark. If instead of considering the three inscribed squares we consider the three inscribed equilateral triangles each with a side parallel to a side of ABC, then we can find similar results; see some of them in [2].

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References

[1] Hatzipolakis, A.P.; Yiu, P.: The Lucas circles of a triangle. *Amer. Math. Monthly* 108 (2001), 444–446.

[2] Herrera, B.: The notable configuration of the inscribed equilateral triangles in a triangle. *Elem. Math.* 57 (2002), 1–6.

[3] Panakis, I.: Plane Trigonometry, Vol II. *Private publication*, Athens (1973).

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